An overdetermined eigenvalue problem in $\mathbb{S}^2$ and the Critical Catenoid conjecture

José M. Espinar
Universidad de Granada

Joint work with D. Marín

Funded by the European Union NextGenerationEU
Plan de Recuperación, Transformación y Resiliencia
30.RP.23.00.04-CONSOLIDACION2022
Ministerio de Ciencia e Innovación
Agencia Estatal de Investigación
PID2020-118137GB-I00

In this talk

We relate solutions to the **overdetermined eigenvalue problem**

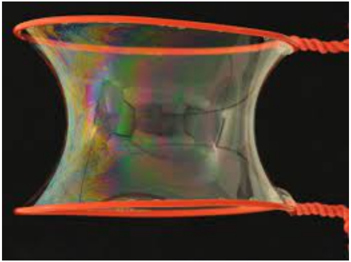
$$\begin{cases} \Delta u + 2u = 0 & \text{in} & \Omega \subset \mathbb{S}^2, \\ u = 0 & \text{along} & \partial\Omega, \\ u > 0 & \text{in} & \Omega \\ |\nabla u| \text{ locally constant} & \text{along} & \partial\Omega. \end{cases}$$

In this talk

We relate solutions to the **overdetermined eigenvalue problem**

$$\begin{cases} \Delta u + 2u = 0 & \text{in} \quad \Omega \subset \mathbb{S}^2, \\ u = 0 & \text{along} \quad \partial\Omega, \\ u > 0 & \text{in} \quad \Omega \\ |\nabla u| \text{ locally constant} & \text{along} \quad \partial\Omega. \end{cases}$$

and **free boundary minimal surfaces in** $\mathbb{B}^3$



Introduction

Let (M,g) be a Riem. mfld., $\Omega \subset M$ bounded domain, $f \in$ Lip$(\mathbb{R})$:

$$\begin{cases} \Delta u + f(u) = 0 & \text{in} \quad \Omega, \\ u = 0 & \text{along} \quad \partial\Omega. \end{cases} \tag{1}$$

$$\frac{\partial u}{\partial \nu} = c \quad \text{along} \quad \partial\Omega, \quad c \in \mathbb{R}^{+}. \tag{2}$$

(1) $+$ (2) is an **overdetermined problem**

Introduction

Let (M, g) be a Riem. mfld., $\Omega \subset M$ bounded domain, $f \in$ Lip$(\mathbb{R})$:

$$\begin{cases} \Delta u + f(u) = 0 & \text{in} \quad \Omega, \\ u = 0 & \text{along} \quad \partial\Omega. \end{cases} \tag{1}$$

$$\frac{\partial u}{\partial \nu} = c \quad \text{along} \quad \partial\Omega, \quad c \in \mathbb{R}^+. \tag{2}$$

(1) $+$ (2) is an **overdetermined problem** and solutions, if they do exist, often determine the shape of Ω.

A classical result

Question: What can we say about the geometry of Ω?

A classical result

Question: What can we say about the geometry of Ω?

J. Serrin considered the case: $(M, g) = (\mathbb{R}^n, \langle , \rangle)$, $f = 1$, $\Omega \in \mathcal{C}^2$:

Theorem (Serrin, 1971)

If $u \in \mathcal{C}^2(\Omega)$ is a solution to the equation $\Delta u + 1 = 0$ with zero Dirichlet data and $\frac{\partial u}{\partial n} = c > 0$ along $\partial\Omega$ then Ω is a metric ball and u a radial function. ($u > 0$ by the Maximum Principle)

A classical result

Question: What can we say about the geometry of Ω?

J. Serrin considered the case: $(M, g) = (\mathbb{R}^n, \langle , \rangle)$, $f = 1$, $\Omega \in \mathcal{C}^2$:

Theorem (Serrin, 1971)

If $u \in \mathcal{C}^2(\Omega)$ is a solution to the equation $\Delta u + 1 = 0$ with zero Dirichlet data and $\frac{\partial u}{\partial n} = c > 0$ along $\partial\Omega$ then Ω is a metric ball and u a radial function. ($u > 0$ by the Maximum Principle)

Two proofs:

- Method of moving planes (Serrin, 1971): Extends Alexandrov reflection method for embedded CMC hypersurfaces
- Method of P-functions (Weingberger, 1971): $P(u) = |\nabla u|^2 + \frac{2}{n}$ is subharmonic and (Ω, u) is the ball solution $\iff P \equiv c$

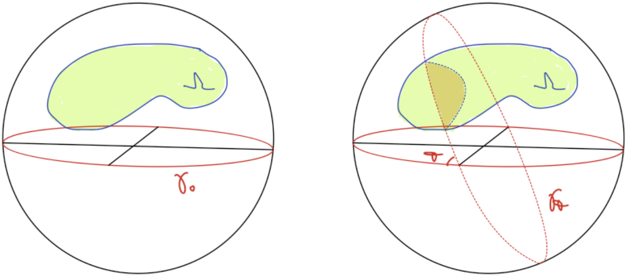
Related results with equation $\Delta u + f(u) = 0$

Moving planes method has been generalized:

- Works for positive solutions if f is a Lipchitz function: Pucci-Serrin.
- Bounded domains in Space forms $\mathbb{S}^n_+$ and $\mathbb{H}^n$: Kumaresan-Prajapat.

Moving planes method has been generalized:

- Works for positive solutions if f is a Lipchitz function: Pucci-Serrin.
- Bounded domains in Space forms $\mathbb{S}^n_+$ and $\mathbb{H}^n$: Kumaresan-Prajapat.



Related results with equation $\Delta u + f(u) = 0$

Geometric methods have been introduced recently:

- Works for positive solutions if f is a Lipchitz function: Pucci-Serrin.
- Bounded domains in Space forms $\mathbb{S}^n_+$ and $\mathbb{H}^n$: Kumaresan-Prajapat.

