Convex Algebraic Geometry

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1 An emerging field

Many geometric objects arising naturally in science and technology possess two desirable properties. They are *convex* and *semialgebraic*. *Convex sets* have the property that one can move between any two of its points along a straight line without leaving the set. *Semialgebraic sets* can be described by combining polynomial inequalities by simple logical operations. The areas of mathematics primarily investigating these objects are *Convex Analysis* and *Real Algebraic Geometry*, respectively. Algorithmically, the property of being convex and a semialgebraic description of a set can both be exploited each on its own. However, at the moment, these methods are totally different and disjoint with huge limitations.

Convexity can lead to very fast numerical algorithms for navigating a geometric object. However, for these algorithms to work, one needs additional structure such as an easily computable self-concordant barrier function on the interior of the set [17]. For semialgebraic sets, very general symbolic algorithms are known to investigate and handle them [4]. However, these algorithms are often not efficient enough for practical purposes.

In spite of their ubiquity, the investigation of the special features of convex semialgebraic sets have been neglected for a long time. Only in recent years have new results and methods come up that have resulted in these geometric objects receiving attention from a wide range of areas including Classical Algebraic Geometry, Complexity Theory, Control Theory, Convex Geometry, Functional Analysis, Optimization Theory and Real Algebraic Geometry [22, 13, 15, 24, 1, 27]. Starting less than a decade ago, there have been more and more meetings where people from some of these areas have come together, with convex semialgebraic sets serving as a central tool of common interest.

The motivation behind organizing this meeting was the realization that it is now time for the emergence of an area of research where convex semialgebraic sets are the central objects of study rather than supporting tools. We call this area *Convex Algebraic Geometry* and it is devoted to the systematic study of convex semialgebraic sets.

2 The objects of study

Our objects of study live most of the the time in an n-dimensional Euclidean space, i.e., a space spanned by n axes, any two of which are perpendicular. One-dimensional space consists just of one axis, and its convex subsets are intervals (which also happen to be semialgebraic).

The first non-trivial, but still very special case, is that of two-dimensional space. This is a plane spanned by two axes which meet in an *origin*. Figures in this plane can often be nicely visualized by drawing them on a sheet of paper. Examples of convex semialgebraic subsets of the plane include single points, line segments, open and closed discs (more generally, an open disc together with a *finite* number of connected subsets of its boundary), the closed or open area circumscribed by a triangle, a trapezoid or an octagon (or more generally, a convex polygon). It is also possible to round the corners of such shapes. As another example, the set of all points (x, y) with $y \geq x^2$ (the area above a parabola) is a convex semialgebraic set, but we cannot replace x^2 by $\exp(x)$ here since then we no longer have a semialgebraic set.

Though each of our eyes sees only a two-dimensional picture of our environment, we are used to thinking in three dimensions since three-dimensional space is locally a good model for the space in which we live. Examples of convex semialgebraic subsets of three-dimensional space include balls, cones, pyramids, cylinders and platonic solids like a tetrahedron, a cube, an octrahedron, a dodecahedron, an icosahedron, the small rhombicosidodecahedron or the deltoidal hexecontahedron. Idealized pie slices and houses are also convex and semialgebraic. Again one can round the corners. In reality, an egg is not convex since one can discover little hills on the eggshell by looking at it under a microscope. Also its surface is unlikely to be semialgebraic since it is the result of a biological process. But for all practical purposes we can think of an egg as being convex and semialgebraic. This is also true for the shape of many, but not all, potatoes.

Mathematicians are used to investigating spaces with more than three dimensions. In fact, they carry over almost all geometric notions at least to arbitrary finite dimension. One of the many reasons for this is that our brain has a strong capacity to think in geometric terms, and we want to use this capacity to also understand phenomena which cannot be described by three coordinates only. The most prominent example of this is to think of time as an additional space coordinate. For example, to analyze an ice hockey game, it might be sufficient to think of the positions of the players and the puck at any given time as differently colored points in two dimensional space. Using the third coordinate for time, these positions move along differently colored curves in three-dimensional space which can be seen as a braid. For a football game, it might be more appropriate to start already with three dimensions and add time as a fourth dimension.

By means of analogy (passing from three to four dimensions is much like passing from two to three dimensions) and formal logic, mathematicians manage to extend their geometric intuition to higher dimensions. It is a daily routine for them to think geometrically in high-dimensional spaces. For example, the space of possible states of an engine could consist of many coordinates describing such parameters as the position and speed of the cylinders as well as temperature and pressure inside them. Thinking of it as a geometric object helps in understanding how to steer it from one state to another.

Convexity is highly desirable for many purposes [26, 2, 3]. It is one of the most useful features for navigating a geometric object. The class of semialgebraic sets, on the other hand, is perhaps the most obvious class of nonlinear geometric objects that should, in principle, be amenable to algorithms. Thus convex semialgebraic sets in an arbitrary finite-dimensional space are interesting objects of study especially since techniques which make use of both convexity and the semialgebraic property are ill-developed at present.

