FRG report: The crystal structure of the plethysm of Schur functions

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Our focused research group consisted of Laura Colmenarejo, Rosa Orellana, Franco Saliola, Anne Schilling and Mike Zabrocki. We worked at the BIRS station from April 1 until April 8 (except Rosa Orellana who left on April 6).

1 The problem

The irreducible polynomial representations of GL_n are indexed by partitions with at most n parts. Given such a representation indexed by the partition λ, its character is the Schur polynomial

$$s_\lambda = \sum_{T \in \mathrm{SSYT}(\lambda)} x^{\mathrm{weight}(T)}. \tag{1}$$

Composition of these representations becomes composition of their characters, denoted $s_\lambda[s_\mu]$ and this operation is known as the operation of plethysm. Composition of characters is a symmetric polynomial which Littlewood [Lit44] called the (outer) *plethysm*. The objective of our focused research group project is the resolution of the following well known open problem.

Problem 1 *Find a combinatorial interpretation of the coefficients $a^\nu_{\lambda,\mu}$ in the expansion*

$$s_\lambda[s_\mu] = \sum_\nu a^\nu_{\lambda,\mu} s_\nu. \tag{2}$$

In the last more than a century of research in representation theory, the basic problem of understanding the coefficients $a^\nu_{\lambda,\mu}$ has stood as a measure of progress in the field. A related question that we felt was an important first step in the resolution of Problem 1 is the following second approach to understanding the underlying representation theory.

Problem 2 *Find a combinatorial interpretation for the multiplicity of an irreducible S_n module indexed by a partition λ in an irreducible polynomial GL_n module indexed by a partition μ.*

We will discuss briefly below why the resolution of Problem 1 will also solve Problem 2, but we generally believe that the second question we are considering can be used as a guide for the resolution of the first.

We arrived at Banff with two different approaches to this research. Some preparation for this meeting was carried out by email in the weeks before in a discussion about how the different ways of looking at the underlying combinatorics and representation theory are related.

2 Overview of the direction of research

With five people working intensely on this problem we mainly focused on two different (but ultimately equivalent) approaches. Working as two groups, we followed the main ideas of each others progress with the intention that ideas about what would be successful from one approach could be translated to the other .

The first approach was to use the restriction coefficients from GL_n to S_n as a guide to computing the coefficients $a_{\lambda,(r)}$ (a subset of the full plethysm problem, but we expect that this base case could serve as a guide to the more general plethysm problem). The Schur function s_μ is the character of an irreducible polynomial GL_n module and, since the symmetric group S_n is embedded in GL_n, it can also be considered as the character of an S_n module. We are then looking for the multiplicity of an irreducible S_n module indexed by a partition λ in the irreducible GL_n module indexed by the partition μ. There is a theorem due to Littlewood [Lit50] that this multiplicity is equal to the coefficient of s_μ in the plethysm expression $s_\lambda[1 + s_1 + s_2 + s_3 + \ldots]$. Due to other combinatorial considerations, it suffices to understand the plethysm $s_\lambda[s_r]$ to resolve this question.
An idea from a talk by Arun Ram [Ram2017] that we had discussed before arriving at BIRS encouraged us to look closely at an insertion algorithm that combinatorially described the decomposition of $V_n^{\otimes k}$ into S_n irreducibles. The decomposition of $V_n^{\otimes}$ into GL_n irreducible modules is well understood combinatorially through crystals and the Robinson-Schensted-Knuth insertion algorithm. We began with a new insertion algorithm (similar to, but not the same as, the insertion algorithm Ram described in his talk) and then found that the dynamic reading word of Loehr and Warrington [LW12] could be used as a guide to make it compatible with the usual crystal structure of $V^{\otimes k}$. This approach looks extremely promising and it has been the subject of two follow-up research meetings in April.

The second approach that we considered in parallel with the first was to put a crystal structure on combinatorial objects representing the monomial expansion of plethysms. The papers of Marc van Leuwen [vL99] extending the work of Carré and Leclerc [CarLec95] describe the crystal structure of combinatorial objects representing the monomial expansion of $s_2[s_\lambda]$ and $s_{11}[s_\lambda]$ known as domino tableaux. Our main focus here was to extend the known crystal structure on these domino tableaux to ribbon tableaux to compute $s_\mu[s_\lambda]$ for partitions of μ larger or equal to 3. A note at the end of [vL99] indicated that this idea was tried and was known to be difficult, but we had hoped that almost two decades of research and understanding of crystals might make this approach amenable. In particular, the technique of tableaux switching, which recently also has played an crucial role in Schubert calculus, seems to be an important ingredient in this approach.