Related results with equation $\Delta u + f(u) = 0$

Geometric methods have been introduced recently:

- Works for positive solutions if f is a Lipchitz function: Pucci-Serrin.
- Bounded domains in Space forms $\mathbb{S}^n_+$ and $\mathbb{H}^n$: Kumaresan-Prajapat.
- Non-compact domains in Space Forms $\mathbb{R}^n$ and $\mathbb{H}^n$, extending techniques from CMC hypersurfaces: Scibaldi-Ros-Ruíz, E.-Mao, E.-Farina-Mazet.
- Simply-connected domains in $\mathbb{S}^2$, **Index method**: E.-Mazet

Related results with equation $\Delta u + f(u) = 0$

Geometric methods have been introduced recently:

- Works for positive solutions if f is a Lipchitz function: Pucci-Serrin.
- Bounded domains in Space forms $\mathbb{S}^n_+$ and $\mathbb{H}^n$: Kumaresan-Prajapat.
- Non-compact domains in Space Forms $\mathbb{R}^n$ and $\mathbb{H}^n$, extending techniques from CMC hypersurfaces: Scibaldi-Ros-Ruíz, E.-Mao, E.-Farina-Mazet.
- Simply-connected domains in $\mathbb{S}^2$, **Index method**: E.-Mazet

P-function method has been generalized:

- Weak solutions of divergence-form equations: Garofalo-Lewis.
- Serrin's result in product manifolds: Farina-Roncoroni.

Disclaimer:

Serrin's result is not true in general. There are non-rotationally symmetric domains that support a solution to an overdetermined problem:

Disclaimer:

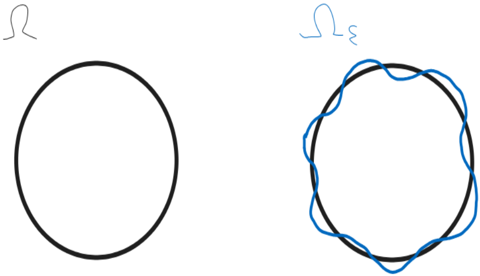
Serrin's result is not true in general. There are non-rotationally symmetric domains that support a solution to an overdetermined problem:

- For the first eigenvalue of the laplacian in a general compact manifold (Delay-Sicbaldi, 2013).

Disclaimer:

Serrin's result is not true in general. There are non-rotationally symmetric domains that support a solution to an overdetermined problem:

- For the first eigenvalue of the laplacian in a general compact manifold (Delay-Sicbaldi, 2013).
- Sign-changing solutions to equation $\Delta u + f(u) = 0$ (Ruiz, 2023).



More boundary components

In Serrin's case ($\Delta u + 1 = 0$, $\Omega \subset \mathbb{R}^n$)

$$\frac{\partial u}{\partial \nu} = c \text{ (equal!)} \quad \text{along} \quad \partial\Omega \implies \partial\Omega \quad \text{connected.}$$

More boundary components

In Serrin's case ($\Delta u + 1 = 0$, $\Omega \subset \mathbb{R}^n$)

$$\frac{\partial u}{\partial \nu} = c \text{ (equal!)} \quad \text{along} \quad \partial\Omega \implies \partial\Omega \quad \text{connected.}$$

But there are rotationally symmetric solutions to the Dirichlet problem defined in annular domains:

$$u(|x|) = \frac{1 - A \cdot |x|^2}{4} + B \cdot \Gamma(|x|), \quad |x| \in [r_1(A,B), r_2(A,B)].$$

More boundary components

In Serrin's case ($\Delta u + 1 = 0$, $\Omega \subset \mathbb{R}^n$)

$$\frac{\partial u}{\partial \nu} = c \ \text{(equal!)} \quad \text{along} \quad \partial\Omega \implies \partial\Omega \quad \text{connected.}$$

But there are rotationally symmetric solutions to the Dirichlet problem defined in annular domains:

$$u(|x|) = \frac{1 - A \cdot |x|^2}{4} + B \cdot \Gamma(|x|), \quad |x| \in [r_1(A,B), r_2(A,B)].$$

Question: Can we classify more general rotationally symmetric solutions? (different Neumann boundary condition!)

Let (Ω, u) be a solution to the Dirichlet problem. $|\pi_0(\partial\Omega)| \geq 2$

$$\frac{\partial u}{\partial \nu} \quad \text{is locally constant along} \quad \partial\Omega \tag{3}$$

Let (Ω, u) be a solution to the Dirichlet problem. $|\pi_0(\partial\Omega)| \geq 2$

$$\frac{\partial u}{\partial \nu} \quad \text{is locally constant along} \quad \partial\Omega \tag{3}$$

There are some results in exterior domains:

Theorem (Reichel, 1995), (Sirakov, 2001)

Let (Ω, u) a positive solution to

$$\begin{cases} \Delta u + f(u) = 0 & \text{in} & \Omega \subset \mathbb{R}^n, \\ u = a > 0 & \text{in} & \Gamma_i, \\ u = 0 & \text{along} & \Gamma_o, \end{cases}$$

where $\partial\Omega = \Gamma_i \cup \Gamma_o$, being Γ_i the inner component and Γ_o the outter component. Suppose that u satisfies (3) and that $\frac{\partial u}{\partial \nu} \leq 0$ along Γ_i.
Then Ω is a rotationally symmetric annulus and u is radially symmetric.