3 Spectrahedra and linear matrix inequalities

Symbolic computation with semialgebraic sets is a classical subject. Extensive work has been done on such problems such as, effective real quantifier elimination, computing the connected components of the set, polynomial system solving, and computing the dimension [4]. In the presence of convexity, it should however be possible to solve many of these algorithmic issues in a much more effective way.

Traditionally there are also a lot of techniques, mainly in numerical computation (and here in Convex Optimization [17]) that take advantage of convexity. Perhaps the most prominent example is Linear Programming (LP) which is used in a lot of real world applications.

Until recently, there were very few techniques combining the convex and the semialgebraic points of view. A very interesting new line of research tries to exploit Semidefinite Programming (SDP) for handling convex semialgebraic sets. SDP is an increasingly well-known generalization of LP which still has nice theory and for which good software packages exist. Whereas LP is optimization of a linear function on a polyhedron (i.e., a solution set of a system of linear inequalities), SDP is optimization of a linear function on a *spectrahedron*, i.e., a solution set of a *linear matrix inequality* (LMI). An LMI is an inequality of the form

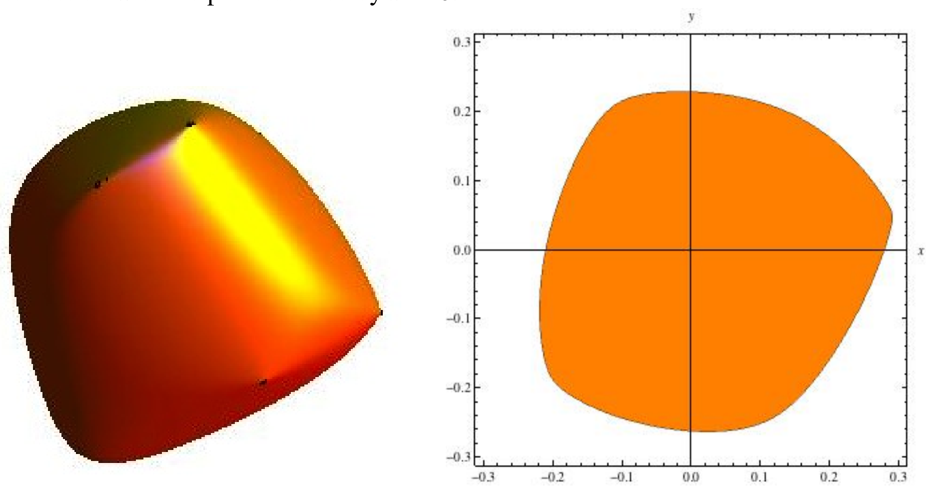
$$L(x) \succeq 0 \qquad (x \in \mathbb{R}^n) \tag{1}$$

where L is a symmetric linear matrix polynomial, i.e.,

$$L = A_0 + X_1 A_1 + \cdots + X_n A_n$$

where each A_i is a symmetric $s \times s$-matrix, the X_i are variables and $\succeq 0$ means positive semidefinite (i.e., all eigenvalues are nonnegative). When one restricts the A_i to be diagonal matrices, then (1) is just a linear system of inequalities. In some vague sense, spectrahedra and SDP generalize polyhedra and LP in much of the same way that symmetric matrices generalize diagonal matrices. Because symmetric matrices can be diagonalized, much of the theory of LP (such as interior point methods, see [17]) goes through for SDP. On the other hand, SDP is much more expressive than LP as can be seen in Figure 1.

Figure 1: A spectrahedron defined by the linear matrix inequality $I + xA + yB + zC \succeq 0 \quad (x, y, z \in \mathbb{R}^3)$ with 10×10 matrices A, B and C whose entries were uniformly and independently chosen among -1, 0 and 1, and its intersection with the plane defined by $z = 0$.



An LMI is likely to be a good representation of a convex semialgebraic set. It makes convexity an obvious feature of the set whereas in a semialgebraic description (a logical formula involving polynomial inequalities) the convexity is usually hidden. One of the current core questions in Convex Algebraic Geometry is which convex semialgebraic sets are defined by an LMI, i.e., are spectrahedra, see Section 4. Another extremely important question is what can be modeled by SDP using slack variables, i.e., which sets are projections (or equivalently, linear images) of spectrahedra, see Section 5.

4 Rigidly convex sets and real zero polynomials

For trivial reasons not every convex semialgebraic set is a spectrahedron. An important question is what makes a convex semialgebraic set a spectrahedron? For example spectrahedra are always closed. It is also known that spectrahedra share other special properties with polyhedra (e.g., they are basic closed and all their faces are exposed). All properties of spectrahedra known at the moment are subsumed by a crucial notion introduced by Helton and Vinnikov called *rigid convexity* [11]. To explain this notion, we need to introduce the notion of real zero (RZ) polynomials.

