Recent Advances in Hydrodynamics

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1 Recent Developments and Open Problems[1]

Recent developments in the theory, numerics, and experimental aspects of fluid mechanics have centered on a more quantitative understanding and modeling of fluid phenomena—understanding that goes beyond universal laws and dimensional scaling, even though these still retain an important role.

In particular, the fundamental role that boundaries play in modeling incompressible fluid flows has been underscored by a renewed impetus in studying boundary layers, from both an analytical and modeling point of view. On the analytical side, significant progress has been made in understanding boundary layer thickness and the importance of boundaries, even for addressing regularity. Conditions that guarantee well-posedness of the Prandtl equations have been determined, in tandem with studies of the resulting instabilities and ill-posedness where these conditions are violated [8, 18, 21, 29, 30]. Progress has also been made in justifying the Prandtl approximation under non-classical slip boundary conditions, such as Navier-slip boundary conditions [17] or for special classes of flows, such as flow with symmetry [35, 34] and linear flows [16], and in seeking physically motivated criteria for the vanishing viscosity limit to hold [10, 27, 28, 32]. By the vanishing viscosity limit we mean convergence strongly in the energy norm of solutions of the Navier-Stokes equations to solutions of the Euler equations.

On the computational side, new methods, such as wavelet and penalization methods [36, 37], have emerged that allow for a more accurate and stable resolution of the small scales associated with boundary layers and the modeling of vorticity near walls. Yet, modeling the behavior of viscous fluids near rigid walls remains one of the most outstanding open problems in hydrodynamics today with an impact on many real-life problems, such as skin friction and drag reduction. More specifically, from a mathematical point of view, it is still unknown whether the vanishing viscosity

[1]In view of the large literature on the subject, most of the references cited in this and other sections pertain to relevant results by participants in the workshop and their co-authors.

limit holds for regular data and domains for short time, and if, in this case, the Prandtl approximation to the Navier-Stokes solution is valid; i.e., if it converges to the actual solution as the viscosity vanishes. Furthermore, while it is clear that such an approximation is no longer valid when boundary layer separation occurs, the precise relation between layer separation, the vanishing viscosity limit, and the Prandtl approximation is still not understood, except in special cases.

There are still several open question related to the well-posedness of the Prandtl equations. It is known that they are ill-posed in Sobolev spaces and well-posed for analytic and nearly analytic (Gevrey class) data [19], but the precise Gevrey regularity needed is still open. Boundary layers arise in several other contexts in hydrodynamics. For example, for the Euler-α model, which is a reduced model for an inviscid second-grade fluid, convergence to the Euler solution can be shown under Navier boundary conditions and used for an improved existence result of strong Euler solutions in 3D [4]. Boundary layers are of fundamental importance in modeling flow through porous media, as they impact the resulting filtration law at the macroscale. Modeling of multi-phase flow has improved significantly recently, especially in non-standard geometries, such as karstic geometry [25]. However, a rigorous justification of the filtration laws is still lacking in many regimes.

A related area that has seen a resurgence of interest is flow stability and controllability, from the perspective of both partial differential equations (PDEs) and dynamical systems (including the existence of attractors, such as inertial manifolds). Significant work has been done recently on determining modes and forms for various hydrodynamics equations, also in the context of uncertainty quantification ([15] and references therein). Instabilities around suitably chosen flow profiles, in particular shear flows, have been key to establishing ill-posedness of the Prandtl equations. Fast rotation is recognized as a stabilizing mechanism in three-dimensional flows, by making the flow nearly two-dimensional, leading in this case to global well-posedness of the Navier-Stokes equation. However other stabilizing mechanisms have been recognized, both for inviscid and viscous flows, involving oscillations in time [11] as well as space. In the latter case, vorticity may be transferred to small scales and lead to asymptotic stability even in inviscid flows. This phenomenon is called inviscid damping, akin to Landau damping in kinetic equations [3].

Recent progress has been made on controllability of the Navier-Stokes equation, which has practical applications in engineering problems, but many questions remain unanswered. Controllability is especially relevant in fluid-structure interaction problems, which have been the focus of significant work recently, especially regarding the motion of solid objects in an incompressible fluid [20]. A main open question in this area is the existence of solutions past collision in the inviscid case and when slip is present in the viscous case.

Flow instabilities connect naturally to perhaps the most fundamental open question in mathematical hydrodynamics; that is, the question of finite-time blowup for solutions to the Navier-Stokes and Euler equations. Recent careful numerical simulations seems to suggest that regular Euler solutions may blow up in finite time [23]. In this context, inviscid regularizations of the Euler equations, such as the Euler-α and Euler-Voigt models may provide criteria for blow up that are more easily monitored numerically [31]. In Ref. [23], the possible blow up occurs at the boundary of the domain. In fact, the arguments in Ref. [23] have informed analytical work on the growth of vorticity in two-dimensional Euler flows, for which global existence and uniqueness is known. Examples have been constructed of solutions that saturate the double exponential growth of gradients of vorticity, the theoretical upper bound, again at points on the boundary [26]. The question of whether such growth can occur at interior points remains open. For viscous flows, global well-posedness has been established for reduced, but physically relevant models, in particular for the primitive equations of the oceans and atmosphere, for which it holds under no additional smallness assumptions on the initial data [22]. At the same time, finite-time blow up can occur in this model in the inviscid case [7].

