Workshop in Analytic and Probabilistic Combinatorics BIRS-16w5048

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1 Overview of the Field

In analytic combinatorics, the objects of study often come from enumerative or algebraic combinatorics. Applied problems of interest are drawn from classic combinatorics (lattice paths, permutations, integer partitions, combinatorics on words), graph theory, information theory (data compression), number theory, probability (random walks, branching processes such as trees), theoretical computer science (space and time complexity, sorting, searching, hashing), and applied areas, including biological sciences, information sciences, mathematical and statistical physics, and so on. The methods include analytic (complex-valued) approaches, such as analyzing the singularities of the relevant generating functions; symbolic computation (e.g., in Maple, Mathematica, or Sage); multivariate methods; mathematical transforms (Fourier, Laplace, Mellin); etc. One of the main goals of analytic combinatorics is the precise characterization of exact or asymptotic information about the enumeration of combinatorial objects, or about the mean, variance, distribution, etc., of randomly distributed objects. Since modern-day computing platforms allow researchers throughout the sciences to routinely study very large objects, the study of asymptotic properties of objects is more relevant today than ever before.

In probabilistic combinatorics, the objects of study often come from extremal combinatorics or graph theory, or computational complexity theory. The methods used can come from classic or modern probability theory, including the classic "Probabilistic Method" introduced by Paul Erdös in the 1930s. Existence proofs are a common feature in the work of this group, and so are constructive proofs and efficient algorithms.

Topological dynamical systems and ergodic theory were also relevant to the workshop. Important areas in combinatorics like arithmetic combinatorics and combinatorial number theory have benefited and grown from techniques borrowed from ergodic theory, e.g., the celebrated Furstenberg's proof of Szemerédi's theorem. In these contexts, combinatorial structures are often associated to *shift spaces*, the objects of study in symbolic dynamics. The structure of shift spaces is combinatorially rich in itself for they can be defined by forbidding sets of finite configurations (or words) in configurations on lattices. Analytic methods are often applied when studying shift spaces, e.g., dynamic zeta functions, and probabilistic methods come into play when invariant probability measures are associated to shifts spaces, in which case we are in the context of ergodic theory.

2 Open Problems and Recent Developments

Several open problems were presented and discussed, during both the open problem session and the open discussions throughout the workshop. We describe some of these.

2.1 Open Problem Session

2.1.1 D. Bevan. *Irrational asymptotic counting.*

The procedure for the asymptotic enumeration of a combinatorial class $\mathcal{A}$ is well-known: First, construct the formal power series, $A(z) = \sum_{\alpha \in \mathcal{A}} z^{|\alpha|}$. Then, by considering $A(z)$ to be the Taylor series of a function of a complex variable, extract the asymptotics of the coefficients, $|\mathcal{A}_n| = [z^n]A(z)$, from the location and nature of the dominant (least positive real) singularity of $A(z)$, by using results such as those of Flajolet & Odlyzko [9] and Flajolet & Sedgewick [10].

Suppose now that we have an *irrational combinatorial class* $\mathcal{J}$, the sizes of whose objects are not necessarily integers. Under modest restrictions, we can still construct an *irrational power series*, $J(z) = \sum_{\eta\in\mathcal{J}} z^{|\eta|}$, as before, where now $|\eta| \in \mathbb{R}_{\geqslant 0}$. For example, the series for Motzkin paths, in which the up- and down-steps have length $\sqrt{2}$, rather than 1, is

$$1 + z + z^2 + z^{2\sqrt{2}} + z^3 + 3z^{1+2\sqrt{2}} + z^4 + 6z^{2+2\sqrt{2}} + z^5 + z^{4\sqrt{2}} + \dots$$

Every such irrational power series has a radius of convergence in $[0, \infty]$ and is analytic within its disk of convergence, except for a branch cut, which is usually taken to be the negative real axis.

For an irrational power series, $J(z)$, attempting to extract the asymptotics of $|\mathcal{J}_\tau| = [z^\tau]J(z)$ makes little sense, due to the presence of wild fluctuations. However, the partial sums, $|\mathcal{J}_{\leqslant t}| = [z^{\leqslant t}]J(z)$, can be expected to have amenable asymptotic behaviour.