A polynomial p is a real zero polynomial at $a \in \mathbb{R}^n$ (is RZ at a, for short) if $p(a) > 0$ and all complex zeros of the univariate polynomial obtained by restricting p to a straight line passing through a are real. In other words, a polynomial of degree d is RZ at a point a if it has d real zeros counted with multiplicity on each generic line through a. It can be shown that a polynomial that is RZ at a is also RZ at any point in a small neighborhood of a. We refer to this by saying that the RZ property *spreads out*.

A subset $C \subseteq \mathbb{R}^n$ is called *rigidly convex* if there is a point $a \in \mathbb{R}^n$ and a polynomial p with the real zero property at a such that C equals the closure of the connected component of $\{x \in \mathbb{R}^n \mid p(x) > 0\}$ at a. Note that being convex is not part of the definition of "rigidly convex". However, it can be shown in an elementary way that each rigidly convex set is indeed convex (cf. [25]).

Each spectrahedron is rigidly convex inside its affine hull (i.e., identifying its affine hull with $\mathbb{R}^d$ where d is the dimension of the spectrahedron). To see this, we suppose that we are given a full-dimensional spectrahedron in $\mathbb{R}^n$. Then it can be seen easily that it can be written as $\{x \in \mathbb{R}^n \mid L(x) \succeq 0\}$ for a symmetric linear matrix polynomial L having the additional property that there is $a \in \mathbb{R}^n$ with $L(a) \succ 0$. Here $\succ 0$ stand for *positive definite*, i.e., all eigenvalues are (strictly) positive. Now the determinant of L is easily seen to be RZ at a (essentially because symmetric matrices have all its eigenvalues real) and the given spectrahedron is the closure of the connected component at a of the positivity set of this determinant.

Rigidly convex sets share all of the currently known properties of spectrahedra [25, 20]. In particular, they are semialgebraic sets which are *basic closed*, i.e., can be described by a finite system of weak polynomial inequalities (by means of the so-called *Renegar derivatives* which were the subject of many discussions during the workshop). Also they are convex sets all of whose faces are exposed. Rigid convexity is the strongest property of spectrahedra known so far. If one wants to show that some basic closed semialgebraic set with exposed faces is not a spectrahedron, then at the current state of the art, *the* thing to do, is to show that it is not rigidly convex.

To this end, it is useful to introduce another slight reformulation of rigid convexity based on the notions of algebraic interiors and their minimal polynomials, going back to Helton and Vinnikov as well. An *algebraic interior* in $\mathbb{R}^n$ is the closure of a connected component of the positivity set $\{x \in \mathbb{R}^n \mid p(x) > 0\}$ of a polynomial p (note that it is always closed, and despite the word "interior", never open except if it is the whole space). By definition, rigidly convex sets (and in particular spectrahedra) are algebraic interiors. Such a polynomial p of smallest possible degree is uniquely defined up to a positive constant factor and we call it the *minimal polynomial* of this algebraic interior. A crucial observation is that the minimal polynomial is a factor of every other such polynomial p.

It follows that an algebraic interior is rigidly convex if and only if its minimal polynomial is a real zero polynomial at some of its interior points, or equivalently at *any* of its interior points (since the RZ property spreads out as mentioned above). For example, the television screen like set $\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^4 + x_2^4 \le 1\}$ is an algebraic interior with minimal polynomial $1 - X_1^4 - X_2^4$. This polynomial is not RZ at the origin. Hence the television screen is a convex basic closed semialgebraic set with only exposed faces which is not rigidly convex and therefore not a spectrahedron. On the other hand, the disc $\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \le 1\}$ is an algebraic interior whose minimal polynomial $1 - X_1^2 - X_2^2$ is RZ at the origin. Therefore the disc is rigidly convex. In fact, it is even a spectrahedron since

$$\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1^2 + x_2^2 \le 1\} = \left\{ (x_1, x_2) \in \mathbb{R}^2 \mid \begin{pmatrix} 1 & x_1 & x_2 \\ x_1 & 1 & 0 \\ x_2 & 0 & 1 \end{pmatrix} \succeq 0 \right\}.$$

Starting from spectrahedra which are intrinsically *real* objects, we defined rigidly convex sets and see now that the Zariski closure of their boundaries, seen as a *complex* algebraic varieties are important. This is only one of the many points where classical complex algebraic geometry comes into play. To visualize this thread of thinking, we ask the reader to look again at Figure 1 above and then compare it with the derived Figures 2, 3 and 4 below. Topologically, what you see is a set of nested ovals (which might touch), the innermost of them being the boundary of the convex set we started with.

