Report on the activities of the focused research group on Computability and Complexity of statistical behavior of Dynamical Systems

For all practical purposes, the world around us is not a deterministic one. Even if a simple physical system can be described deterministically, say by the laws of Newtonian mechanics, the differential equations expressing these laws typically cannot be solved explicitly. Computers are generally not much help either: of course, a system of ODEs can be solved numerically, but the solution will inevitably come with an error due to round-offs of computations and inputs. Commonly, solutions of dynamical systems are very sensitive to such small errors (the phenomenon known as "Chaos"), so the same computation can give wildly different numerical results.

Of course, this difficulty is well known to the practitioners, who analyze chaotic dynamical systems in the language of statistics, based on what is broadly known as *Monte Carlo* technique, pioneered by Ulam and von Neumann in 1946 [6]. Informally speaking, we can throw lots of random darts to select a large number of sets of initial values; run our simulation for the desired duration for each of them; then statistically average the outcomes. We then expect these averages to reflect the true statistics of our system. To set the stage more formally, let us assume that we have a discrete-time (a continuous-time case will require an obvious adjustment) dynamical system

$$f : D \to D, \text{ where } D \text{ is a finite domain in } \mathbb{R}^n$$

that we would like to study. Let $\bar{x}_1, \ldots, \bar{x}_k$ be k points in D randomly chosen, for some $k1$ and consider the probability measure

$$\mu_{k,n} = \frac{1}{kn} \sum_{l=1}^{k} \sum_{m=1}^{n} \delta_{f^{\circ m}(\bar{x}_l)}, \tag{1}$$

where $\delta_{\bar{x}}$ is the delta-mass at the point $\bar{x} \in \mathbb{R}^n$. The mapping f can either be given by mathematical formulas, or stand for a computer program we wrote to simulate our dynamical system. We then postulate that for $k, n \to \infty$ the probabilities $\mu_{k,n}$ converge to a limiting statistical distribution of our system and thus we can use them to make meaningful long-term *statistical* predictions of its behavior.

Let us say that a measure μ on D is a *physical measure* of f if it is the weak limit of *Birkhoff sums* $\frac{1}{n}\sum_{m=1}^{n} \delta_{f^{\circ m}(\bar{x})}$ for a set of initial values $\bar{x} \in A \subset D$ with positive Lebesgue measure. This means that the limiting statistics of such points will appear in the averages (1) with a non-zero probability. If there is a unique physical measure in our dynamical system, then one random dart in (1) will suffice. Of course, there are systems with many physical measures. For instance, Newhouse [4] showed that a polynomial map f in dimension 2 can have infinitely many attracting basins, on each of which the dynamics will converge to a different stable periodic regime. This in itself, however, is not necessarily an obstacle to the Monte-Carlo method, and indeed, the empirical belief is that it still succeeds. Let us say that a map is *non-statistical* if Birkhoff sums do not converge to a well-defined limit on a positive measure set of initial values (D. Ruelle in [5] called such maps "historic").

The empirical belief in Monte Carlo method appears to be unfounded in some cases. C. Rojas and M. Yampolsky have considered the simplest examples of non-linear dynamical systems: quadratic maps of the interval $[-1,1]$ of the form

$$f_a(x) = ax(1-x),\ a \in (0,4]$$

and found values of a for which:

(1) there exists a *unique* physical measure μ which is the weak limit of

$$\frac{1}{n}\sum_{m=1}^{n} \delta_{f_a^{\circ m}(x)}$$

for Lebesgue almost all $x \in [0,1]$;

(2) the measure μ is not computable.

Thus, the Monte-Carlo computational approach may fail spectacularly for truly simple maps – not because there are no physical measures, or too many of them, but because the "nice" unique limiting statistics cannot be computed, and thus the averages (1) will not converge to anything meaningful in practice.

Furthermore, P. Berger [1] introduced the concept of *emergence*, which is the exponential growth rate of the size of the set of finite Birkhoff sums in the space of probability measures. Positive emergence means that the set of numerical observations for a given dynamical system is "all over the place", depending on the number of steps of iteration and the initial condition. With co-authors [2, 3], Berger showed that positive emergence is a common phenomenon in natural spaces of dynamical systems.

The two approaches: emergence and non-computability have been developed by distinct groups of researchers in parallel to address the same question: *how hard is it to describe a typical dynamical system statistically?* Working together at BIRS led to an exciting synthesis of techniques and ideas. We have answered several important open questions and formulated specific directions of future work. As is seen from the following incomplete list, the answer to the above question could be *very hard indeed* for a given map.

Refining the above result of Rojas-Yampolsky, we have shown:

Theorem. *There exist computable parameters $a \in [0,1]$ for which the map f_a has a unique physical measure μ_a which is not computable.*

Amusingly, using similar techniques we also proved:

Theorem. *For each of the following questions there exist computable parameters $a \in [0,1]$ such that the answers cannot be obtained in ZFC:*

- *is f_a statistical?*
- *is f_a chaotic?*

Since a has a finite description, namely a computer program which upon input n outputs de n-th decimal digit of a, this is quite striking.

We also asked whether it may be possible to construct examples in which μ_a is non-computable, and yet, absolutely continuous, which is commonly seen as a nice characteristic that natural measures should have. This is unknown at present and would be very surprising, if true. However, we mapped out a conjectural approach to producing such examples.

We showed that our techniques can be combined to show non-computability or unprovability in other settings, such as rational maps of the Riemann sphere and area-preserving maps.

We asked whether positive emergence can be shown in families of dissipative maps in which Newhouse phenomenon occurs generically, in particular, the family of dissipative complex quadratic Hénon maps with semi-Siegel points (based on the work of Yampolsky and Yang).

We cannot yet describe a general model for the dynamics that produces emergence or non-computability (similar to how Smale's horseshoe gives a general model for chaotic dynamics). We feel that such a description may be possible, and would be crucial for understanding the occurence and typicality of these phenomena.

The week of brainstorming in Banff has led us to a synthesis of approaches, which has already resulted in breakthroughs, and will undoubtedly lead to more. We are grateful to BIRS for bringing us together for this meeting and for creating an ideal working environment.

References

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