If a class consists of objects whose sizes are all rational with a common denominator d, then a change of variable $z \mapsto z^d$ in the corresponding Puiseux series yields a normal power series. It follows that asymptotics can be extracted from the location and nature of the dominant singularity of the Puiseux series, analogously to the integer-sized scenario. Furthermore, by approximating irrational power series with Puiseux series, it can be shown that the (first-order, exponential) growth rate of an irrational combinatorial class is the reciprocal of the dominant singularity of its irrational power series.

Therefore, it is natural to conjecture that full asymptotics of the partial sums of the coefficients of an irrational power series can always be extracted directly from the location and nature of its dominant singularity, in a manner exactly analogous to the situation with Taylor series.

2.1.2 M. Bóna. *Parking functions.*

Let us assume there are n parking spaces denoted $1, 2, \dots, n$, in that order, on a one way street. Cars $C_1, C_2, \dots, C_n$ sequentially enter the street and try to park. Each car C_i has its preferred parking space $f(i)$. A car will drive to its preferred parking space and try to park there. If the space is occupied, the car will park in the next available space. If the car must leave the street without parking, then the process fails. If $(f(1), f(2), \dots, f(n))$ is a sequence of preferences that allows every car to park, then we call f a *parking function*.

It is well-known that $f\colon [n] \to [n]$ is a parking function if and only if for all $i \in [n]$, there are at most $i-1$ distinct elements $j \in [n]$ for which $i < f(j)$.

Let f be a parking function on $[n]$, and let us say that $\sum_{i=1}^{n} f(i)$ is the *sum* of f.

Now fix n, and let $a_{n,k}$ be the number of parking functions on $[n]$ that have sum k. Miklós Bóna conjectures that the sequence $a_{n,n}, a_{n,n+1}, \dots, a_{n,n(n+1)/2}$ is log-concave.

This conjecture was verified to be true by Richard Stanley on a computer for up to $n = 50$. The real zeros property does not hold. The presenter showed one reason for which he believes that the conjecture holds.

2.1.3 D. Galvin. *Centipedes.*

For a graph G use $i_t(G)$ to denote the number of independent sets (sets of pairwise non-adjacent vertices) of size t in G. Alavi, Malde, Schwenk and Erdős [1] studied the independent set sequence $(i_t(G))_{t\geq 0}$, and showed that it can in general exhibit arbitrary patterns of rises and falls. This is in sharp contrast to the behavior of the sequence $(m_t(G))_{t\geq 0}$, whose tth term is the number of matchings (1-regular subgraphs) of size t in G. By a famous theorem of Heilmann and Lieb [16], the sequence $(m_t(G))_{t\geq 0}$ is always unimodal.

There are families of graphs for which the independent set sequence is always unimodal; for example, Hamidoune [15] showed this for claw-free graphs (graphs without an induced star on four vertices). Alavi et al. [1] posed the following intriguing question:

Question 1 *Is the independent set sequence of every tree unimodal?*

Paths, being claw-free, have unimodal independent set sequences, and it is easy to verify that the same is true for stars; the intuition that what is true for stars and paths is usually true for all trees is probably what lead Alavi et al. to raise their question.

Despite substantial effort, not much progress has been made on this question, with just a few sporadic families of trees being dealt with; see [17, 18, 28, 29] for some results. A partial result for all trees was obtained by Levit and Mandrescu [19]: if G is a tree (and, more generally, if G is bipartite), then the final one third of its independent set sequence is decreasing.

Trees that have not yet been dealt with include:

- caterpillars (paths with pendant stars; these interpolate between paths and stars);
- binary trees; and
- the uniform random tree (for which we simply seek an asymptotic almost sure result).

2.1.4 K. Petersen. *Multidimensional Eulerian numbers.*

Consider a walk on the graph with two loops, L and R, based at the same vertex. In simple opposite reinforcement, when L is chosen another loop is added to R, and when R is chosen another loop is added to L. The set of all possible walks is described by the set of all possible infinite paths in an infinite, directed graph (Bratteli duagram) with vertices (i, j), $i, j \geq 0$, with $j + 1$ edges from (i, j) to $(i + 1, j)$ and $i + 1$ edges from (i, j) to $(i, j + 1)$. The number $i + j$ is called the *level* of (i, j). The number of paths from the root $v_0 = (0, 0)$ to a vertex (i, j) is the *Eulerian number* $A(i + j, j)$. These numbers satisfy the recurrence $A(0, 0) = 1$, $A(n, k) = 0$ for $k \notin \{0, 1, \ldots, n\}$, and $A(n, k) = (n - k + 1)A(n - 1, k - 1) + (k + 1)A(n - 1, k)$ for $n = 1, 2, \ldots, k = 0, 1, \ldots, n$. On the set X of infinite paths in this graph which start at the root there is an interesting map T, called the *adic transformation* or *Vershik map*. For details see [8, 14, 11, 12, 25, 24].