Helton and Vinnikov showed in their seminal article [11] that each rigidly convex set of dimension at most two is a spectrahedron. As a quite trivial example, we remark that this is a way of seeing that the disc mentioned above is a spectrahedron without explicitly writing down an LMI defining it. Their result relies on the theory of *determinantal representations*. In fact, they even showed that each RZ polynomial, say RZ at the origin, in two variables has a *positive determinantal representation*, i.e. is the determinant of a linear symmetric matrix polynomial $L = A_0 + X_1 A_1 + \cdots + X_n A_n$ where each A_i is a real matrix and A_0 is *positive definite* (in our case $n = 2$). Then the associated rigidly convex set, namely the closure of the connected component of $\{x \in \mathbb{R}^n \mid p(x) > 0\}$ at the origin, equals $\{x \in \mathbb{R}^n \mid L(x) \succeq 0\}$ and therefore is a spectrahedron. This result of Helton and Vinnikov on positive determinantal representation of RZ polynomials in two variables is equivalent to an old conjecture of Peter Lax (who was awarded the Abel Prize in 2005) originally formulated for homogeneous polynomials in three variables, see [14].

One of the most prominent open problems in Convex Algebraic Geometry, subject to many discussions at the workshop, is whether the results of Helton and Vinnikov can be extended to more than two variables. In fact, Helton and Vinnikov conjectured that each rigidly convex set (of any dimension) is a spectrahedron.

Figure 2: The real zero set of the minimal polynomial of the spectrahedron from Figure 1 intersected with the cube $[-1,1]^3$, and its intersection with the plane defined by $z = 0$.

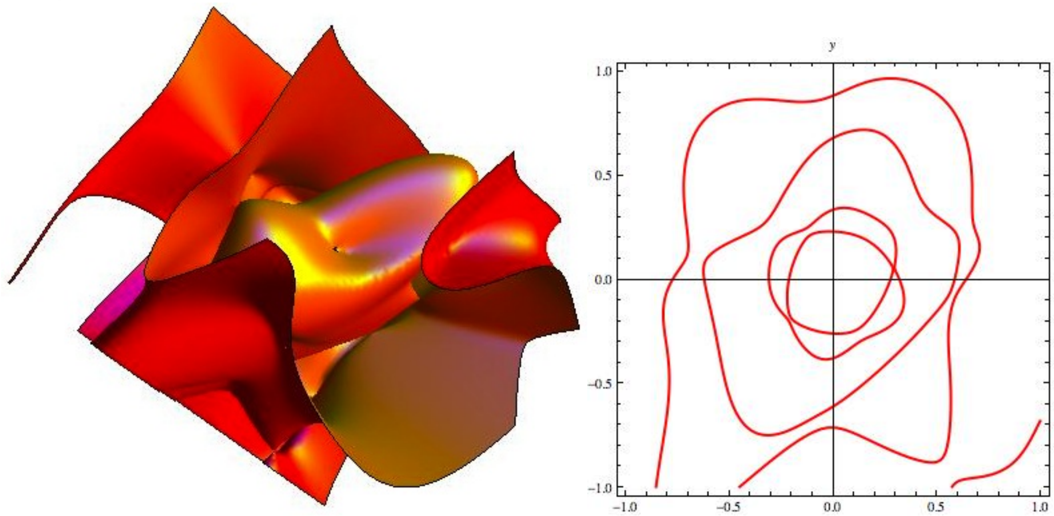
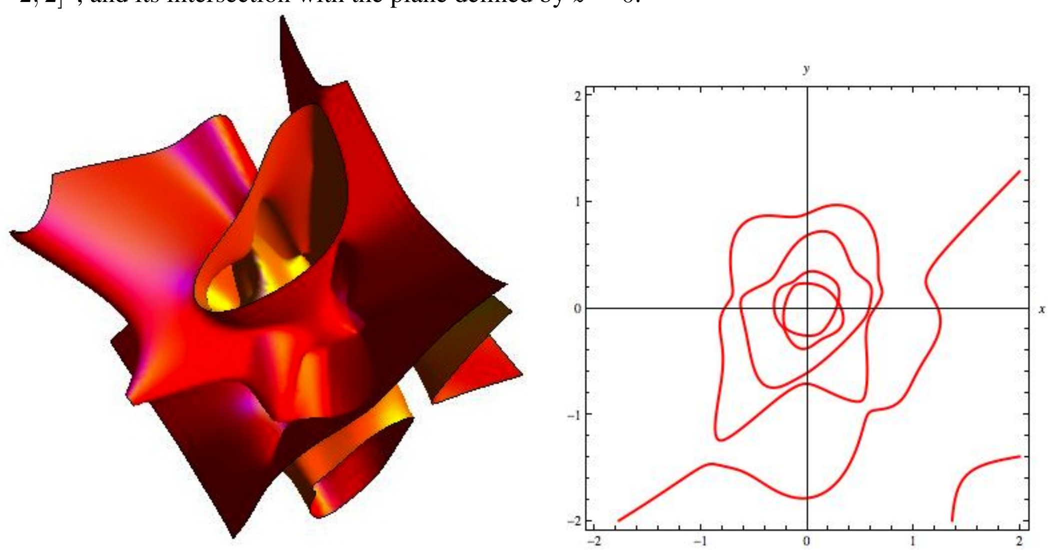
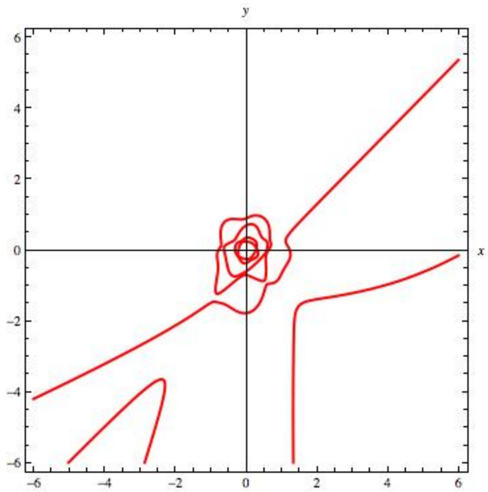