Case of Agostiniani-Borghini-Mazzieri

Consider problem

$$\begin{cases} \Delta u + 1 = 0 & \text{in} \quad \Omega \subset \mathbb{R}^2, \\ u = 0 & \text{along} \quad \partial\Omega, \end{cases} \tag{4}$$

$$|\nabla u| \quad \text{is constant along some} \quad \Gamma \in \pi_0(\partial\Omega). \tag{5}$$

Case of Agostiniani-Borghini-Mazzieri

Consider problem

$$\begin{cases} \Delta u + 1 = 0 & \text{in} \quad \Omega \subset \mathbb{R}^2, \\ u = 0 & \text{along} \quad \partial\Omega, \end{cases} \tag{4}$$

$$|\nabla u| \quad \text{is constant along some} \quad \Gamma \in \pi_0(\partial\Omega). \tag{5}$$

Theorem (Agostiniani-Borghini-Mazzieri, 2021)

If (Ω, u) is a solution to (4),(5) and u has infinitely many maximum points, then Ω is a rotationally symmetric annulus and u depends on the distance to the center of the annulus.

Case of Agostiniani-Borghini-Mazzieri

Consider problem

$$\begin{cases} \Delta u + 1 = 0 & \text{in} \quad \Omega \subset \mathbb{R}^2, \\ u = 0 & \text{along} \quad \partial\Omega, \end{cases} \tag{4}$$

$$|\nabla u| \quad \text{is constant along some} \quad \Gamma \in \pi_0(\partial\Omega). \tag{5}$$

Theorem (Agostiniani-Borghini-Mazzieri, 2021)

If (Ω, u) is a solution to (4),(5) and u has infinitely many maximum points, then Ω is a rotationally symmetric annulus and u depends on the distance to the center of the annulus.

Theorem (Agostiniani-Borghini-Mazzieri, 2021)

There exist non-rotationally symmetric solutions to problem (4), with $\frac{\partial u}{\partial n}$ locally constant along $\partial\Omega$.

Our case

We study the equation

$$\begin{cases} \Delta u + 2u = 0 & \text{in} \quad \Omega \subset \mathbb{S}^2, \\ u = 0 & \text{along} \quad \partial\Omega, \\ u > 0 & \text{in} \quad \Omega. \end{cases} \tag{6}$$

$$|\nabla u| \quad \text{is locally constant along} \quad \partial\Omega. \tag{7}$$

Our case

We study the equation

$$\begin{cases} \Delta u + 2u = 0 & \text{in} & \Omega \subset \mathbb{S}^2, \\ u = 0 & \text{along} & \partial\Omega, \\ u > 0 & \text{in} & \Omega. \end{cases} \tag{6}$$

$$|\nabla u| \quad \text{is locally constant along} \quad \partial\Omega. \tag{7}$$

Objective: Classify rotationally symmetric solutions to (6) + (7).
Difficulty: The moving plane method is not available.
Observation: $P(u) = |\nabla u|^2 + u^2$ is a $P-$function, $\Delta P \geq 0$

Statement of the result

Theorem (Espinar-M., 2023)

Let (Ω, u) a solution to $\Delta u + 2u = 0$, where $\Omega \subset \mathbb{S}^2$ is a ring-shaped domain with $\mathcal{C}^2$-boundary. Suppose that

1. $u = 0 \quad$ along $\quad \partial\Omega$
2. $|\nabla u| \quad$ is locally constant along $\quad \partial\Omega$
3. u has infinitely many maximum points.

Then Ω is a rotationally symmetric neighborhood of an equator and u exhibits rotational symmetry with respect to the axis perpendicular to the plane defining this equator.

Sketch of the proof

Based on the approach of Agostiniani-Borghini-Mazzieri in “On Serrin problem for ring-shaped domains”.

1. Describe a 1-parameter family of rotationally symmetric model solutions.

Sketch of the proof

Based on the approach of Agostiniani-Borghini-Mazzieri in “On Serrin problem for ring-shaped domains”.

1. Describe a 1-parameter family of rotationally symmetric model solutions.
2. Given $\mathcal{U} \in \pi_0(\Omega \setminus \text{Max}(u))$, construct a correspondence between $(\mathcal{U}, u)$ and $(\mathcal{U}_R, u_R)$, with $\mathcal{U}_R \in \pi_0(\Omega_R \setminus \text{Max}(u_R))$.

Sketch of the proof

Based on the approach of Agostiniani-Borghini-Mazzieri in “On Serrin problem for ring-shaped domains”.

1. Describe a 1-parameter family of rotationally symmetric model solutions.
2. Given $\mathcal{U} \in \pi_0(\Omega \setminus \text{Max}(u))$, construct a correspondence between $(\mathcal{U}, u)$ and $(\mathcal{U}_R, u_R)$, with $\mathcal{U}_R \in \pi_0(\Omega_R \setminus \text{Max}(u_R))$.
3. Compare the geometry of the level sets of $(\mathcal{U}, u)$ with those of $(\mathcal{U}_R, u_R)$:

Sketch of the proof

Based on the approach of Agostiniani-Borghini-Mazzieri in “On Serrin problem for ring-shaped domains”.

1. Describe a 1-parameter family of rotationally symmetric model solutions.
2. Given $\mathcal{U} \in \pi_0(\Omega \setminus \text{Max}(u))$, construct a correspondence between $(\mathcal{U}, u)$ and $(\mathcal{U}_R, u_R)$, with $\mathcal{U}_R \in \pi_0(\Omega_R \setminus \text{Max}(u_R))$.
3. Compare the geometry of the level sets of $(\mathcal{U}, u)$ with those of $(\mathcal{U}_R, u_R)$: norm of the gradient, curvature, length.

Sketch of the proof

Based on the approach of Agostiniani-Borghini-Mazzieri in “On Serrin problem for ring-shaped domains”.

1. Describe a 1-parameter family of rotationally symmetric model solutions.
2. Given $\mathcal{U} \in \pi_0(\Omega \setminus \text{Max}(u))$, construct a correspondence between $(\mathcal{U}, u)$ and $(\mathcal{U}_R, u_R)$, with $\mathcal{U}_R \in \pi_0(\Omega_R \setminus \text{Max}(u_R))$.
3. Compare the geometry of the level sets of $(\mathcal{U}, u)$ with those of $(\mathcal{U}_R, u_R)$: norm of the gradient, curvature, length.
4. $\mathcal{U}$ or $\Omega \setminus \mathcal{U}$ contained on an hemisphere $\implies$ moving planes.