For vertices P, Q in the diagram, denote by $\dim(P, Q)$ the number of paths from P to Q. Cylinder sets determined by initial paths terminating at a common vertex are mapped to one another by powers of T and so they must be assigned equal measure by any invariant measure. For a path $x \in X$, denote by x_n the vertex of x at level n. If μ is a T-invariant ergodic Borel probability measure on X and C is any cylinder set terminating at a vertex P, then

$$\mu(C) = \lim_{n\to\infty} \frac{\dim(P, x_n)}{\dim(v_0, x_n)} \quad \text{for } \mu\text{-almost every } x \in X. \tag{1}$$

This makes it important to study asymptotics of *generalized Eulerian numbers*: $A_{p,q}(i, j)$ is the number of paths in the graph from (p, q) to $(p + i, q + j)$. (The Eulerian number is $A(i + j, j) = A_{0,0}(i, j)$.)

In papers cited below, it was proved that the natural symmetric measure on X is ergodic for T, and indeed it is the unique fully supported ergodic invariant measure. Key ingredients of the proof in [**?**] included an explicit formula for the generalized Eulerian numbers,

$$A_{p,q}(i, j) \;=\; \sum_{t=0}^{i} (-1)^{i-t} \binom{p + q + t + 1}{t} \binom{p + q + i + j + 2}{i - t} (p + 1 + t)^{i+j}, \tag{2}$$

and a monotonicity property of ratios,

$$\frac{A_{p,q}(i, j + 1)}{A_{p,q-1}(i, j + 1)} \;\leq\; \frac{A_{p,q}(i, j)}{A_{p,q-1}(i, j)} \;\leq\; \frac{q + j}{q + 1 + j}\, \frac{A_{p,q}(i + 1, j)}{A_{p,q-1}(i + 1, j)}\;. \tag{3}$$

It is possible that a similar approach could be used on multidimensional Eulerian numbers.

Problem. Consider the Bratteli diagram and adic transformation determined by opposite reinforcement on a graph consisting of m loops based at the same vertex: when a loop is chosen, each of the other loops is reinforced by 1. The *multidimensional generalized Eulerian numbers* are defined to be the path counts in the downward directed infinite graph. The recurrence relation is written easily. (i) Find an exact formula for these numbers, analogous to (2). (ii) Prove monotonicity of ratios, analogous to (3). (iii) Prove that there is a unique fully supported ergodic measure. (iv) Determine the dynamical properties of the adic transformation with respect to this measure. (v) Are there combinatorial consequences, for example to the counting of permutations or other arrangements?

2.1.5 S. Wagner. *The maximum agreement subtree problem.*

The maximum agreement subtree problem arises in the theory of phylogenetics. Consider two leaf-labeled (rooted or unrooted) binary trees T_1 and T_2 with n leaves (i.e., the leaves are labeled $1, 2, \dots, n$). A subset of leaves induces a binary tree in a natural way, by taking the smallest subtree that contains all those leaves and suppressing all vertices of degree 2 (except the root, if rooted trees are considered) afterwards.

If a certain subset S of $\{1, 2, \dots, n\}$ induces the same tree in both T_1 and T_2 (i.e., there is a label-preserving isomorphism), then this tree is called an *agreement subtree* of T_1 and T_2. An agreement subtree of maximum size is called a *maximum agreement subtree*, and the size of such a tree is denoted $\mathrm{MAST}(T_1, T_2)$.

The study of the distribution of $\mathrm{MAST}(T_1, T_2)$ if T_1 and T_2 are randomly generated trees was initiated by Bryant, McKenzie, and Steel [6]. Two different models of randomness were considered: the uniform model (T_1 and T_2 are chosen uniformly at random from all binary trees with n leaves), and the Yule-Harding model (see, e.g., [26, Section 2.5]), which is essentially equivalent to random binary increasing trees and random binary search trees. Simulations suggest that the expected value of $\mathrm{MAST}(T_1, T_2)$ is of order $\Theta(n^a)$ for an exponent $a \approx 1/2$ in both models.