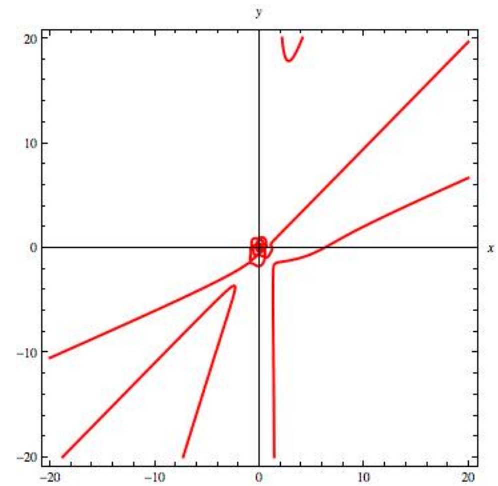
Figure 3: The real zero set of the minimal polynomial of the spectrahedron from Figure 1 intersected with the cube $[-2,2]^3$, and its intersection with the plane defined by $z = 0$.



This very important conjecture is still open. Furthermore, Helton and Vinnikov even conjectured that each RZ polynomial (in any number of variables) has a positive determinantal representation though their proof which uses deep Algebraic Geometry clearly could not be extended to more than two variables. After some discussion among workshop participants, Petter Brändén (Royal Institute of Technology, Stockholm) was able to solve this major problem in the negative during the workshop. This gave rise to an extra talk that Petter Brändén gave on Thursday morning in addition to his regular talk on Wednesday. This special talk was one of the highlights of the workshop since he even gave an extremely sophisticated argument, based on matroid theory, that even a weaker conjecture is false, namely that *some power* of each RZ polynomial has a positive determinantal representation (which would also imply the characterization of spectrahedra by rigid convexity). See [5] for these results, the proof for the stronger conjecture has been simplified since by Tim Netzer [19] who also attended the workshop.

Figure 4: The real zero set of the minimal polynomial of the spectrahedron from Figure 1 intersected with $[-6,6] \times [-6,6] \times \{0\}$ and $[-20,20] \times [-20,20] \times \{0\}$, respectively.





However, there remain many open questions concerning the existence of positive determinantal representations. Some of these would still imply a full characterization of spectrahedra via rigid convexity. Others would work towards it. For example, it is known that the Renegar derivatives [25, 20] of RZ polynomials are again RZ at the same point. The real zero set of the Renegar derivative of a polynomial interlaces the real zero set of the polynomial. More precisely, between any of the two ovals (cf. Figures 2 to 4) and outside of the outermost oval of the real zero set of the RZ polynomial there is an oval of the Renegar derivative. If you draw the ovals of a polynomial and of its Renegar derivative, then the two innermost ovals are boundaries of convex sets, the innermost coming from the polynomial and the second innermost one from its Renegar derivative. Now define the Renegar derivative of a spectrahedron as the rigidly convex set defined by the Renegar derivative of its minimal polynomial. Even the following very special case of the conjecture of Helton and Vinnikov is open: Is the Renegar derivative of a spectrahedron (or at least of a polyhedron) again a spectrahedron?

Also largely open is the question of how to decide whether positive determinantal representations of RZ polynomials exist and how to produce them in an effective way. See [8] for a recent related result and for an overview of what has been done in this direction.

5 Projections of spectrahedra and semidefinite representations

As discussed above, perhaps the most natural class of convex sets going beyond polyhedra that is accessible to effective manipulation consists of spectrahedra. However, many convex semialgebraic sets one would like to deal with in an effective way are not spectrahedra. Whereas the projection (or linear image) of a polyhedron remains a polyhedron, the class of spectrahedra is not closed under projections. As a trivial example, the open half line $\mathbb{R}_{>0}$ of positive real numbers can be written as

$$\mathbb{R}_{>0} = \left\{ x \in \mathbb{R} \mid \exists y \in \mathbb{R} : \begin{pmatrix} x & 1 \\ 1 & y \end{pmatrix} \succeq 0 \right\}$$

but it is not a spectrahedron since it is not closed. During the workshop several propositions were made for naming projections of spectrahedra, including *spectrahedral shadow* and *umbrahedron* (from *umbra*, the latin word for shadow). Here we call projections of a spectrahedron *semidefinitely representable*. A set $S \subseteq \mathbb{R}^n$ obviously is semidefinitely representable if and only if there is a symmetric linear matrix polynomial L in the original variables $X_1, \dots, X_n$ and finitely many additional variables $Y_1, \dots, Y_m$ such that

$$S = \{x \in \mathbb{R}^n \mid \exists y \in \mathbb{R}^m : L(x,y) \succeq 0\}.$$

We call such an L a *semidefinite representation* of S (in the literature it is sometimes also called a "lifted LMI representation").