Model solutions

Return to equation

$$(*) \begin{cases} \Delta u + 2u = 0,\ u > 0 & \text{in} \quad \Omega \subset \mathbb{S}^2, \\ u = 0 & \text{along} \quad \partial\Omega. \end{cases}$$

Consider cylindrical coordinates in $\mathbb{S}^2$:

$$\mathbb{S}^2 = \left\{ (\sqrt{1-r^2}\cos\theta, \sqrt{1-r^2}\sin\theta, r) \ : \ r \in [-1,1],\ \theta \in [0,2\pi) \right\}.$$

Model solutions

Return to equation

$$(*) \begin{cases} \Delta u + 2u = 0,\ u > 0 & \text{in} \quad \Omega \subset \mathbb{S}^2, \\ u = 0 & \text{along} \quad \partial\Omega. \end{cases}$$

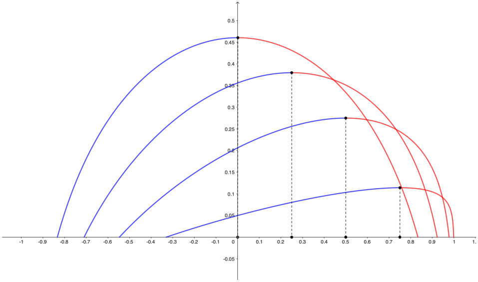
Consider cylindrical coordinates in $\mathbb{S}^2$:

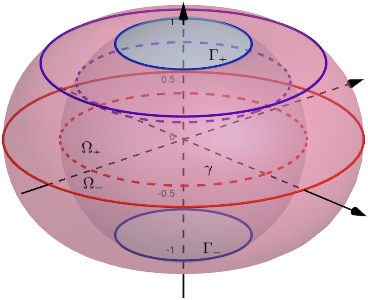
$$\mathbb{S}^2 = \left\{ (\sqrt{1-r^2}\cos\theta, \sqrt{1-r^2}\sin\theta, r) \ : \ r \in [-1,1],\ \theta \in [0, 2\pi) \right\}.$$

Rotationally symmetric solutions to $(*)$:

$$\Omega_R = \{ r \in [r_-(R), r_+(R)] \}, \quad u_R(r) = \alpha(R)\left(1 - r\,\text{arctanh}(r) + \omega(R) r\right),$$

$\omega(R)$ is such that $\text{Max}(u_R) = \{r = R\}$, $\alpha(R) = r_+(R)\sqrt{1 - r_+(R)^2} > 0$.
$R :=$critical height.





Left: Graphs of some of the model functions.
Right: Radial graph of the model solution of critical radius $R = 0$.

$$\partial\Omega_R = \{r = r_-(R)\} \cup \{r = r_+(R)\} = \Gamma_-(R) \cup \Gamma_+(R)$$

$$\Omega_R \setminus \text{Max}(u_R) = \Omega_R \setminus \gamma_R = \Omega_-(R) \cup \Omega_+(R)$$

The $\overline{\tau}$-function

Let (Ω, u) be a solution to $(*)$ and $\mathcal{U} \in \pi_0(\Omega \setminus \mathsf{Max}(u))$. Define the *Normalised Wall Shear Stress* (NWSS) of the region as

$$\overline{\tau}(\mathcal{U}) := \max \left\{ \max_{\Gamma}(|\nabla u| / u_{\mathsf{max}}) \ : \ \Gamma \in \pi_0(\partial\Omega \cap \mathsf{cl}(\mathcal{U})) \right\}.$$

The $\overline{\tau}$-function

Let (Ω, u) be a solution to $(*)$ and $\mathcal{U} \in \pi_0(\Omega \setminus \text{Max}(u))$. Define the *Normalised Wall Shear Stress* (NWSS) of the region as

$$\overline{\tau}(\mathcal{U}) := \max\left\{\max_{\Gamma}(|\nabla u| / u_{\text{max}}) \; : \; \Gamma \in \pi_0(\partial\Omega \cap \text{cl}(\mathcal{U}))\right\}.$$

- It is a scale-invariant quantity.
- If $\overline{\tau}_{\pm}(R) := \overline{\tau}(\Omega_{\pm}(R)), \quad \forall R \in [0,1)$, then $\overline{\tau}_{+}(0) = \overline{\tau}_{-}(0) = \tau_0 > 1$.
- $\overline{\tau}_{+} : [0,1) \to [\tau_0, +\infty)$ and $\overline{\tau}_{-} : [0,1) \to (1, \tau_0]$ are monotone functions.

Theorem

Let (Ω, u) be a solution to the Dirichlet problem and let $\mathcal{U} \in \pi_0(\Omega \setminus \text{Max}(u))$. If $\overline{\tau}(\mathcal{U}) \leq 1$, then Ω is an open hemisphere and $u(r, \theta) = \alpha r$ for some $\alpha > 0$.

Theorem

Let (Ω, u) be a solution to the Dirichlet problem and let $\mathcal{U} \in \pi_0(\Omega \setminus \text{Max}(u))$. If $\overline{\tau}(\mathcal{U}) \leq 1$, then Ω is an open hemisphere and $u(r, \theta) = \alpha r$ for some $\alpha > 0$.