It was shown that the expected value is $O(\sqrt{n}\,)$ under both models [4]. On the other hand, [4] also provides lower bounds of $\Omega(n^{1/8})$ (uniform model) and $\Omega(n^{0.384})$ (Yule-Harding). This raises the natural question for the "correct" exponent in both cases.

We remark that this question is connected to a famous problem of a similar nature: Take two random permutations σ and π of $\{1, 2, \dots, n\}$, and consider the largest subset of $\{1, 2, \dots, n\}$ that induces the same permutation in both σ and π. This is equivalent to determining the largest subset that induces the same permutation in $\sigma^{-1}\pi$ and the identity permutation. This, in turn, is nothing but the *longest increasing subsequence* in $\sigma^{-1}\pi$, and if σ and π are both uniformly random permutations, then so is $\sigma^{-1}\pi$.

The celebrated work of Baik, Deift and Johansson [2] showed that the limiting distribution of the longest increasing subsequence of a uniformly random permutation is the Tracy-Widom distribution. It was known much earlier (and it is much easier to prove) that its typical order of magnitude is $\Theta(\sqrt{n}\,)$. This gives some hope that the correct order of magnitude for the maximum agreement subtree can be determined as well—it might even be as high as $\Theta(\sqrt{n}\,)$.

2.1.6 M. D. Ward. *Toral automorphisms*.

Toral automorphisms $T\colon \mathbb{T}^d \to \mathbb{T}^d$ are defined by square integral matrices $A \in \mathrm{GL}_d(\mathbb{Z})$ satisfying $\det(A) = \pm 1$. They are classified according to the spectra of the matrices defining them, so if the spectra of A is written like

$$|\lambda_1| \geq \cdots \geq |\lambda_s| > 1 = |\lambda_{s+1}| = \cdots = |\lambda_{s+2t}| > |\lambda_{s+2t+1}| \geq \cdots \geq |\lambda_d|,$$

then $T = T_A$ is *ergodic* if no eigenvalue of A of norm 1 is a root of unity, *hyperbolic* if the eigenvalues of A lay outside the unit circle (i.e., $t = 0$), and *quasihyperbolic* if it is ergodic and $t > 0$. The *topological entropy* of T is $h = \sum_{j=1}^{s} \log |\lambda_j|$. Consider periodic orbits of cardinality k, $\tau = \{x, T(x), \dots, T^k(x) = x\}$. In [21], it is shown that for quasihyperbolic toral automorphisms the following analogue of Mertens' theorem holds:

$$M_T(N) := \sum_{|\tau| \leq N} \frac{1}{e^{h|\tau|}} \sim m \log N + C_1 + O(N^{-1}),$$

for some $m = m(A) \in \mathbb{N}$ given by $m = \displaystyle\int_X \prod_{i=1}^{t} (2 - 2\cos(2\pi x_i)\, dx_1 \dots dx_t$, where $X \subset \mathbb{T}^d$ is the closure of $\{(n\theta_1, \dots, n\theta_t) \ : \ n \in \mathbb{Z}\}$, and where $e^{\pm 2\pi i \theta_1}, \dots, e^{\pm 2\pi i \theta_t}$ are the eigenvalues with unit modulus. Let

$$A = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 8 \end{pmatrix} \qquad \text{and} \qquad A^{(n)} := \underbrace{A \oplus A^2 \oplus \cdots \oplus A^n}_{n \text{ times}} \qquad \forall n \geq 1.$$

In such a setup, $A^{(n)}$ defines a quasihyperbolic toral automorphism for each $n \geq 1$, so there is a sequence of integer numbers $m(n) = m(A^{(n)})$, which is `A133871` in [27]. In [13], the precise asymptotic growth of this sequence was obtained together, and several combinatorial properties were discussed.
Problem. Extend the work in [13] to more general classes of quasihyperbolic toral automorphisms.

