A higher order scalar curvature

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IASM-BIRS, Hangzhou, Feb.29.2024

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A higher order scalar curvature

$1 \le k \le n/2$, (M^n, g) *k-th Gauss-Bonnet-Chern curvature*:

$$R_k := \frac{1}{2^k} \delta^{i_1 i_2 \cdots i_{2k-1} i_{2k}}_{j_1 j_2 \cdots j_{2k-1} j_{2k}} R_{i_1 i_2}{}^{j_1 j_2} \cdots R_{i_{2k-1} i_{2k}}{}^{j_{2k-1} j_{2k}},$$

1. $R_1 = R$, scalar curvature.
2. $R_2 = |Riem|^2 - 4|Ric|^2 + R^2 = |W|^2 + 8(n-2)(n-3)\sigma_2$
3. $k = \frac{n}{2}$, it is the Euler density. Gauss-Bonnet-Chern theorem.

$$\int_M R_{\frac{n}{2}} = c\chi(M),$$

The Gauss-Bonnet-Chern curvature was first appeared in the paper of Lanczos in 1938 for $n = 4$ and $k = 2$.

Gauss-Bonnet-Chern curvature has been intensively studied in Gauss-Bonnet gravity, as a generalization of Einstein gravity.

Scalar curvature:

$$R_k := \frac{1}{2^k}\delta^{i_1 i_2 \cdots i_{2k-1} i_{2k}}_{j_1 j_2 \cdots j_{2k-1} j_{2k}} R_{i_1 i_2}{}^{j_1 j_2} \dots R_{i_{2k-1} i_{2k}}{}^{j_{2k-1} j_{2k}},$$

- k-Ricci Tensor:

$$\begin{aligned} \mathcal{R}ic^a_b &:= \tfrac{k}{2^k}\delta^{i_1 i_2 \cdots i_{2k-1} i_{2k}}_{b\, j_2 \cdots j_{2k-1} j_{2k}} R_{i_1 i_2}{}^{a j_2} R_{i_3 i_4}{}^{j_3 j_4} \dots . R_{i_{2k-1} i_{2k} d_k}{}^{j_{2k-1} j_{2k}} \\ &= k R_{i_1 i_2}{}^{a j_2} P_{b j_2}{}^{i_1 i_2} \end{aligned}$$

- k-Einstein tensor

$$\begin{aligned} \mathcal{E}^a_b &= -\tfrac{1}{2^{k+1}}\delta^{a i_1 i_2 \cdots i_{2k-1} i_{2k}}_{b j_1 j_2 \cdots j_{2k-1} i_{2k}} R_{i_1 i_2}{}^{j_1 j_2} \dots R_{i_{2k-1} i_{2k}}{}^{j_{2k-1} j_{2k}} \\ &= \mathcal{R}ic^a_b - \tfrac{1}{2} R_k \delta^a_b \end{aligned}$$

We write R_k red

$$R_k = P^{ij}{}_{kl} R_{ij}{}^{kl},$$

where

$$P^{ij}{}_{kl} := \frac{1}{2^k} \delta^{i_1 i_2 \cdots i_{2k-3} i_{2k-2} ij}_{j_1 j_2 \cdots j_{2k-3} j_{2k-2} kl} R_{i_1 i_2}{}^{j_1 j_2} \cdots R_{i_{2k-3} i_{2k-2}}{}^{j_{2k-3} j_{2k-2}}.$$

It is important that P is divergence-free, i.e.

$$P^{ij}{}_{kl,i} = 0$$

and P has the same symmetry as $Riem$. It implies that

$$\mathcal{E}_{ij,i} = 0$$

.

Variation of total R_k-curvature

Lemma (Variation of total R_k-curvature)

The first variation of $\mathcal{F}_k = \int L_k(g)dv(g)$ *is given by*

$$\delta\mathcal{F}_k(g)[h] = \int_M \langle \mathcal{E}_k, h\rangle dv(g). \qquad (\delta g = h)$$