Idea of the proof:

1. If $\text{cl}(\mathcal{U}) = \text{cl}(\Omega)$ ($\text{Max}(u)$ does not separate): Condition $\overline{\tau}(\mathcal{U}) \leq 1$ implies that $P(u) = |\nabla u|^2 + u^2$ attains its maximum inside Ω $\Longrightarrow$ rigidity.
2. The function

$$E(t) = \frac{1}{u_{\text{max}}^2 - t^2} \int_{\text{cl}(\mathcal{U}) \cap \{u=t\}} |\nabla u|.$$

   is non-incresing if $\overline{\tau}(\mathcal{U}) \leq 1$.
3. If $\text{cl}(\mathcal{U}) \neq \text{cl}(\Omega)$ ($\text{Max}(u)$ does separate): $E(t) \to +\infty$ as $t \to u_{max}$ and $E(t) \leq E(0) < +\infty$; contradiction!

The correspondence

Definition (Expected critical height)

Let (Ω, u) be a solution to the Dirichlet problem and $\mathcal{U} \in \pi_0(\Omega \setminus \mathrm{Max}(\xi))$. Suppose Ω is not a topological disk. Then:

- if $\overline{\tau}(\mathcal{U}) < \tau_0$, we set

$$\bar{R}(\mathcal{U}) = \overline{\tau}_-^{-1}\left(\overline{\tau}(\mathcal{U})\right), \tag{8}$$

- if $\overline{\tau}(\mathcal{U}) \geq \tau_0$, we set

$$\bar{R}(\mathcal{U}) = \overline{\tau}_+^{-1}\left(\overline{\tau}(\mathcal{U})\right). \tag{9}$$

The correspondence

Definition (Expected critical height)

Let (Ω, u) be a solution to the Dirichlet problem and $\mathcal{U} \in \pi_0(\Omega \setminus \text{Max}(\xi))$. Suppose Ω is not a topological disk. Then:

- if $\overline{\tau}(\mathcal{U}) < \tau_0$, we set

$$\bar{R}(\mathcal{U}) = \overline{\tau}_{-}^{-1}\left(\overline{\tau}(\mathcal{U})\right), \tag{8}$$

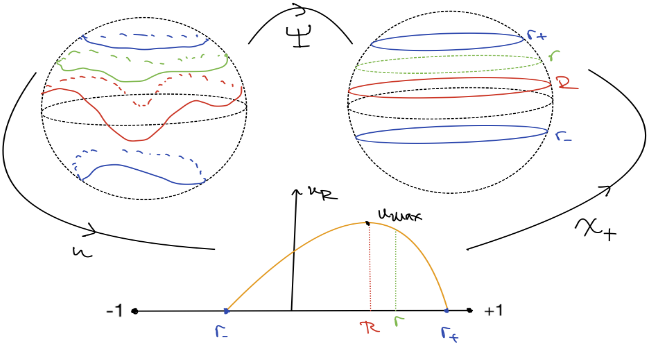
- if $\overline{\tau}(\mathcal{U}) \geq \tau_0$, we set

$$\bar{R}(\mathcal{U}) = \overline{\tau}_{+}^{-1}\left(\overline{\tau}(\mathcal{U})\right). \tag{9}$$

Remark

$R(\mathcal{U}) \in [0, 1)$ is well defined because of the previous result.

Pseudo-radial functions



Remark

By definition: $\Psi(r,\theta) = r$ if $(\mathcal{U}, u) = (\Omega_{\pm}(R), u_R)$.

Pseudo-radial functions

Let $R(\mathcal{U}) = R$, and suppose that $u_{\max} = (u_R)_{\max}$. Define the function $F : [0, u_{\mathsf{max}}] \times [r_-(R), r_+(R)] \to \mathbb{R}$ by

$$F(u, r) = u - \alpha(R)(1 - r \operatorname{arctanh}(r) + \omega(R)r).$$

$\frac{\partial F}{\partial r} = 0$ if, and only if, $r = R \implies$ Implicit function theorem:

Pseudo-radial functions

Let $R(\mathcal{U}) = R$, and suppose that $u_{\max} = (u_R)_{\max}$. Define the function $F : [0, u_{\text{max}}] \times [r_-(R), r_+(R)] \to \mathbb{R}$ by

$$F(u, r) = u - \alpha(R)(1 - r \operatorname{arctanh}(r) + \omega(R) r).$$

$\frac{\partial F}{\partial r} = 0$ if, and only if, $r = R \implies$ Implicit function theorem:

$$\chi_- : [0, u_{\max}] \to [r_-(R), R] \quad \text{and} \quad \chi_+ : [0, u_{\max}] \to [R, r_+(R)]$$

such that

$$F(u, \chi_\pm(u)) = 0 \quad \text{for all} \quad u \in [0, u_{\text{max}}].$$

Pseudo-radial functions

Let $R(\mathcal{U}) = R$, and suppose that $u_{\max} = (u_R)_{\max}$. Define the function $F : [0, u_{\text{max}}] \times [r_-(R), r_+(R)] \to \mathbb{R}$ by

$$F(u, r) = u - \alpha(R)(1 - r \operatorname{arctanh}(r) + \omega(R)r).$$

$\frac{\partial F}{\partial r} = 0$ if, and only if, $r = R \implies$ Implicit function theorem:

$$\chi_- : [0, u_{\max}] \to [r_-(R), R] \quad \text{and} \quad \chi_+ : [0, u_{\max}] \to [R, r_+(R)]$$

such that

$$F(u, \chi_\pm(u)) = 0 \quad \text{for all} \quad u \in [0, u_{\text{max}}].$$

Definition (Pseudo-radial functions)

- If $\overline{\tau}(\mathcal{U}) \geq \tau_0$, define $\Psi(p) = \chi_+(u(p))$ for all $p \in \mathcal{U}$.
- If $\overline{\tau}(\mathcal{U}) < \tau_0$, define $\Psi(p) = \chi_-(u(p))$ for all $p \in \mathcal{U}$.

Gradient estimates

Theorem

It holds

$$|\nabla u|^2(p) \leq |\nabla u_R|^2 \circ \Psi(p) \text{ for all } p \in \mathcal{U}.$$

Moreover, if the equality holds at one single point of $\mathcal{U}$, then $(\Omega, \xi) \equiv (\Omega_R, u_R)$ up to rotation and change of scale.