2.2 Open Discussions

2.2.1 R. Gómez. *Markov shifts and loop systems.*

Let G be a (countable) digraph[1] with a vertex set $V = V(G)$ and edge set $E = E(G) \subset V^2$; its adjacency matrix $A = A(G)$ is therefore a $\{0,1\}$-matrix indexed by V. The *Markov shift* associated to G (or A) is the dynamical system $X = (X, \sigma)$, where $X = X_G = X_A$ is the *space of bi-infinite paths* on G,

$$X := \{\mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in V^{\mathbb{Z}} \ : \ (x_n, x_{n+1}) \in E \ \ \forall n \in \mathbb{Z}\},$$

and $\sigma\colon X \to X$ is the *shift map* defined by the rule $\sigma(\mathbf{x})_n = x_{n+1}$ for every $\mathbf{x} = (x_n)_{n\in\mathbb{Z}} \in X$ and $n \in \mathbb{Z}$. The directed closed paths (or *cycles*) of G correspond to the periodic points of X (i.e., the finite orbits). Let

$$p_n := \#\{\mathbf{x} \in X \ : \ \sigma^n(\mathbf{x}) = \mathbf{x}\} \text{ and } q_n = \#\{\mathbf{x} \in X \ : \ \sigma^n(\mathbf{x}) = \mathbf{x} \text{ and } \sigma^k(\mathbf{x}) \neq \mathbf{x} \ \forall k = 1, \ldots, n-1\}$$

for every $n \geq 1$. The counting sequence $\{p_n\}_{n=1}^{\infty}$ of the periodic points is encoded in an *exp-log* generating function, the so called *dynamical* or *Artin-Mazur zeta function* $\zeta = \zeta_X = \zeta_A$,

$$\zeta(z) := \exp\left(\sum_{n=1}^{\infty} \frac{p_n}{n} z^n\right) = \prod_{n=1}^{\infty} \frac{1}{(1-z^n)^{q_n/n}} = \frac{1}{\det(I - zA)}.$$

For a fixed vertex $v \in V$, consider the class $\mathcal{Q}^{(v)}$ of the *first return loops to* v, i.e., all the (non-empty) closed paths that start and end at v and do not visit v otherwise (other vertices can occur several times). If $\mathcal{Q}^{(v)}$ is a combinatorial class (i.e., if it has a finite number of elements of any given size), then X has *well defined local zeta functions* (a property that does not depend on the vertex v). In this case, the ordinary generating function of $\mathcal{Q}^{(v)}$ satisfies

$$Q^{(v)}(z) := \sum_{n=1}^{\infty} q_n^{(v)} z^n = 1 - \frac{\zeta_{A^{(v)}}(z)}{\zeta_A(z)} = 1 - \frac{\det(I - zA)}{\det(I - zA^{(v)})},$$

where $A^{(v)}$ results from A by removing the row and column corresponding to v. Let $\mathcal{P}^{(v)} = \textsc{Seq}\left(\mathcal{Q}^{(v)}\right)$ be the combinatorial class of cycles in v (not necessarily first return). The sequence schema yields the ordinary generating function of $\mathcal{P}^{(v)}$,

$$P^{(v)}(z) := \sum_{n=0}^{\infty} p_n^{(v)} z^n = \frac{1}{1 - Q^{(v)}(z)}.$$

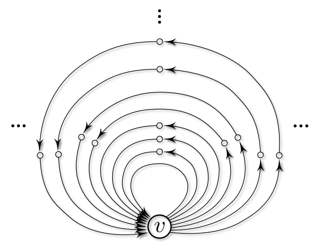
The (*topological*) *entropy* of X is $h(X) = \log \lambda$, where $\lambda := \lim_{n\to\infty} \left(p_n^{(v)}\right)^{1/n}$ (again, it does not depend on the chosen vertex v). Having equal entropy is an equivalence relation h between Markov shifts, which henceforth are assumed to have finite entropy. Recall the probabilistic regime classification for X:

Transient (T)	*Null Recurrent* (NR)	*Positive Recurrent* (PR)	*Strong Positive Recurrent* (SPR)
$Q^{(v)}(1/\lambda) < 1$	$Q^{(v)}(1/\lambda) = 1$and $\sum_{n=1}^{\infty} nq_n^{(v)}/\lambda^n = \infty$	$Q^{(v)}(1/\lambda) = 1$and $\sum_{n=1}^{\infty} nq_n^{(v)}/\lambda^n < \infty$	$\lim\limits_{n\to\infty} \left(q_n^{(v)}\right)^{1/n} < \lambda$

