Connections in Infinite Dimensional Dynamics

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1 Overview of the Field

Understanding dramatic changes in the dynamics of physical systems is a critical part of describing real world phenomena like the formation of hurricanes, booms and busts in the stock market, and even the progression of global pandemics. Such dynamical transitions are described by connecting orbits in nonlinear dynamical systems, mainly in the form of ordinary differential equations, partial differential equations and delay differential equations. This workshop focused on combining computational techniques with abstract mathematics to improve our fundamental understanding of transitions in dynamical systems.

2 Open Problems

The meeting was moved to an online, zoom-enabled workshop in response to travel restrictions. Since participants were dispersed through multiple time zones we decided to have a very small number of short talks, leaving as much time for open discussion as possible and making the best use of the roughly 5 hours per day work window.

The idea was to start the meeting with short 15 minutes talks on major open problems in the area, each followed by 15 minutes of conversation. The slogan for the talks was “don’t tell us what you know how to do, tell us what you don’t know how to do”. Six volunteers were found to give such presentations:

1. Connecting orbits for strongly indefinite problems/ill posed PDEs (Jonathan Jaquette)
2. DSGRN and Hill Function Continuation (Konstantin Mischaikow)
3. Stability of nonlinear waves for delayed PDEs (Blake Barker)
4. Forcing and topology from partial information in infinite dimensions (Jan Bouwe van den Berg)
5. Hopf bifurcations in the FitzHugh-Nagumo PDE (Elena Queirolo)
6. Traveling waves for a fourth order problem with exponential nonlinearity (Michael Plum)

3 Scientific Progress Made

While progress was made on all six open problems during the week, for brevity we have opted to report here on three of these.

Concerning *Open Problem 1*, an approach was proposed for computing the connecting orbits between critical points which comprise the boundary operator in Hamiltonian Floer homology. While a semiflow in this problem cannot be defined, there is a well defined notion of stable and unstable eigenvectors about a critical point. By using a variation of constants formula, a solution to the connecting orbit problem may be defined as the fixed point of an integral equation which integrates forward along the stable eigendirections, and backwards along the unstable eigendirections. This approach could be applied to extend recent work on stable manifolds in parabolic PDEs (a topic of discussion at the BIRS 14w5098 workshop) to the strongly indefinite case. Again, by integrating forward along stable directions and backwards along unstable directions, the infinite length connecting orbit problem may be formulated as a finite length boundary value problem, imposing Dirichlet boundary conditions to connect the infinite dimensional (un)stable manifolds at either ends. During the conference, the feasibility of this approach was discussed. It turns out that recently developed techniques for rigorously integrating parabolic PDEs, developed as a result of discussions at another previous BIRS workshop, namely 17w5141, are expected to provide some of the tools we need to rigorously carry out those forward and backward integrations between the manifolds.

Concerning *Open Problem 3*, in recent years rigorous numerics have been used to prove existence and uniqueness of solutions to boundary value problems in DDEs, and also to prove stability of traveling waves in non-delay PDEs with a single spatial dimension. To prove stability of traveling waves with one spatial dimension, the Evans function is a very useful tool because zeros of the Evans function correspond to eigenvalues of the linearized PDE problem. Samaey and Sandstede have recently proposed a strategy for determining stability of pulses for partial differential equations with time delays using the Evans function. During the BIRS workshop, it was proposed to combine this Evans function approach with rigorous numerics to prove stability of traveling wave solutions to PDEs with delay. Through discussions at this workshop, a group of five participants began working and made significant progress on this problem for a particular traveling wave system. By the end of the workshop, they already had the outline of a paper written up and several steps of the numerical proof completed or in progress.

Concerning *Open Problem 5*, the main challenge is to prove that the relevant derivative operator has a simple eigenvalue crossing the imaginary axes, satisfies a shifted Fredholm operator index condition, and a non-resonance condition is met. Promising ideas were put forwards by the participants for each of the three issues mentioned. First, concerning eigenvalue counting, one could attempt to obtain precise information on the eigenvalues of an approximation of the derivative operator, interpret them as approximations of the eigenvalues of the derivative operator and derive analytic bounds on the error. Second, regarding the Fredholm index condition, it was proposed to deduce the operator's Fredholm index from the adjoint operator's properties, possibly combined with using compact perturbations and the continuation property of the Fredholm index. Third, concerning the non-resonance condition, one approach is to construct rough analytic bounds on the position of the eigenvalues and rigorously verify using computer-assistance the non-resonance condition only within the finite region provided by these bounds.

4 Outcome of the Meeting

It was fantastic and extremely valuable to have this opportunity to focus on open problems and meet with collaborators during the Covid-19 pandemic when travel was not feasible. We made substantial progress on a number of problems that would not have otherwise been possible, as discussed above. The BIRS staff were incredibly helpful and the meeting would not have worked at all without their diligent support. What we missed in this online format, as opposed to an in person meeting, were all the side conversations and new contacts that develop naturally in a typical Banff meeting.