Proposition

1. A critical point of $\mathcal{F}_k(g) = \int R_k(g)dv(g)$ in the class of fixed volume is a k-Stein metric, i.e.

$$\mathcal{R}ic_k = \lambda g.$$

2. A critical point of $\mathcal{F}_k(g) = \int R_k(g)dv(g)$ in a conformal class of fixed volume is a k-metric, i.e.

$$R_k = const..$$

$$R_1 = R, \quad \mathcal{E} = \mathrm{Ric} - \frac{1}{2} R g, \quad P_{(1)}{}^{ij}_{kl} = \frac{1}{2}(\delta^i_k \delta^j_l - \delta^i_l \delta^j_k).$$

$$\begin{aligned} R_2 &= \tfrac{1}{4}\delta^{i_1 i_2 i_3 i_4}_{j_1 j_2 j_3 j_4} R^{j_1 j_2}{}_{i_1 i_2} R^{j_3 j_4}{}_{i_3 i_4} = |Rm|^2 - 4|Ric|^2 + R^2 \\ &= |W|^2 + 8(n-2)(n-3)\sigma_2 \end{aligned}$$

$$\mathcal{E}^i_j = 2RR^i_j - 4R^i_l R^l{}_j - 4R_{kl}R^{ki}{}_{lj} + 2R^i{}_{klm} R_j{}^{klm} - \frac{1}{2}\delta^i_j R_2$$

$$(P_{(2)})^{ij}_{kl} = R^{ij}_{kl} - R^i_k \delta^j_l + R^i_l \delta^j_k - R^j_l \delta^i_k + R^j_k \delta^i_l + \frac{1}{2} R(\delta^i_k \delta^j_l - \delta^i_l \delta^j_k).$$

$$\begin{aligned} \mathrm{Riem} &= P_{(2)} + \mathrm{Ric} \owedge g + \frac{1}{4} Rg \owedge g \\ &= W + \frac{1}{n-2}\mathrm{Ric} \owedge g - \frac{R}{(n-1)(n-2)} \frac{1}{2} g \owedge g \\ &= W + S \owedge g, \end{aligned}$$

Schouten tensor: $S = \frac{1}{n-2}\left(\mathrm{Ric} - \frac{R}{2(n-1)} g\right)$

Examples: σ_k-scalar curvatures

- Schouten Tensor $S = \frac{1}{n-2}(Ric - \frac{R}{2(n-1)})g$.
- σ_k-scalar curvature:

$$\sigma_k(g) := \sigma_k(g^{-1} \cdot S).$$

$$\sigma_1(g) = R_g/(n-1)$$

$\mathrm{Riem} = W + S \owedge g$, i.e,

$$R^{ij}{}_{kl} = W^{ij}{}_{kl} + S^i_k \delta^j_l - S^i_l \delta^j_k + S^j_l \delta^i_k - S^j_k \delta^i_l.$$

If $W = 0$, then

$$R_k = k! 2^k \sigma_k$$

Mean curvature in $\mathbb{R}^{n+1}$

- Let $\Sigma \subset \mathbb{R}^{n+1}$. $R^{ij}{}_{kl} = h^i_k h^j_l - h^i_l h^j_k$.

$$\begin{aligned} R_k &= \frac{1}{2^k} \delta^{i_1 i_2 \cdots i_{2k-1} i_{2k}}_{j_1 j_2 \cdots j_{2k-1} j_{2k}} R_{i_1 i_2}{}^{j_1 j_2} \cdots R_{i_{2k-1} i_{2k}}{}^{j_{2k-1} j_{2k}} \\ &= \delta^{i_1 i_2 \cdots i_{2k-1} i_{2k}}_{j_1 j_2 \cdots j_{2k-1} j_{2k}} h^{j_1}_{i_1} h^{j_2}_{i_2} \cdots h^{j_{2k-1}}_{i_{2k-1}} h^{j_{2k}}_{i_{2k}} = H_{2k} \end{aligned}$$

- Let $\Sigma \subset \mathbb{H}^{n+1}$. $R^{ij}{}_{kl} = -(\delta^i_k \delta^j_l - \delta^i_l \delta^j_k) + h^i_k h^j_l - h^i_l h^j_k$

$$R_k = \sum_{i=0}^{k} (-1)^i H_{2k-2i}$$

$R_1 = H_2 - 1$, and $R_2 = H_4 - 2H_2 + 1$

one may ask:

- Geroch type conjecture: No metric on T^n with $R_k > 0$?

  When $k = 1$, it is Schoen-Yau, Gromov-Lawson

  When $k = n/2$, it is trivially true from the Gauss-Bonnet-Chern theorem.

- Llarull type Theorem? $(k < n/2)$ Any (M, g) with $g \geq g_{\mathbb{S}^n}$, $R_k(g) \geq R_k(\mathbb{S}^n)$ with a nonzero degree map from M to $\mathbb{S}^n$ is isometric to $\mathbb{S}^n$?