Having a semidefinite representation of a convex semialgebraic set is very advantageous [16]. For instance, it allows you to optimize a linear function on the set via SDP by using the Y_i as slack variables. Also it turns out that more and more operations on semialgebraic convex sets (like the taking the interior for example) can be done in a very efficient way by using semidefinite representations, see for instance [19].

Large classes of convex semialgebraic sets are known to be semidefinitely representable [28, 7, 6, 23, 29]. In their seminal articles [9, 10], Helton and Nie conjecture that *each* convex semialgebraic set is semidefinitely representable. Note that the converse is clear since the properties of being convex and of being semialgebraic are preserved under projections (for trivial reasons and because of Tarski's real quantifier elimination, respectively). The conjecture of Helton and Nie is still open and is certainly one of the main questions in Convex Algebraic Geometry. More and more results seem to work in its favor.

First, there are results showing that a lot of basic closed semialgebraic sets are semidefinitely representable. The basic method for obtaining these results go back to Lasserre [12] and links semidefinitely representable sets to sums of squares representations of positive polynomials. The main idea is as follows. Start with a finite system of weak polynomial inequalities. The idea is to *linearize* it. Very naively, one could try to replace each monomial which is a product of at least two variables by a new variable Y_i. One would end up with a finite system of linear inequalities. The projection of its solution set on the X-space would clearly contain the solution set of the original system of inequalities. On the other hand this projection would be a polyhedron and therefore in general cannot be equal to the original solution set and not even to its convex hull. Lasserre's idea was to generate a whole infinite family of inequalities which are obviously redundant before the linearization but add valuable information after linearization. The infinite family is chosen in a way such that it becomes an LMI after linearization. As an example, the inequality $-X_1^4+2X^2-X+1\geq 0$ could give rise to the family of additional redundant inequalities $(aX_1+bX_2+c)^2(-X_1^4+2X^2-X+1)\geq 0$ where $a,b,c\in\mathbb{R}$ are parameters. This family can now be rewritten as

$$\begin{pmatrix}a & b & c\end{pmatrix}\begin{pmatrix} 1-X_1-X_1^4+2X_2^2 & X_1-X_1^2-X_1^5+2X_1X_2^2 & X_2-X_1X_2-X_1^4X_2+2X_2^3 \\ X_1-X_1^2-X_1^5+2X_1X_2^2 & X_1^2-X_1^3-X_1^6+2X_1^2X_2^2 & X_1X_2-X_1^2X_2-X_1^5X_2+2X_1X_2^3 \\ X_2-X_1X_2-X_1^4X_2+2X_2^3 & X_1X_2-X_1^2X_2-X_1^5X_2+2X_1X_2^3 & X_2^2-X_1X_2^2-X_1^4X_2^2+2X_2^4 \end{pmatrix}\begin{pmatrix}a\\ b\\ c\end{pmatrix}=$$

$$(-X_1^4+2X_2^2-X_1+1)\begin{pmatrix}a & b & c\end{pmatrix}\begin{pmatrix}1\\ X_1\\ X_2\end{pmatrix}\begin{pmatrix}1 & X_1 & X_2\end{pmatrix}\begin{pmatrix}a\\ b\\ c\end{pmatrix}\geq 0$$

where $a,b,c\in\mathbb{R}$. After linearization this becomes the LMI

$$\begin{pmatrix} 1-X_1-Y_1+2Y_2 & X_1-Y_3-Y_4+2Y_5 & X_2-Y_6-Y_7+2Y_8 \\ X_1-Y_3-Y_4+2Y_5 & Y_3-Y_9-Y_{10}+2Y_{11} & Y_6-Y_{12}-Y_{13}+2Y_{14} \\ X_2-Y_6-Y_7+2Y_8 & Y_6-Y_{12}-Y_{13}+2Y_{14} & Y_2-Y_5-Y_{15}+2Y_{16} \end{pmatrix}\succeq 0.$$

Now in this example the set of all $(x_1,x_2)\in\mathbb{R}^2$ such that there are $y_1,\ldots,y_{16}\in\mathbb{R}$ satisfying this inequality clearly is all of $\mathbb{R}^2$ since it contains the solution set of the original solution set of the original inequality $-X_1^4+2X^2-X+1\geq 0$ whose convex hull is $\mathbb{R}^2$.

