HOCHSCHILD HOMOLOGY AND GLOBAL DIMENSION RESEARCH IN TEAMS 22 - 29 NOVEMBER 2009 REPORT

PETTER ANDREAS BERGH & DAG MADSEN

Given an algebra A over a commutative ring k, we may form the sequence

$$\cdots \to A^{\otimes 4} \xrightarrow{d_3} A^{\otimes 3} \xrightarrow{d_2} A^{\otimes 2} \xrightarrow{d_1} A \to 0$$

of k-modules (the tensor products are taken over the ground ring k) and maps, where d_n is defined by

$$\begin{aligned} a_0 \otimes \cdots \otimes a_n \quad \mapsto \quad & \sum_{i=0}^{n-1} (-1)^i a_0 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n \\ + \quad & (-1)^n a_n a_0 \otimes a_1 \otimes \cdots \otimes a_{n-1}. \end{aligned}$$

This is a complex, i.e. the composition of two consecutive maps is zero, and it is known as the *Hochschild complex* of A. The nth homology group $\operatorname{Ker} d_n / \operatorname{Im} d_{n+1}$, denoted $\mathrm{HH}_n(A)$, is the nth *Hochschild homology group* of A.

The aim of this Research in Teams meeting was to establish a conjecture due to Han for certain classes of algebras. The conjecture, stated in [Han], says that if all the higher Hochschild homology groups of a finite dimensional algebra vanish, then the algebra is of finite global dimension. The motivation for this conjecture was a remark made by Happel in [Hap], a remark which evolved into a cohomological version of the above conjecture, and which became known as "Happel's question". However, a counterexample to this cohomological version was found by Buchweitz, Green, Madsen and Solberg in [BGMS].

Han's conjecture has been establish for some classes of algebras. In [AV-P], Avramov and Vigué-Poirrier showed that it holds for finite dimensional commutative algebras, whereas Solotar and Vigué-Poirrier showed in [SV-P] that it holds for certain classes of finitely generated and graded algebras. We, Bergh and Madsen, established the conjecture for Koszul algebras, local graded algebras and cellular algebras in [BeM], provided the ground field has characteristic zero. Our cooperation began during the workshop "Hochschild Cohomology of Algebras: Structure and Applications", held at BIRS in September 2007.

During this Research in Teams meeting, we studied Hochschild homology basically in two settings. First, we looked at split pairs of algebras, that is, algebras A and B related via a pair

$$B \xrightarrow{i} A \xrightarrow{\pi} B$$

of algebra homomorphisms satisfying $\pi \circ i = 1_B$. Since Hochschild homology is functorial, vanishing of $\mathrm{HH}_n(A)$ implies vanishing of $\mathrm{HH}_n(B)$, hence one obtains information on the Hochschild homology of A through studying that of B. Using this very elementary concept, we obtained results on the Hochschild homology of the following classes of algebras:

(1) fibre products of algebras,
(2) trivial extensions,
(3) monomial algebras,
(4) path algebras whose underlying quivers contain loops,

(5) graded-commutative algebras,
(6) quantum complete intersections.

For the quantum complete intersections, we also computed lower bounds for the dimensions of the Hochschild cohomology groups.

During the second part of our meeting, we studied the Hochschild homology of path algebras containing *oriented cycles* of a special form. The importance of path algebras is established in the classical result of Gabriel: every finite dimensional basic algebra over an algebraically closed field k is isomorphic to a factor kQ/I of a path algebra, where Q is some finite quiver and $I \subseteq kQ$ is an ideal. Moreover, every finite dimensional algebra over a field is Morita equivalent to a finite dimensional basic algebra over the same field. If A is an algebra of the form kQ/I, where Q and I are as above, then an oriented cycle in A is a sequence

$$i_1 \xrightarrow{\alpha_1} i_2 \xrightarrow{\alpha_2} i_3 \to \cdots \to i_t \xrightarrow{\alpha_t} i_1$$

of arrows in Q, starting an ending in the same vertex. Such sequences appear very naturally when one studies finite dimensional algebras, due to the following fact: a path algebra (modulo relations) has infinite global dimension if and only if the underlying quiver contains an oriented cycle.

We showed that if there exists such an oriented cycle in A with the property that

$$\alpha_2\alpha_1 = \alpha_3\alpha_2 = \cdots = \alpha_t\alpha_{t-1} = \alpha_1\alpha_t = 0,$$

then A has infinitely many nonzero Hochschild homology groups. In other words, Han's conjecture holds for algebras containing such oriented cycles. Applying this to graded algebras, we generalized one of the results in [SV-P], and also obtained a result on the Hochschild homology of a fibre product of local rings. Moreover, we recovered the result which establishes Han's conjecture for algebras whose radicals square to zero.

References

[AV-P] L.L. Avramov, M. Vigué-Poirrier, *Hochschild homology criteria for smoothness*, Internat. Math. Res. Notices (1992), no. 1, 17-25.

[BeM] P.A. Bergh, D. Madsen, *Hochschild homology and global dimension*, Bull. London Math. Soc. 41 (2009), no. 3, 473-482.

[BGMS] R.-O. Buchweitz, E. Green, D. Madsen, Ø. Solberg, *Finite Hochschild cohomology without finite global dimension*, Math. Res. Lett. 12 (2005), no. 5-6, 805-816.

[Han] Y. Han, *Hochschild (co)homology dimension*, J. London Math. Soc. (2) 73 (2006), no. 3, 657-668.

[Hap] D. Happel, *Hochschild cohomology of finite-dimensional algebras*, in *Séminaire d'Algèbre Paul Dubreil et Marie-Paul Malliavin, 39ème Année (Paris, 1987/1988)*, Lecture Notes in Mathematics 1404 (1989), 108-126.

[SV-P] A. Solotar, M. Vigué-Poirrier, *Two classes of algebras with infinite Hochschild homology*, Proc. Amer. Math. Soc. 138 (2010), no. 3, 861-869.

Petter Andreas Bergh, Institutt for matematiske fag, NTNU, N-7491 Trondheim, Norway

E-mail address: `bergh@math.ntnu.no`

Dag Madsen, Mathematics department, 215 Carnegie, Syracuse University, Syracuse, NY 13244-1150, USA