At the level of weak solutions, significant progress has been made in the inviscid case. Using convex integration, weak solutions of the Euler equations known as *wild solutions*, with interesting properties, have been constructed in both two and three dimensions [14]. Strong non-uniqueness holds for these solutions, in part because the behavior of the associated energy can be arbitrarily prescribed. Wild solutions have provided examples of energy-dissipative Euler solutions, which considerably narrow the existing gap in proving Onsager's conjecture. Onsager's conjecture states that flows with velocities that are Hölder continuous of exponent $1/3$ or higher are energy conservative, while there exists energy-dissipative solutions if the exponent is less than $1/3$. The conservative part of the conjecture has been established, with the minimal regularity needed measured in certain Besov spaces (see [9] and references therein). The dissipative part is still open, though the theory of wild solutions allows one to construct energy-dissipative Euler solutions with regularity close to Hölder $1/3$; these solutions are integrable in time but have space regularity less than $1/3$ [13].

Onsager's conjecture is utilized to justify employing irregular, energy-dissipative Euler solutions to model turbulence, as numerically and experimentally it is observed that the rate of energy dissipation does not vanish as the viscosity approaches zero. A major open problem in this context is whether it is possible to construct such dissipative solutions as limits of Navier-Stokes solutions as the viscosity vanishes (at least in the absence of boundaries). While the understanding of turbulence from a rigorous and quantitative point of view is still lacking in many respects, important progress has been made in obtaining rigorous bounds on quantities of interest, such as structure functions, as well as in modeling and experimental observation, even for inhomogeneous and anisotropic turbulence, such as wall-bounded turbulence. On the computational side, progress has been made in the efficiency of pseudospectral methods, such as the method of *implicit dealiasing* [5]. Penalty methods have been developed to exploit the efficiency and high spectral accuracy of the fast Fourier transform in the presence of arbitrary, nonperiodic boundaries (e.g. [36]). For homogeneous Dirichlet boundary conditions, a convergence proof of the penalty method for the Navier-Stokes equations was given by Angot [2]. Advances have also been made in Lagrangian methods, including numerical techniques [33] and experimental procedures for bridging from Eulerian to Lagrangian statistics [24]. Experimentally, improved particle velocimetry allows accurate tracking of particles in turbulent flows, even close to walls. There has also been a renewed interest in Lagrangian methods on the analytical side. One advantage of using a Lagrangian formulation for the Euler equations is that Lagrangian paths are analytic even for rough data.

Lastly, the interplay between geometry, analysis, and mechanics is becoming ever more relevant. On the computational side, improved pseudospectral and wavelet methods, based on tessellation, such as Voronoi cells, can naturally handle curved geometries, as in global circulation and climate models (see [1] for the sphere). On the analytical side, recent work on the fluid equations in hyperbolic geometries and in the relativistic setting [6, 12] has elucidated some of the obstructions created by the Euclidean structure, for example in resolving non-uniqueness of weak Navier-Stokes solutions.

2 Presentation Highlights

The first two talks set the stage for much of what was to come, introducing two of the main themes of the workshop: stability/instability of fluid flow and the role that boundaries play in turbulence.

The opening talk, by Edriss Titi, explained how dispersion can act to stabilize or destabilize the solution to PDEs. He started with the example of rotation, acting as a dispersive mechanism, regularizing the solution to Burger's equation and the Navier-Stokes equations. He contrasted this with the example of the Kuramoto-Sivashinsky equation, in which dispersion acts as a destabilizing mechanism.

In the second talk, Kai Schneider presented, in a very graphical manner, recent, detailed numerical calculations of a (2D) vortex dipole interacting with a (flat) wall with no-slip boundary conditions. The numerics suggest that the Prandtl expansion holds in such a situation up to a critical time, after which the vorticity detaches from the boundary, the Prandtl equations become singular, and the expansion breaks down. The calculations also suggest that beyond the critical time, a new asymptotic description of the flow appears possible. The calculations also indicate energy dissipation following detachment of the boundary layer.

Here are some highlights from a few other notable talks at the workshop:

Nicholas Kevlahan explained his method of using dynamically adaptive wavelets for the solution of the shallow water equations. The wavelets provide both adaptivity of discretization and a way to interpolate between scales. This allows the finest level of resolution to be determined dynamically by local error estimates, and decouples the method of solution from the manner in which the discretization is adapted. This results in a robust, flexible framework for computation that can be efficiently parallelized.

Franck Sueur gave a talk on the controllability of the Navier slip-with-friction boundary condition. This is a recent work where the speaker together with his collaborators showed that given a square integrable and divergence-free vector field, there is a time and a boundary control that makes a weak solution trivial at that time. The strategy of the proof is elegant and seeks solutions of the rescaled problem with vanishing viscosity, and small data. It consists of the following three steps:

- High Reynolds number control on a time scale of order 1.
- Well-prepared dissipation of the boundary layer on a time scale $O(T/\varepsilon)$.
- Low Reynolds number control on a time scale $O(T/\varepsilon)$.