Lasserre showed that using a procedure that systematizes this approach leads to LMI relaxations whose solution sets give arbitrarily good approximations to the convex hull of the solution set of the original system of polynomial inequalities in the case that the latter is compact. This uses machinery from Real Algebraic Geometry.

Using much more machinery, Helton and Nie showed that in a lot of cases you get under the same compactness assumption that a sufficiently high relaxation gives exactly the convex hull. See [9, 10] for their celebrated results. Some of their results use just Lasserre's construction together with an ingenious proof bounding the degree of certain sums of squares representations. Their strongest results, which make very few assumptions apart from compactness, use the Lasserre construction locally and glue together the "local" LMIs. This glueing approach is not completely constructive yet.

Netzer and others (see [19]) gave several constructions of how to obtain new semdefinitely representable sets from old ones. These constructions are explicit and can easily be implemented.

Using all these results, one can show that surprisingly many convex semialgebraic sets are semidefinitely representable. For example, the television screen from Section 4 has a semidefinite representation

$$\{(x_1,x_2)\in\mathbb{R}^2 \mid x_1^4+x_2^4\le 1\} = \{(x_1,x_2)\in\mathbb{R}^2 \mid \exists y_1,y_2\in\mathbb{R} : 1-y_1^2-y_2^2\ge 0 \;\&\; y_1\ge x_1^2 \;\&\; y_2\ge x_2^2\}$$
$$= \left\{(x_1,x_2)\in\mathbb{R}^2 \mid \exists y_1,y_2\in\mathbb{R} : \left(\begin{smallmatrix}1+y_1 & y_2\\ y_2 & 1-y_1\end{smallmatrix}\right)\succeq 0 \;\&\; \left(\begin{smallmatrix}y_1 & x_1\\ x_1 & 1\end{smallmatrix}\right)\succeq 0 \;\&\; \left(\begin{smallmatrix}y_2 & x_2\\ x_2 & 1\end{smallmatrix}\right)\succeq 0\right\}.$$

Are all semialgebraic convex sets semidefinitely representable? Before one tries to show this, one might try to work out more examples. For example, the cone of copositive matrices of fixed size is clearly a semialgebraic convex set in the vector space of symmetric matrices. Except for small sizes, it seems to be a hard problem to find a semidefinite representation for it [18].

6 Talks

During this workshop, stimulated by discussions among the workshop participants after his talk, Petter Brändén (Royal Institute of Technology, Stockholm) found sophisticated counterexamples [5] to one of the most outstanding generalizations of the famous Lax Conjecture (proved by Helton and Vinnikov in [11], see [14]) on hyperbolic polynomials. This affects in a direct way one of the mainstreams in current research on semidefinite representability (see Section 4 above). His “bränd-new” result was presented in his spontaneously given second talk. See [5], cf. also [21].

In his video-taped talk, Victor Vinnikov made very accessible the basic ideas behind constructing LMI representations of spectrahedra. He also referred to Petter Brändén’s counterexample (presented in a spontaneous special talk the same morning) and showed that there is some hope for other generalizations of the Lax conjecture to hold (still having the desired consequences). Here is a complete list of talks.

1. **Basu, Saugata** Toda’s theorem – real and complex (joint work with Thierry Zell)
2. **Blekherman, Greg** Convex forms and faces of the cone of sums of squares
3. **Brändén, Petter** Tropicalization of hyperbolic polynomials
4. **Brändén, Petter** A counterexample to the generalized Lax conjecture
5. **Derksen, Harm** (Poly)Matroid Polytopes
6. **Hauenstein, Jonathan** Numerical algebraic geometry
7. **Henk, Martin** Representing Polyhedra by Few Polynomials
8. **Kaltofen, Erich** Certifying the radius of positive semidefiniteness via our ArtinProver package (joint work with Sharon Hutton and Lihong Zhi)
9. **Labs, Oliver** Towards visualization tools for convex algebraic geometry
10. **Laurent, Monique** Error and degree bounds for positivity certificates on the hypercube
11. **Marshall, Murray** Lower bounds for a polynomial in terms of its coefficients
12. **Netzer, Tim** Spectrahedra and their projections
13. **Parrilo, Pablo** Nuclear norm minimization
14. **Plaumann, Daniel** Exposed faces and projections of spectrahedra
15. **Putinar, Mihai** Optimization of non-polynomial functions and applications
16. **Ranestad, Kristian** The convex hull of a space curve
17. **Renegar, Jim** Optimization over hyperbolicity cones
18. **Reznick, Bruce** The cones of real convex forms
19. **Rostalski, Philipp** SDP Relaxations for the Grassmann orbitope
20. **Scheiderer, Claus** Bounded polynomials and stability of preorderings
21. **Smith, Gregory** Determinantal equations
22. **Sottile, Frank** Orbitopes
23. **Theobald, Thorsten** Amoebas of genus at most 1
24. **Vallentin, Frank** Approximation algorithms for SDPs with rank constraints
25. **Vinnikov, Victor** Positive determinantal representations (joint work with Dmitry Kerner)
26. **Vinzant, Cynthia** Faces of the Barvinok-Novik orbitope

7 Acknowledgments

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References

[1] G. Averkov, Representing elementary semi-algebraic sets by a few polynomial inequalities: A constructive approach, preprint. http://arxiv.org/abs/0804.2134

[2] G. Averkov, L. Bröcker, Minimal polynomial descriptions of polyhedra and special semialgebraic sets, preprint. http://arxiv.org/abs/1002.0921

[3] G. Averkov. Henk, Three-dimensional polyhedra can be described by three polynomial inequalities, Discrete Comput. Geom. **42**, no. 2, 166–186 (2009).

