Multi-Banach algebras and Fourier algebras

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1 Overview of the Field

The theory of multi-norms has been developed over several years.

We first recall the definition of a multi-normed space. Let E be a Banach space, and denote by E^n the n-fold Cartesian product of E; $\mathfrak{S}_n$ is the set of permutations of $\{1 \ldots, n\}$.

A *power-norm* based on E is a sequence $(\|\cdot\|_n : n \in \mathbb{N})$ such that $\|\cdot\|_n$ is a norm on E^n and such that the following Axioms (A1)–(A3) are satisfied for each $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in E$:

(A1) $\left\|(x_{\sigma(1)}, \ldots, x_{\sigma(n)})\right\|_n = \|(x_1, \ldots, x_n)\|_n \quad (\sigma \in \mathfrak{S}_n)$;

(A2) $\|(\alpha_1 x_1, \ldots, \alpha_n x_n)\|_n \leq (\max_{i=1,\ldots,n} |\alpha_i|) \|(x_1, \ldots, x_n)\|_n \quad (\alpha_1, \ldots, \alpha_n \in \mathbb{C})$;

(A3) $\|(x_1, \ldots, x_n, 0)\|_{n+1} = \|(x_1, \ldots, x_n)\|_n$.

A *multi-norm* is a power-norm that satisfies the additional axiom:

(A4) $\|(x_1, \ldots, x_{n-1}, x_n, x_n)\|_{n+1} = \|(x_1, \ldots, x_n)\|_n$.

A *dual multi-norm* is a power-norm that satisfies the additional axiom:

(B4) $\|(x_1, \ldots, x_{n-1}, x_n, x_n)\|_{n+1} = \|(x_1, \ldots, x_{n-1}, 2x_n)\|_n$

for each $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in E$.

These notions were introduced by Dales and Polyakov in [7]; there is a substantial theory of the equivalences of multi-norms in [5] and, more recently, in [2].

There is a variant of the notion of multi-norms, that of p–multi-norms. This is the topic of the memoir [6]. Indeed, let E be a linear space, and take p with $1 \leq p \leq \infty$. A *p–multi-norm based on E* is a sequence $(\|\cdot\|_n : n \in \mathbb{N})$ such that $\|\cdot\|_n$ is a norm on E^n for each $n \in \mathbb{N}$ and such that

$$\|T\boldsymbol{x}\|_m \leq \|T : \ell^p_n \to \ell^p_m\| \, \|\boldsymbol{x}\|_n \quad (T \in \mathbb{M}_{m,n}, \, \boldsymbol{x} \in E^n, \, m, n \in \mathbb{N}) \, .$$

Each p–multi-norm is a power-norm; characterizations given in [7] show that ∞–multi-norms and 1–multi-norms in the above sense are exactly multi-norms and dual multi-norms, respectively, and so this new definition generalizes the old one given for the cases $p = 1$ and $p = \infty$.

The memoir [6] defines the 'canonical p–multi-norm' based on a Banach lattice, and shows that, subject to certain additional, necessary conditions, every p–multi-norm can be represented as the restriction of a canonical p–multi-norm on a Banach lattice to a closed subspace of the lattice.

The definition of a p–multi-norm can be reformulated in terms of tensor products, and so the theory can be regarded as a study of certain Banach-space tensor products. The theory is also related to that of absolutely summing operators.

Let G be a locally compact group. We recall that the group algebra $L^1(G)$ is a Banach algebra (see [3] for the theory of Banach algebras); when G is abelian, $L^1(G)$ is identified via the Fourier transform with the Fourier algebra $A(\Gamma)$, where Γ is the dual group of G. There is a generalization of the definition of $A(\Gamma)$ to the case where Γ is an arbitrary (non-abelian) locally compact group; see [8] for the seminal paper. There are several related algebras, including $B(\Gamma)$, the non-abelian analogue of the measure algebra on a locally compact group. For recent papers on these algebras, see [9, 10], for example. One of our goals is to form a deeper connection between the theories of multi-norms and of Fourier algebras.

A new notion that we are considering is that of a 'multi-Banach algebra'. Thus let A be a Banach algebra, and suppose that $(\|\cdot\|_n)$ is a multi-norm based on A (with $\|\cdot\|_1$ the given norm on A). Then $(A^n, (\|\cdot\|_n))$ is a *multi-Banach algebra* if there is a constant $C > 0$ such that

$$\|(a_1b_1, \ldots, a_nb_n)\|_n \leq C\,\|(a_1, \ldots, a_n)\|_n\;\|(b_1, \ldots, b_n)\|_n$$

for each $n \in \mathbb{N}$ and each $a_1, \ldots, a_n, b_1, \ldots, b_n \in A$.

2 Recent Developments and Open Problems

As mentioned, a recent development in the theory of multi-norms is the memoir [6]. This memoir contains a rather extensive study of p–Banach spaces, including famous theorems of Herz and Kwapień, which are given new proofs.

There are further generalizations of these notions in the paper [1].

3 Presentation Highlights

Since this was a workshop for three people assembled for 'Research in Teams', there were no formal presentations.

4 Scientific Progress Made

The major contribution made at this RIT was the completition of a book and a memoir that had been in progress for some time.

Participants Dales and Troitsky at the present RIT are two of the four authors of the memoir [6]. The timing of the week was such that just before the meeting an alleged final version of the memoir was presented. This was studied by Dales and Troitsky at BIRS, and various errors and improvements were discovered; changes were made. Shortly after this week, the memoir was submitted for publication.

A preprint of this memoir is available from any of the four authors.

Participants Dales and Lau at RIT are two of the four authors of the book [4]. This monograph, which was born at an earlier BIRS meeting of the four authors in July 2010 and developed at a subsequent BIRS meeting in May 2012 and other meetings, studies the space $C_0(K)$ of continuous functions that vanish at infinity on a locally compact space K. It considers when $C_0(K)$ is the dual of another Banach space, either isomorphically or isometrically, and how unique such a predual is. It also gives various new constructions of the bidual of such a space; this bidual space has the form $C(\widetilde{K})$, where $\widetilde{K}$ is the compact space that is the 'hyper-Stonean envelope' of the space K, and the constructions describe $\widetilde{K}$ in several different ways, some new. Finally [4] considers whether a space $C(X)$ for a compact space X such that $C(X)$ is the bidual of a Banach space E is necessarily of the form $C(\widetilde{K})$ for some compact space K. This is shown to be the case when E is separable, and indeed there are then only two such spaces K up to homeomorphism; the question remains open for general E.

The final pre-submission version of this book was also available a little before the week in BIRS, and so Dales and Lau were able to study this and resolve several remaining issues (with the help of Troitsky). This book will be published by Springer in June 2106.

The fact that we could discuss changes to these two substantial works in person at BIRS was very valuable.

Further progress was also made in the study of multi-Banach algebras by the three participants, and a paper is being prepared. However certain points remain open; we were able to crystallize these to the study of one or two key examples. The application of the theory of L-decompositions and p-variants of these decompositions continues to underlie the theory.

5 Outcome of the Meeting

We expect that the two works mentioned above will both appear in 2016.

Professor Lau has been appointed as a 'Distinguished Faculty Visitor' at Lancaster University for one month in 2016, and we hope to make substantial further progress on our work in that month; we look forward to another meeting at BIRS in 2017 or 2018.

It is a pleasure to thank BIRS for the opportunity to advance our work in such a pleasant setting.

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