Gradient estimates

Theorem

It holds

$$|\nabla u|^2(p) \leq |\nabla u_R|^2 \circ \Psi(p) \text{ for all } p \in \mathcal{U}.$$

Moreover, if the equality holds at one single point of $\mathcal{U}$, then $(\Omega, \xi) \equiv (\Omega_R, u_R)$ up to rotation and change of scale.

Idea of the proof:

1. Define a function $\beta = \beta(\Psi) > 0$ such that $\beta \cdot (|\nabla u|^2 - |\nabla u_R|^2 \circ \Psi)$ satisfies an elliptic inequality.
2. As $R = R(\mathcal{U})$ then $|\nabla u|^2 \leq |\nabla u_R|^2 \circ \Psi$ along cl$(\mathcal{U}) \cap \partial\Omega \implies |\nabla u|^2 \leq |\nabla u_R|^2 \circ \Psi$ in $\mathcal{U}$ by the maximum principle.
3. Equality at one single point $\implies |\nabla u|^2 = |\nabla u_R|^2 \circ \Psi$ in $\mathcal{U}$. Then the level sets have constant curvature.

Curvature and length estimates

Proposition

Let $p \in \partial\Omega$ such that $|\nabla \xi|^2 (p) = \max_{\partial\Omega \cap \mathrm{cl}(\mathcal{U})} |\nabla \xi|^2$, $\bar{r}_\pm := r_\pm(R)$, and $\kappa(p)$ curvature with respect to the inner orientation. Then

- $\kappa(p) \leq -\frac{\bar{r}_+}{\sqrt{1-\bar{r}_+^2}}$ if $\overline{\tau}(\mathcal{U}) \geq \tau_0$
- $\kappa(p) \leq \frac{\bar{r}_-}{\sqrt{1-\bar{r}_-^2}}$ if $\overline{\tau}(\mathcal{U}) < \tau_0$.

Curvature and length estimates

Proposition

Let $p \in \partial\Omega$ such that $|\nabla\xi|^2(p) = \max_{\partial\Omega\cap \mathrm{cl}(\mathcal{U})} |\nabla\xi|^2$, $\bar{r}_\pm := r_\pm(R)$, and $\kappa(p)$ curvature with respect to the inner orientation. Then

- $\kappa(p) \leq -\frac{\bar{r}_+}{\sqrt{1-\bar{r}_+^2}}$ if $\overline{\tau}(\mathcal{U}) \geq \tau_0$
- $\kappa(p) \leq \frac{\bar{r}_-}{\sqrt{1-\bar{r}_-^2}}$ if $\overline{\tau}(\mathcal{U}) < \tau_0$.

Theorem

Suppose that $\mathrm{cl}(\mathcal{U}) \cap \mathrm{Max}(u) = \gamma^{\mathcal{U}}$ and $\mathrm{cl}(\mathcal{U}) \cap \partial\Omega = \Gamma^{\mathcal{U}}$ are sets of analytic closed curves. Then

- $|\gamma^{\mathcal{U}}| \leq \frac{\sqrt{1-R^2}}{\sqrt{1-\bar{r}_+^2}} |\Gamma^{\mathcal{U}}|$ if $\overline{\tau}(\mathcal{U}) \geq \tau_0$,
- $|\gamma^{\mathcal{U}}| \leq \frac{\sqrt{1-R^2}}{\sqrt{1-\bar{r}_-^2}} |\Gamma^{\mathcal{U}}|$ if $\overline{\tau}(\mathcal{U}) < \tau_0$,

Overdetermined problem

Now we can prove the main result:

Theorem (Espinar-M., 2023)

Let $\Omega \subset \mathbb{S}^2$ ring shaped domain with $\mathcal{C}^2$-boundary, $u \in \mathcal{C}^2(\Omega)$ solution to

$$\begin{cases} \Delta u + 2u = 0,\, u > 0 & \text{in} \quad \Omega \subset \mathbb{S}^2, \\ u = 0 & \text{along} \quad \partial\Omega. \end{cases}$$

Suppose that $|\nabla u|$ is locally constant along $\partial\Omega$, and also that u has infinitely many maximum points. Then $(\Omega, u) \equiv (\Omega_R, u_R)$ for some $R \in [0, 1)$ up to a rotation and a change of scale.

Sketch of the proof

1. $\exists\, \gamma \in \mathrm{Max}(u)$ analytic curve such that $\Omega \setminus \gamma = \Omega_1 \cup \Omega_2$ with $\Omega_1 \cap \partial\Omega = \Gamma_1$ and $\Omega_2 \cap \partial\Omega = \Gamma_2$.

Sketch of the proof

1. $\exists\, \gamma \in$ Max(u) analytic curve such that $\Omega \setminus \gamma = \Omega_1 \cup \Omega_2$ with $\Omega_1 \cap \partial\Omega = \Gamma_1$ and $\Omega_2 \cap \partial\Omega = \Gamma_2$.
2. Set $\mathcal{U} \in \{\Omega_1, \Omega_2\}$, $\Gamma = \partial\Omega \cap$ cl$(\mathcal{U})$, $R(\mathcal{U}) = R$:
   - $|\Gamma| \leq 2\pi\sqrt{1 - \bar{r}_+^2}$ if $\overline{\tau}(\mathcal{U}) \geq \tau_0,$
   - $|\Gamma| \leq 2\pi\sqrt{1 - \bar{r}_-^2}$ if $\overline{\tau}(\mathcal{U}) < \tau_0.$