References

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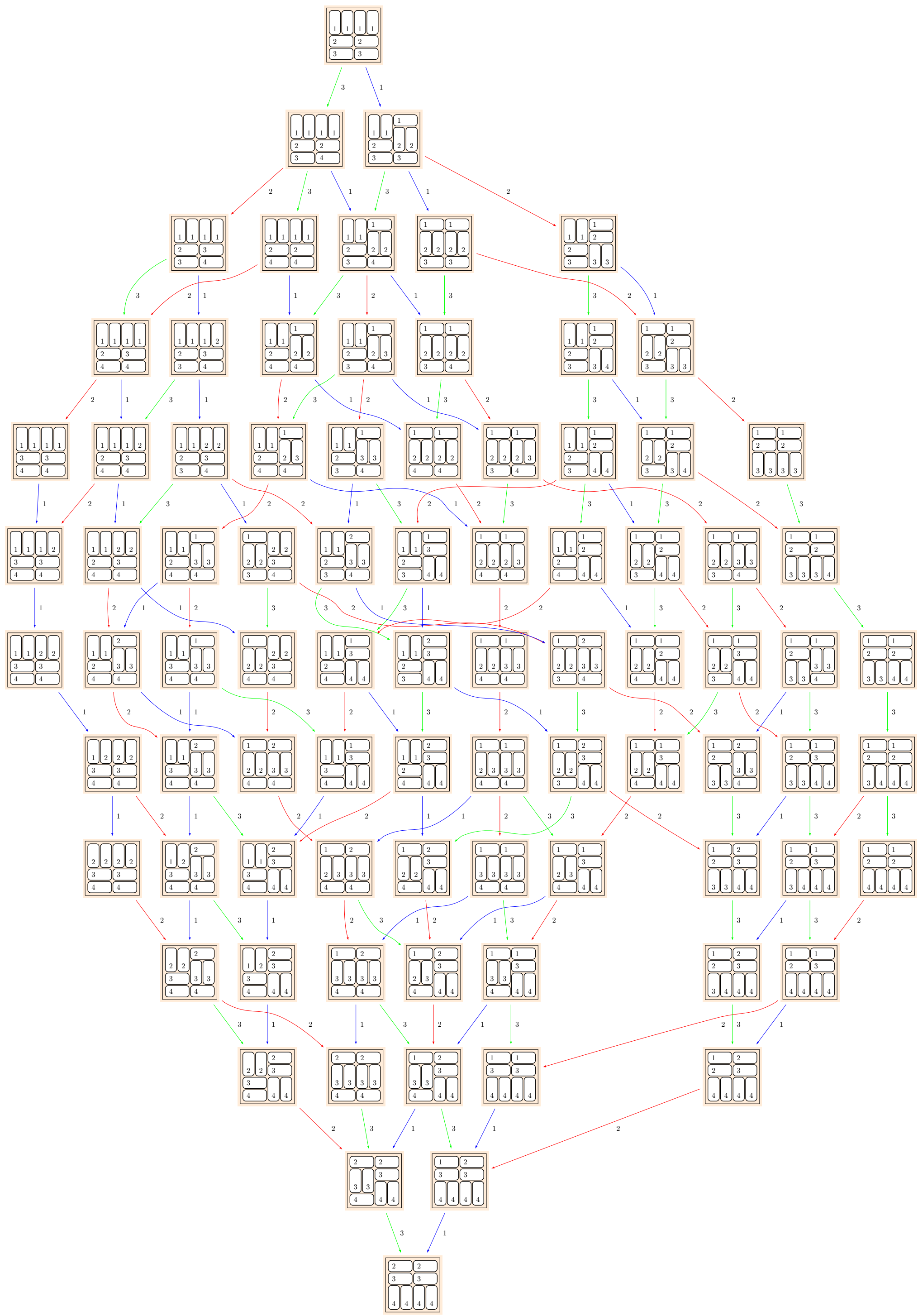

Figure 1: A connected component of the crystal structure on the set of domino tableaux of shape 4444. This component corresponds to the Schur function s_{422} appearing in the plethysm $s_2[s_{22}] = s_{2222} + s_{3311} + s_{422} + s_{44}$.