[4] S. Basu, R. Pollack.-F. Roy, Algorithms in real algebraic geometry, Algorithms and Computation in Mathematics **10**, Springer-Verlag, Berlin, 2003.

[5] P. Brändén, Obstructions to determinantal representability, preprint. http://arxiv.org/abs/1004.1382

[6] J. Gouveia, T. Netzer, Positive polynomials and projections of spectrahedra, preprint. http://arxiv.org/abs/0911.2750

[7] J. Gouveia, R. Thomas, Convex hulls of algebraic sets, preprint. http://arxiv.org/abs/1007.1191

[8] B. Grenet, E.L. Kaltofen, P. Koiran, N. Portier, Symmetric determinantal representation of formulas and weakly skew circuits, preprint. http://arxiv.org/abs/1007.3804

[9] J.W. Helton, J. Nie, Semidefinite representation of convex sets, Math. Program. **122**, no. 1, Ser. A, 21–64 (2010).

[10] J.W. Helton, J. Nie, Sufficient and necessary conditions for semidefinite representability of convex hulls and sets, SIAM J. Optim. **20**, no. 2, 759–791 (2009).

[11] J.W. Helton, V. Vinnikov, Linear matrix inequality representation of sets, Comm. Pure Appl. Math. **60**, no. 5, 654–674 (2007).

[12] J.B. Lasserre, Convex sets with semidefinite representation, Math. Program. **120**, no. 2, Ser. A, 457–477 (2009).

[13] M. Laurent, Sums of squares, moment matrices and optimization over polynomials, Emerging applications of algebraic geometry, 157–270, IMA Vol. Math. Appl. **149**, Springer, New York, 2009. Updated version available at: http://homepages.cwi.nl/~monique/files/moment-ima-update-new.pdf

[14] A. Lewis, P. Parrilo, M. Ramana: The Lax conjecture is true, Proc. Amer. Math. Soc. **133**, no. 9, 2495–2499 (2005)

[15] M. Marshall, Positive polynomials and sums of squares, Mathematical Surveys and Monographs **146**, American Mathematical Society, Providence, RI, 2008.

[16] A. Nemirovski, Advances in convex optimization: conic programming, International Congress of Mathematicians.,Vol. I, 413–444, Eur. Math. Soc., Zürich, 2007.

[17] Y. Nesterov and A. Nemirovskii, Interior-point polynomial algorithms in convex programming, SIAM Studies in Applied Mathematics **13**, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1994.

[18] T. Netzer: On semidefinite representations of non-closed sets, Linear Algebra Appl. **432**, no. 12, 3072–3078 (2010)

[19] T. Netzer, On semidefinite representations of non-closed sets, preprint. http://arxiv.org/abs/0907.2764

[20] T. Netzer, D. Plaumann, M. Schweighofer, Exposed faces of semidefinitely representable sets, SIAM J. Optim. **20**, no. 4, 1944–1955 (2010).

[21] T. Netzer, A. Thom, Polynomials with and without determinantal representations, preprint. `http://arxiv.org/abs/1008.1931`

[22] J. Nie, K. Ranestad, B. Sturmfels, The algebraic degree of semidefinite programming, Math. Program. **122**, no. 2, Ser. A, 379–405 (2010).

[23] K. Ranestad, B. Sturmfels, On the convex hull of a space curve, preprint. `http://arxiv.org/abs/0912.2986`

[24] K. Ranestad, B. Sturmfels, The convex hull of a variety, preprint. `http://arxiv.org/abs/1004.3018`

[25] J. Renegar, Hyperbolic programs, and their derivative relaxations, Found. Comput. Math. **6**, no. 1, 59–79 (2006).

[26] R.T. Rockafellar, Convex analysis, Princeton Mathematical Series **28**, Princeton University Press, Princeton, NJ, 1970.

[27] P. Rostalski, B. Sturmfels, Dualities in convex algebraic geometry, preprint. `http://arxiv.org/abs/1006.4894`

[28] R. Sanyal. Sottile, B. Sturmfels, Orbitopes, preprint. `http://arxiv.org/abs/0911.5436`

[29] C. Scheiderer, Convex hulls of curves of genus one, preprint. `http://arxiv.org/abs/1003.4605`