Sketch of the proof

1. $\exists\,\gamma \in \mathrm{Max}(u)$ analytic curve such that $\Omega \setminus \gamma = \Omega_1 \cup \Omega_2$ with $\Omega_1 \cap \partial\Omega = \Gamma_1$ and $\Omega_2 \cap \partial\Omega = \Gamma_2$.
2. Set $\mathcal{U} \in \{\Omega_1, \Omega_2\}$, $\Gamma = \partial\Omega \cap \mathrm{cl}(\mathcal{U})$, $R(\mathcal{U}) = R$:
   - $|\Gamma| \leq 2\pi\sqrt{1-\bar{r}_+^2}$ if $\overline{\tau}(\mathcal{U}) \geq \tau_0$,
   - $|\Gamma| \leq 2\pi\sqrt{1-\bar{r}_-^2}$ if $\overline{\tau}(\mathcal{U}) < \tau_0$.
3. Length estimates:

$$|\gamma| \leq \sqrt{1-R^2}\,\frac{|\Gamma|}{\sqrt{1-\bar{r}_\pm^2}} \leq 2\pi\sqrt{1-R^2}.$$

Sketch of the proof

1. $\exists\, \gamma \in$ Max(u) analytic curve such that $\Omega \setminus \gamma = \Omega_1 \cup \Omega_2$ with $\Omega_1 \cap \partial\Omega = \Gamma_1$ and $\Omega_2 \cap \partial\Omega = \Gamma_2$.
2. Set $\mathcal{U} \in \{\Omega_1, \Omega_2\}$, $\Gamma = \partial\Omega \cap$ cl$(\mathcal{U})$, $R(\mathcal{U}) = R$:
   - $|\Gamma| \leq 2\pi\sqrt{1-\bar{r}_+^2}$ if $\overline{\tau}(\mathcal{U}) \geq \tau_0$,
   - $|\Gamma| \leq 2\pi\sqrt{1-\bar{r}_-^2}$ if $\overline{\tau}(\mathcal{U}) < \tau_0$.
3. Length estimates:

$$|\gamma| \leq \sqrt{1-R^2}\,\frac{|\Gamma|}{\sqrt{1-\bar{r}_\pm^2}} \leq 2\pi\sqrt{1-R^2}.$$

4. $|\gamma| \leq 2\pi \implies \gamma$ contained on an hemisphere $\mathbb{S}^2_+$ or γ an equator.

Sketch of the proof

1. $\exists\, \gamma \in$ Max(u) analytic curve such that $\Omega \setminus \gamma = \Omega_1 \cup \Omega_2$ with $\Omega_1 \cap \partial\Omega = \Gamma_1$ and $\Omega_2 \cap \partial\Omega = \Gamma_2$.
2. Set $\mathcal{U} \in \{\Omega_1, \Omega_2\}$, $\Gamma = \partial\Omega \cap$ cl$(\mathcal{U})$, $R(\mathcal{U}) = R$:
   - $|\Gamma| \leq 2\pi\sqrt{1 - \bar{r}_+^2}$ if $\overline{\tau}(\mathcal{U}) \geq \tau_0$,
   - $|\Gamma| \leq 2\pi\sqrt{1 - \bar{r}_-^2}$ if $\overline{\tau}(\mathcal{U}) < \tau_0$.
3. Length estimates:

$$|\gamma| \leq \sqrt{1 - R^2}\,\frac{|\Gamma|}{\sqrt{1 - \bar{r}_\pm^2}} \leq 2\pi\sqrt{1 - R^2}.$$

4. $|\gamma| \leq 2\pi \implies \gamma$ contained on an hemisphere $\mathbb{S}^2_+$ or γ an equator.
   - First case: $\Omega_1 \subset \mathbb{S}^2_+$ or $\Omega_2 \subset \mathbb{S}^2_+ \implies$ moving planes.
   - Second case: Cauchy-Kovalevskaya.

Minimal surfaces with free boundaries

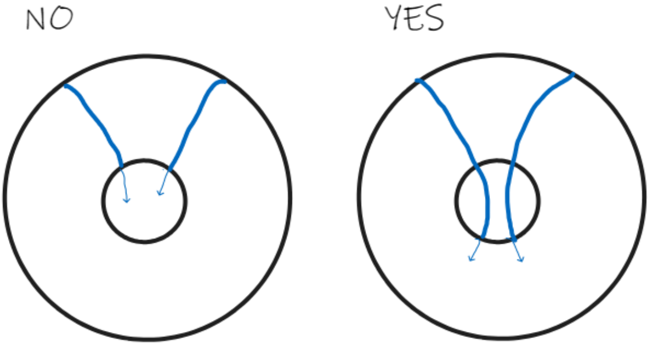
Definition

Let $\Sigma \subset \mathbb{B}^3$ be an open immersed minimal surface with boundary. We will say that Σ has free boundaries if each boundary component of Σ meets orthogonally a sphere centered at the origin (*from the inside*), possibly of different radii.

Minimal surfaces with free boundaries

Definition

Let $\Sigma \subset \mathbb{B}^3$ be an open immersed minimal surface with boundary. We will say that Σ has free boundaries if each boundary component of Σ meets orthogonally a sphere centered at the origin (*from the inside*), possibly of different radii.