No any idea! The big problem: No analytic methods to study R_k.

1. Minimal surfaces method of Schoen-Yau?
2. Spin method?
3. Harmonic functions (or maps) of Stern?

Minimal surfaces? Very naive idea:

Area functional $A(\Sigma) = \int_\Sigma 1 = \int_S R_0$.

The minimal surface method can be viewed: when we study problems related to the scalar curvature R_1, we use the functional $\int_\Sigma R_0$.

When we consider problems related to R_2, we may use the functional $\int_{\mathbb{S}} R_1$. Its Euler-Lagrange equation is:

$$E_g \cdot A = 0, \tag{0.1}$$

where A is the 2nd Fundamental form.

Hence the 2-minimal surface is defined by (0.1).

If $\Sigma \subset \mathbb{R}^n$, then (0.1) is

$$H_3 = 0.$$

It is fully nonlinear. The ellipticity requires restrictive conditions.

Aasymptotically flat (AF) of decay order τ if there is a compact set K such that $M \setminus K$ is diffeomorphic to $\mathbb{R}^n \setminus B_R(0)$

$$g_{ij} = \delta_{ij} + \sigma_{ij}, \text{ with } |\sigma_{ij}| + r|\partial\sigma_{ij}| + r^2|\partial^2\sigma_{ij}| = O(r^{-\tau})$$

ADM mass (Arnowitt-Deser-Misner):

$$m_1(g) := m_{ADM} := \frac{1}{2(n-1)\omega_{n-1}} \lim_{r\to\infty} \int_{S_r} (g_{ij,i} - g_{ii,j})\nu_j dS,$$

Bartnik: m_{AMD} is well-defined and a geometric invariant, if

$$\tau > \frac{n-2}{2} \quad \text{and} \quad R \in L^1(M).$$

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The Gauss-Bonnet-Chern mass

We expend

$$R_k = c(n,k)\partial_i\left(g_{jk,l}P^{ijkl}\right) + O(r^{-(k+1)\tau-2k})$$

and define

$$m_k(g) := m_{GBC}(g) = c_k(n)\lim_{r\to\infty}\int_{S_r} P^{ijkl}\partial_l g_{jk}\nu_i dS,$$

Theorem (Ge-W.-Wu Adv Math (2014))

Suppose that (M^n, g) $(k < \frac{n}{2})$ is AF of decay order $\tau > \frac{n-2k}{k+1}$ and R_k is integrable on (M^n, g). Then the Gauss-Bonnet-Chern mass m_k is well-defined and invariant.

Li-Nguyen had a similar mass

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Positive mass theorem

Positive mass theorem is true for m_k, if
(1) $(\mathbb{R}^n, e^{2u}|dx|^2)$ (Ge-W.-Wu, IMRN (2014))
(2) graphical AF manifolds (Ge-W.-Wu)

Theorem (Positive Mass Theorem (Ge-W.-Wu))

Let $(M^n, g) = (\mathbb{R}^n, \delta + df \otimes df)$ and $L_k \in L^1(M)$, then

$$m_k = \frac{c_k(n)}{2} \int_{M^n} \frac{L_k}{\sqrt{1 + |\nabla f|^2}} dV_g,$$

In particular, $L_k \geq 0$ yields $m_k \geq 0$.

$k = 1$, Lam (2010), de Lima-Girao, Huang-Wu.
Key Lemma. $L_k(g) = c(n)\partial_i(P^{ijkl}\partial_l g_{jk})$.

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Penrose inequality

Theorem (Penrose Inequality ($k = 1$ Lam, $k \geq 2$ Ge-W.-Wu))

$\Omega \subset \mathbb{R}^n$, $\Sigma = \partial\Omega$. $f : \mathbb{R}^n \setminus \Omega \to \mathbb{R}$, $(M, g) = (\mathbb{R}^n \backslash \Omega, \delta + df \times df)$.
Σ is in a level set of f and $|\nabla f(x)| \to \infty$ as $x \to \Sigma$. Then

$$m_k = c_k(n) \int_{M^n} \frac{L_k}{\sqrt{1 + |\nabla f|^2}} dV_g + c(n) \int_\Sigma H_{2k-1}$$