Two Markov shifts defined by two digraphs G and H are *shift dominant equivalent* (SDE) if for some (equiv. every) pair of vertices $v \in V(G)$ and $w \in V(H)$, there exists an integer $K = K(v,w) \geq 0$ such that for every $n \geq 1$,

$$p_n^{(v)} \leq p_{n+K}^{(w)} \qquad \text{and} \qquad p_n^{(w)} \leq p_{n+K}^{(v)}.$$

[1]Henceforth all digraphs are assumed to be strongly connected and *aperiodic*, i.e., the greatest common divisor of the lengths of their cycles is 1.



$$Q^{(v)}(z) = \sum_{n=1}^{\infty} q_n^{(v)} z^n$$

$$q_n^{(v)} = \text{number of first return cycles to } v \text{ of length } n$$

Figure 1: Loop system generated by a power series.

Almost isomorphism (AI) is an equivalence relation introduced in [5], and AI $\Rightarrow$ SDE $\Rightarrow$ h (i.e., two Markov shifts which are AI are necessarily SDE, and being SDE implies that they have equal entropy). For every pair of metasymbols $\mathfrak{X}, \mathfrak{Y} \in \{h, SDE, AI\}$ such that $\mathfrak{X} \Rightarrow \mathfrak{Y}$, write $\mathfrak{X} \gg \mathfrak{Y}$ if there exists uncountably many non $\mathfrak{X}$-equivalent elements in each $\mathfrak{Y}$-equivalence class. We know the following:

SPR	PR	NR	T
AI $\Leftrightarrow$ SDE $\Leftrightarrow h$	AI $\gg$ SDE $\Leftrightarrow h$	AI $\Rightarrow$ SDE $\gg h$	AI $\Rightarrow$ SDE $\gg h$

There is evidence that the symbols $\Rightarrow$ in the NR and T regimes could be replaced by $\gg$. *Loop systems* are Markov shifts defined by generating functions, like $Q^{(v)}$ for example, the corresponding digraph would consist of $q_n^{(v)}$ cycles of length n based on a distinguished vertex v and otherwise vertex disjoint (illustrated in Figure 1). Hence we seek to exhibit an uncountable family of generating functions that induce loop systems which are SDE but not AI. The exponential growth of their coefficients must be the same for it corresponds to the entropy. It is the subexponential growth that must be controlled. We encountered the following asymptotic problem:
Problem. Let $\zeta(s, z) := \sum_{n=1}^{\infty} z^n/n^s$ for every $s > 1$ and $|z| \leq 1$. For each $a \in (1, 2)$, let $F(z) := \zeta(a, z)/\zeta(a)$ and $S(z) := \sum_{n=0}^{\infty} S_n z^n := 1/(1 - F(z))$. Determine the precise asymptotics of the coefficients $\{S_n\}_{n \geq 0}$.

3 Presentation Highlights

The speakers addressed the main areas relevant to the workshop and were chosen with wide diversity. The week began in a notable way, with a motivating main talk by **Helmut Prodinger**. He surveyed 40 years of work on tree enumeration, from the early days of asymptotic analysis to recent developments, showing with great expertise the most important contributions, including some of his own discoveries, and many joint works with his collaborators, some of whom were participants of the workshop. Other topics related to this area include automorphism groups and profiles of random trees, Pólya trees, fringe subtrees, Galton-Watson trees and applications to cancer data.

Special integer sequences like Stirling, Euler, Lucas, Schröder and Lah numbers were the subject of several of the talks, many of which presented studies on parameters like the number of factors, efficient computations and generalizations. Asymptotic behavior was common to many of the investigations, no matter the technical context. Asymptotic themes were a common thread, woven through analytic, dynamic, and probabilistic analysis:

- *Analytic.* Singularity analysis [9, 10], saddle point approach, integral transformations, Mellin-Perron summation formulas, dynamic zeta functions, analytic combinatorics in several variables, etc.

- *Probabilistic.* Method of moments, Pólya urn models, coupling with branching Markov chains, Lévy processes, martingales, Kernel method, etc.

James Fill delivered a main talk on the analysis of the Quicksort algorithm. He discussed recent results on the existence of a local limit theorem for key comparisons, a joint work with B. Bollobás and O. Riordan. He also presented a survey that highlighted the importance of the algorithm, showing the work of many people as well as his own contributions with several collaborators. He also discussed several open problems, e.g., on unimodal densities and infinitely divisible distributions.