Model Catenoids

Consider catenoid $C_{\alpha,\omega}$ parametrized by

$$\psi_{\alpha,\omega}(r,\theta) = \alpha\left(\frac{\cos\theta}{\sqrt{1-r^2}}, \frac{\sin\theta}{\sqrt{1-r^2}}, \operatorname{arctanh}(r) - \omega\right),$$

$r \in (-1,1)$ and $\theta \in [0,2\pi)$, with outward Gauss map

$$N(r,\theta) = \left(\sqrt{1-r^2}\,\cos\theta, \sqrt{1-r^2}\,\sin\theta, -r\right) \in \mathbb{S}^2 \setminus \{\mathbf{s},\mathbf{n}\}.$$

Model Catenoids

Consider catenoid $C_{\alpha,\omega}$ parametrized by

$$\psi_{\alpha,\omega}(r,\theta) = \alpha\left(\frac{\cos\theta}{\sqrt{1-r^2}}, \frac{\sin\theta}{\sqrt{1-r^2}}, \operatorname{arctanh}(r) - \omega\right),$$

$r \in (-1,1)$ and $\theta \in [0,2\pi)$, with outward Gauss map

$$N(r,\theta) = \left(\sqrt{1-r^2}\,\cos\theta, \sqrt{1-r^2}\,\sin\theta, -r\right) \in \mathbb{S}^2 \setminus \{\mathbf{s},\mathbf{n}\}.$$

The support function $u(r,\theta) = \langle \psi(r,\theta), N((r,\theta)\rangle$ is given by

$$u(r,\theta) = \alpha(1 - r\operatorname{arctanh}(r) + \omega r).$$

Then u solves $\Delta^{\mathbb{S}^2} u + 2u = 0$ in $\mathbb{S}^2 \setminus \{\mathbf{s},\mathbf{n}\}$.

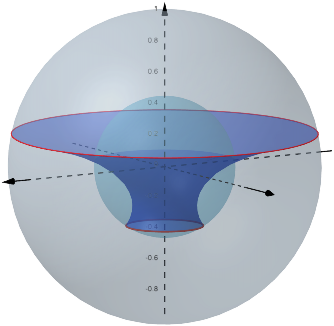
$$C_R := \left\{\psi_{\alpha(R),\omega(R)}(r,\theta) \,:\, r \in (r_-(R), r_+(R)),\ \theta \in [0,2\pi)\right\}$$

Up to reflection with respct to $\{z = 0\}$, $\forall R \in [0, 1)$:

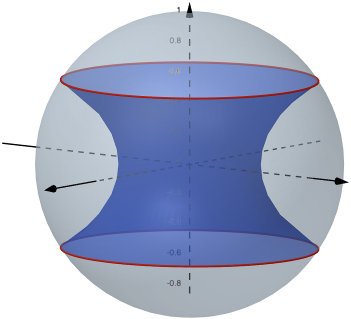
C_R Model catenoid $\longleftrightarrow (\Omega_R, u_R)$ Model solution

Up to reflection with respct to $\{z = 0\}$, $\forall R \in [0, 1)$:

C_R Model catenoid $\longleftrightarrow$ (Ω_R, u_R) Model solution



(a) $R = 1/2$



(b) $R = 0$ (Critical Catenoid)

Souam's Correspondence

Proposition (Souam, 2004)

Let Σ be a minimal surface with free boundaries, $\partial\Sigma = \bigcup_{i=1}^{k} \zeta_i$, and injective Gauss map $N : \Sigma \to \mathbb{S}^2$. Set $v(p) := \langle p, N(p) \rangle$. Then

$$u(z) := (v \circ N^{-1})(z) = \left\langle N^{-1}(z), z \right\rangle , \quad \forall z \in N(\Sigma) = \Omega$$

satisfies OEP

$$\begin{cases} \Delta^{\mathbb{S}^2} u + 2u = 0 & \text{in} \quad \Omega, \\ u = 0 & \text{along} \quad \partial\Omega, \\ |\nabla^{\mathbb{S}^2} u|^2 = b_i^2 & \text{along} \quad \Gamma_i \in \pi_0(\partial\Omega),\ i \in \{1, \ldots, k\}, \end{cases}$$

where $\partial\Omega = \bigcup_{i=1}^{k} \Gamma_i$, $N(\zeta_i) = \Gamma_i$, and $|b_i|$ is the radius of the sphere in which $\zeta_i \in \pi_0(\partial\Sigma)$ is contained.

Statement of the results

Theorem (E.-Marín., 2023)

Let $\Sigma \subset \mathbb{B}^3$ be an embedded minimal annulus with free boundaries, $\partial\Sigma = \zeta_1 \cup \zeta_2$, such that $\zeta_1 \subset \mathbb{S}^2$ and $\zeta_2 \subset \mathbb{S}^2(\tilde{r})$ for some $0 < \tilde{r} \leq 1$(always true up to a dilation!). Suppose that its support function has infinitely many critical points. Then, there exists $R \in [0, 1)$ such that $\Sigma \equiv C_R$, up to a rotation around the origin.

Statement of the results

Theorem (E.-Marín., 2023)

Let $\Sigma \subset \mathbb{B}^3$ be an embedded minimal annulus with free boundaries, $\partial\Sigma = \zeta_1 \cup \zeta_2$, such that $\zeta_1 \subset \mathbb{S}^2$ and $\zeta_2 \subset \mathbb{S}^2(\tilde{r})$ for some $0 < \tilde{r} \leq 1$(always true up to a dilation!). Suppose that its support function has infinitely many critical points. Then, there exists $R \in [0, 1)$ such that $\Sigma \equiv C_R$, up to a rotation around the origin.

Corollary (E-Marín, 2023)

Let $\Sigma \subset \mathbb{B}^3$ be an embedded free boundary minimal annulus, and suppose its support function has infinitely many critical points. Then Σ is the critical catenoid.

Proof of the Theorem

We must prove:

Theorem (E.-Marín, 2023)

Let $\Sigma \subset \mathbb{B}^3$ be an embedded minimal annulus with free boundaries, then it has an injective Gauss map and its support function has a constant sign in Σ.

Proof of the Theorem

We must prove:

Theorem (E.-Marín, 2023)

Let $\Sigma \subset \mathbb{B}^3$ be an embedded minimal annulus with free boundaries, then it has an injective Gauss map and its support function has a constant sign in Σ.

Proposition

In the previous case, if u is its support function and $|\mathrm{Crit}(u)| = +\infty$, then $\tilde{\gamma} = \mathrm{Max}(u) = \mathrm{Crit}(u)$ is a closed simple curve.

THANKS FOR THE ATTENTION!