Yasunori Maekawa gave a talk on the Prandtl expansion in the case of the half-space with periodic horizontal boundary conditions. In this work, he is building toward the result that the vanishing viscosity limit holds when the initial data is analytic in a strip near the boundary, improving upon results from the 1990s for data analytic in the whole domain.

Dragos Iftimie talked about the zero-α limit for the α-Euler model. Adam Larios discussed the related Euler-Voigt equations, a regularization of the Euler equations, and a new blowup criteria for the Euler equations based upon the behavior of these solutions (which are globally well-posed) as the parameter α vanishes.

Matthias Hieber's described the extension of global strong well-posedness results for the 3D viscous primitive equations from the L^2 setting (pioneered by Cao and Titi) to the L^p setting.

Jean-Christophe Nave described a very clever and efficient numerical method for calculating the flow map for transport equations as it evolves over time as an iteration of diffeomorphisms. He presented a number of numerical examples based upon passive transport, though the approach is equally applicable to active transport equations like the 2D Euler equations.

Sylvie Monniaux showed how to apply abstract functional analytic techniques to a very practical problem: the solution of the Navier-Stokes equations with time-dependent Navier-type boundary conditions, thereby extending existing (relatively recent) results for the time-independent case.

Sina Ghaemi described particle image velocimetry in near-wall turbulence. Planar and volumetric particle image velocimetry and particle tracking velocimetry was applied to a turbulent channel flow at a Taylor Reynolds number of 190. These state-of-the-art experimental techniques yield accurate measurement of turbulence statistics in the inner layer of wall flows, which is of fundamental importance for the development of passive and active flow control systems. Sina illustrated how passive drag reduction can be achieved using superhydrophobic surfaces or polymeric drag reducers.

3 Scientific Progress Made

The workshop served to promote the development of junior talents in the field and to facilitate the integration of young scientists into the community. The participants exchanged recent research results and discussed challenges and interesting new topics.

In addition, the workshop has enabled several participants (who so far worked on equations without boundaries) to get a state-of-the-art view of boundary effects and their importance to fluid flow. We believe the workshop has provided them with concrete open problems to work on in this context. A similar statement can be made regarding the ever-growing importance of accurate, faithful numerical simulations in modern fluid mechanics.

Moreover, several new collaborations developed between the participants, existing collaborations were advanced, and new directions of research have opened up. In particular:

- Kai Schneider reported that he and Jean-Christophe Nave made significant progress on their paper, “A characteristic mapping method for two-dimensional incompressible Euler equations,” while at the workshop.

- Hantaek Bae worked with Jim Kelliher, and Edriss S. Titi and Slim Ibrahim worked on a continuing common project on primitive equations.

- For Marcelo Disconzi it was an opportunity to finish a manuscript he had been writing with Magdalena Czubak and Chi Hin Chan. He also appreciated the access it gave him to recent developments in areas, such as boundary layers and numerics, that come in contact with his own research, although not specifically topics that he has worked on himself. It also allowed him to exchange ideas about his work with many people who have a different view/perspective on the topic and to interact (and sometimes meet for the first time) with leading researchers in the field (something particularly important to younger researchers).

4 Outcome of the Meeting

On Thursday evening, we held an informal discussion, in which the great majority of the attendees discussed both their experiences at the workshop and where they see the field headed. Some themes that emerged were the following:

- Over the past 10 or 20 years questions related to the regularity of solutions, especially of the Navier-Stokes and Euler equations, have occupied much of the attention and resources of researchers from the pure end of hydrodynamics. There was a trend among the talks at this workshop, reflected in the larger community, toward questions related to the stability of solutions. This was generally applauded for having greater meaning and impact among numerical and experimental researchers, and indeed among engineers and physicists. In addition, instability studies provide a possible means to probe the existence of finite-time singularities in hydrodynamics.

- Another trend observed both in this conference and in the field at large was an increasing focus on the role that boundaries play in affecting the behavior of fluids. This was clearly evident in pure, numerical, and experimental talks at this conference.

- The numerical talks provided what could be viewed as a survey of the state of the art in the computation of inviscid and high Reynolds number fluid flow. These talks were extremely useful for the more pure researchers attending.
- The role of dissipation of energy—Onsager's conjecture—had a prominent place in the conference, including both positive and negative results.
- The α and α-Voigt models, originally conceived as mathematical artifices for regularizing solutions to fluid equations, were of interest to numerical researchers.
- The pure researchers gained useful insight into what issues the numericists find important and what their limitations are.
- The effective use of wavelets in the numerical solution of incompressible fluids equations was seen as real advance (and somewhat surprising since the wavelets cannot be divergence-free if compactly supported).
- Talks also covered control theory (the talk of Franck Sueur) and filtration laws (the talk by Xiaoming Wang).

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