Other topics considered in the talks included asymptotic behavior of coefficients of polynomials with only unit roots, permutations (with striking presentations on enumerative and analytic aspects of sorting with C-machines as well as on staircases), graphs with marked subgraphs, Young tableaux, and the game of memory. Dynamical systems were considered too, from finite dynamics coming from iteration of mappings to multidimensional random shifts of finite type.

Analytic combinatorics in several variables remarkably arose jointly with probability theory in the main talk of **Marni Mishna**, who considered weighted lattice paths, and in particular, the Gouyou-Beauchamps model. Her talk exhibited the power of multivariate asymptotics in the style of Pemantle & Wilson [22, 23]. She showed how the weights intervene and clearly described the multivariate asymptotic behavior along the various parameters' ranges. She posted questions, mentioned recent advances, and discussed several enticing open problems on this interesting subject.

Michael Drmota gave the last main talk, on subgraph counting in series parallel graphs. His presentation demonstrated the strength of analytic combinatorics by exhibiting technical expertise applied to a fascinating problem. Based on block decomposition, it was shown how generating functions can be obtained for series parallel graphs. Then he explained results on Gaussian limiting distributions of additive parameters of subcritical graph classes. He utilized techniques developed by other participants like the Quasi-Power Theorem (by H. K. Hwang). One main ingredient was a lemma obtained by Drmota, in joint work with B. Gittenberger and J. F. Morgenbesser, regarding a local analytic approximation to a system of an infinite number of equations.

4 New Projects and Scientific Progress

The wide variety of specialties present at the workshop led to numerous new collaborations, even new perspectives to tackle old open problems. For instance:

- J. Pantone mentions that he started to work with C. Banderier on examples of combinatorial objects that have functional equations that exhibit the same kind of properties as the open questions from Pantone's talk (namely, that they are not differentially algebraic and their functional equations have cross variable substitution). M. Mishna talked about how the open questions from Pantone's work relate to several of Mishna's results on lattice walks in the plane. She indicated several possible avenues of attack to make progress on the problem.

- A. Morales presented his recent work with G. Panova and I. Pak about the enumeration of skew Young tableaux. In particular, they found an asymptotic form for the enumeration of a particular family of skew Young tableaux up to some unknown constant C. J. Pantone mentioned an algorithmic method to predict such constants, so Morales asked him if his technique would work on the sequence of Morales, Panova, and Pak. Using the 50 terms of the sequence that Morales provided, Pantone was able to provide a rough estimate of C. Since then, Pantone has generated the next 150 terms, and is working to obtain more accurate estimates.

- C. Mailler and G. Uribe Bravo had been working on projects related to random graphs. During the workshop, and (in particular) during the talk of M. Drmota, they came across a very interesting class of random graphs that they had not encountered previously. During the informal discussions that took place on Friday, they explored the possibilities of this class and decided to pursue a joint project on the profiles of series parallel graphs.

- M. D. Ward proposed an approach to the problem in section §2.2.1, namely to apply saddle point method, followed by Mellin transforms and beta integrals. It is promising because there is a saddle point, and the Mellin transform $L^*(s)$ applied to $L(t) := \ln(S(e^t))$ behaves nicely, so that there is an explicit expression for $L^*(s)$. We expect that beta integrals will help to estimate $L^*(s)$ and be able to revert the process to get the desired result: $S_n = \Theta(1/\log n)$.

- To address the problem in section §2.1.6, R. Gómez looked at "divisor matrices" (introduced by Redheffer in 1977): If $A^{(n)}$ is the $\{0,1\}$-matrix defined entrywise for every $i, j \in \{1, \ldots, n\}$ by the rule $A^{(n)}(i,j) = 1$ if and only if $j = 1$ or $i|j$, then $\det(A^{(n)}) = M(n)$, the Merten's function. It is known that $\#M^{-1}(\pm 1) = \infty$, so infinitely many $A^{(n)}$ define toral automorphisms. We learned that none of these can be quasihyperbolic, even in a more general setting in which all the formal manipulation of Dirichlet series can be reproduced [7]. Now we are considering other possible generalizations and also looking at integral orthogonal matrices.