In particular, if $L_k \geq 0$ (dominant energy condition) holds, then the Alexandrov-Fenchel inequality yields a Penrose inequality

$$m_2 \geq \frac{1}{4} \left(\frac{\int_\Sigma R_\Sigma}{(n-1)(n-2)\omega_{n-1}} \right)^{\frac{n-4}{n-3}} \geq \frac{1}{4} \left(\frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{n-4}{n-1}}.$$

Penrose Inequality for AF graphs

$$m_k = m_{GBC} \geq \frac{1}{2^k}\left(\frac{|\Sigma|}{\omega_{n-1}}\right)^{\frac{n-2k}{n-1}}.$$

Optimality: The generalized anti-de Sitter Schwarzschild space-time is given by

$$(1-\frac{2m}{r^{\frac{n}{k}-2}})^{-1}dr^2 + r^2 g_{\mathbb{S}^{n-1}},$$

Penrose conjecture for GBC mass for general AF manifolds could be proposed as:

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Hyperbolic Gauss-Bonnet-Chern mass

$$\mathbb{H}^n, b = dr^2 + \sinh^2 r g_{\mathbb{S}^{n-1}} = \frac{1}{1+\rho^2} d\rho^2 + \rho^2 g_{\mathbb{S}^{n-1}}$$

$$\mathbb{N}_b := \{V \in C^\infty(\mathbb{H}^n) | \mathrm{Hess}^b V = Vb\}.$$

$\gamma = -V^2 dt^2 + b$ is a static solution of the Einstein equation $Ric(\gamma) + n\gamma = 0$.

$$\dim \mathbb{N}_b = n + 1$$

$$V_{(0)} = \cosh r,\ V_{(1)} = x^1 \sinh r,\ \cdots,\ V_{(n)} = x^n \sinh r,$$

where r is the hyperbolic distance from an arbitrary fixed point on $\mathbb{H}^n$ and $x^1, x^2, \cdots, x^n$ are the coordinate functions restricted to $\mathbb{S}^{n-1} \subset \mathbb{R}^n$. We equip the vector space $\mathbb{N}_b$ with a Lorentz metric

$$\eta(V_{(0)}, V_{(0)}) = 1, \quad \text{and} \quad \eta(V_{(i)}, V_{(i)}) = -1 \quad \text{for} \quad i = 1, \cdots, n.$$

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$$H_k^\Phi(V) = \lim_{r\to\infty} \int_{S_r} \left(\left(V\bar{\nabla}_l e_{js} - e_{js}\bar{\nabla}_l V \right) \tilde{P}_{(k)}^{ijsl} \right) \nu_i d\mu$$

Theorem (Ge-W.-Wu)

Suppose $(M^n, g)(2k \leq n)$ is an asymptotically hyperbolic manifold of decay order $\tau > \frac{n}{k+1}$ and for $V \in \mathbb{N}_b$, $V\tilde{L}_k \in L^1$, then the mass functional $H_k^\Phi(V)$ is well-defined.

$k = 1$, X. Wang, Chruściel-Herzlich, Zhang

$$V\tilde{L}_k = 2\bar{\nabla}_i \left((V\bar{\nabla}_l e_{js} - e_{js}\bar{\nabla}_l V)\tilde{P}^{ijsl} \right) + 2(\bar{\nabla}_i\bar{\nabla}_l V - V b_{il}) e_{js}\tilde{P}^{ijsl} + O(e^{(-(k+1)\tau$$

Hyperbolic GBC mass: If $H_k^\Phi(V) > 0 \ \forall V$,

$$m_k^{\mathbb{H}} := c(n,k) \inf_{\mathbb{N}_b \cap \{V>0, \eta(V,V)=1\}} H_k^\Phi(V)$$

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Suppose $(M^n, g)(2k \le n)$ is an asymptotically hyperbolic manifold of decay order $\tau > \frac{n}{k+1}$ and for $V \in \mathbb{N}_b$, $V\tilde{L}_k \in L^1$, then the mass functional $H_k^{\Phi}(V)$ is well-defined.