- M. Boyle's testimonial expresses optimism about finding tools in the work of Pemantle & Wilson [22, 23] to finish a proof of an old open problem. To roughly describe it, assume the context of section §2.2.1 (restricted to finite digraphs). A *Markov chain* is an assignment P of transition probabilities on the outgoing edges of each of the vertices of a digraph G,[2] thus P induces a *Markov measure* on X_G, i.e., a σ-invariant Borel probability measure with the Markov property. Equal topological entropy is not enough to guarantee the existence of a *measure preserving* AI between two Markov chains: new invariants arise in this general measure theoretical context; they are known as the *greeks*[3], (Δ, β). Measure preserving AIs belong to the class of the so called *finitary isomorphisms with finite expected coding time* (FECT). Boyle referred to an old, open problem, to classify the latter (it goes back to the work of W. Parry from 1979); it is expected that the *greeks* form a complete set of invariants for finitary isomorphisms with FECT. What remains to be shown is the existence of such an isomorphism between two Markov chains with the same *greeks*. The AI construction in [5] corresponds to the case when Δ is trivial (single variable). For general Markov chains, Δ is finitely generated (several variables). We are hoping that the analytic single variable arguments used in [5] to estimate growth rates of finite orbits can be adapted to this multivariable setting to control the asymptotic growth of *weighted* finite orbits, with the ultimate goal of constructing an AI that preserves the Markov measures of two Markov chains with the same *greeks*.

- Testimonials from other participants, like those from E. de Panafieu and I. Mező, also point towards learning new problems and techniques as well as initiating new collaborations. For instance, I. Mező was invited by H. K. Hwang for a weeklong visit at Academica Sinica, to work with him, and with Sara Kropf. Another potential collaboration was initiated by H. Prodinger, who showed to M. D. Ward some computations about protected nodes, a rapidly developing area that is also a topic of research by M. Bóna. Finally, R. Gómez and K. Petersen also disscused various problems during the workshop, a possible collaboration between them could rise.

5 Summary of the Meeting

The workshop brought together two main groups of researchers working in analytic and probabilistic combinatorics. It also included researches working in ergodic theory, specifically symbolic dynamical systems. The variety of subjects was enriching and motivating, which gave rise to several new collaborations and projects. In addition to creating synergy to promote teamwork and to facilitate learning different tools, general diversity was achieved:

[2]Hence P is encoded as a stochastic matrix *compatible* with $A = A(G)$ (i.e., $A_{u,v} \neq 0$ if and only if $P_{u,v} \neq 0$ for every $u, v \in V$).

[3]The *greeks* associated to a Markov chain P is the 4-tuple $(\Gamma, \Delta, c\Delta, \beta)$ defined as follows. First, given a group $\mathcal{G}$, let $\langle K \rangle_{\mathcal{G}}$ denote the subgroup generated by a subset $K \subset \mathcal{G}$.

- The *gamma group* is $\Gamma := \left\langle \{wt(\gamma) := P_{x_0,x_1} \ldots P_{x_{n-1},x_n} \ : \ \gamma = x_0, \ldots, x_n \text{ is a closed directed path in } G\} \right\rangle_{(\mathbb{R}^+,\times)}$.
- The *delta group* is $\Delta := \left\{ \frac{wt(\gamma)}{wt(\gamma')} : \gamma \text{ and } \gamma' \text{ are cycles in } G \text{ of equal length} \right\} \leq \Gamma$.
- $c\Delta$ is the *distinguished generator* of the cyclic group Γ/Δ, where

$$c := wt(\gamma)/wt(\gamma') \text{ with } \gamma \text{ and } \gamma' \text{ cycles satisfying length}(\gamma) = \text{length}(\gamma') + 1 \text{ (well defined modulo } \Delta).$$

- The *beta function* is $\beta(t) :=$ Spectral radius (P^t) (the entries of the P^t are the exponential functions $P^t_{u,v}$ for every $u, v \in V$, with $0^0 = 0$, for example $P^0 = A$ by compatibility, in particular $\log \beta(0)$ is the topological entropy).

Technical arguments (see [20]) can reduce the greeks to the pair (Δ, β) by setting $\Gamma = \Delta$.

- Participants came from the US, Canada, México, Scotland, South Africa, China, Taiwan, Austria and France.
- Six participants were women, and four of them gave talks, one of which was a main talk (and the other two female participants were students).
- The participants ranged from consolidated and internationally recognized experts to young researchers and some students.

Figure 2: Participants of the Workshop in Analytic and Probabilistic Combinatorics, Banff 2016.

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