$k = 1$, X. Wang, Chruściel-Herzlich, Zhang

$$V\tilde{L}_k = 2\bar{\nabla}_i \left((V\bar{\nabla}_l e_{js} - e_{js}\bar{\nabla}_l V)\tilde{P}^{ijsl} \right) + 2(\bar{\nabla}_i\bar{\nabla}_l V - V b_{il}) e_{js}\tilde{P}^{ijsl} + O(e^{(-(k+1)\tau}$$

Hyperbolic GBC mass: If $H_k^{\Phi}(V) > 0 \ \forall V$,

$$m_k^{\mathbb{H}} := c(n,k) \inf_{\mathbb{N}_b \cap \{V>0, \eta(V,V)=1\}} H_k^{\Phi}(V)$$

Hyperbolic Gauss-Bonnet-Chern mass

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Theorem (Penrose Inequality for AH graphs (Ge-W.-Wu))

$k \geq 2$. If $f : \mathbb{H}^n \setminus \Omega \to \mathbb{R}$ with $(M^n, g) = (\mathbb{H}^n \setminus \Omega, b + V^2 df \otimes df)$ is AH of decay order $\tau > \frac{n}{k+1}$ and $V\tilde{L}_k \in L^1$. Assume that $\Sigma = \partial\Omega$ is in a level set of f and $|\bar{\nabla} f(x)| \to \infty$ as $x \to \Sigma$.

$$m_k^{\mathbb{H}} = c(n,k)\left(\frac{1}{2}\int_{M^n} \frac{V\tilde{L}_k}{\sqrt{1+V^2|\bar{\nabla} f|^2}} dV_g + \frac{(2k-1)!}{2}\int_{\Sigma} V H_{2k-1} d\mu\right).$$

$$m_k^{\mathbb{H}} \geq \frac{1}{2^k}\left(\left(\frac{|\Sigma|}{\omega_{n-1}}\right)^{\frac{n}{k(n-1)}} + \left(\frac{|\Sigma|}{\omega_{n-1}}\right)^{\frac{n-2k}{k(n-1)}}\right)^k,$$

if $\tilde{L}_k \geq 0$ and $\Sigma \subset \mathbb{H}^n$ is horospherical convex. Moreover, equality is achieved by an anti-de Sitter Schwarzschild type metric.

$k = 1$, Dahl-Gicquaud-Sakovich, de Lima and Girão

Alexandrov-Fenchel inequaliy in $\mathbb{H}^n$

Theorem (Ge-W.-Wu, JDG (2014), W.-Xia, Adv. Math (2014))

Let $1 \le k \le n-1$. Any horospherical convex hypersurface Σ in $\mathbb{H}^n$ satisfies

$$\int_\Sigma H_k d\mu \ge \omega_{n-1}\left\{\left(\frac{|\Sigma|}{\omega_{n-1}}\right)^{\frac{2}{k}} + \left(\frac{|\Sigma|}{\omega_{n-1}}\right)^{\frac{2}{k}\frac{(n-k-1)}{n-1}}\right\}^{\frac{k}{2}}.$$

Equality holds if and only if Σ is a geodesic sphere.

$k=2$ Li-Wei-Xiong, $H_1 > 0$, $H_2 > 0$ and star-shaped.

It solves a conjecture in integral geometry in $\mathbb{H}^n$ proposed by Gao-Hug-Schneider, at least in the case of horospherical convex.

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Weighted Alexandrov-Fenchel inequalites in $\mathbb{H}^n$

Theorem (Ge-W.-Wu)

Let Σ be a horospherical convex hypersurface in $\mathbb{H}^n$, $V = \cosh r$

$$\int_\Sigma V H_{2k+1} d\mu \geq \omega_{n-1} \left(\left(\frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{n}{(k+1)(n-1)}} + \left(\frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{n-2k-2}{(k+1)(n-1)}} \right)^{k+1}.$$

Equality holds if and only if Σ is a centered geodesic sphere in $\mathbb{H}^n$.

$k = 1$ de Lima-Girao, Dahl-Gicquaud-Sakovich motivated by a similar inequality by Brendle-Hung-Wang

Ideas: Inverse curvature flow by Gerhardt, Heintze-Karcher type inequality of Brendle, optimal geometric inequalities on $\mathbb{S}^{n-1}$ of Guan-W. and Alexandrov-Fenchel inequalites in $\mathbb{H}^n$.

Analysis in a conformal class

Analysis of R_k in a conformal class is rich and successful.

- σ_k-Yamabe problem Find a metric is a given conformal class such that σ_k is constant. (Viaclovsky, Chang-Gursky-Yang, Guan-W. Ge-W., Li-Li, Sheng-Trudinger-Wang, ...)

- A conformal spherical theorem of Chang-Gursky-Yang: (M^4, g) with positive Yamabe constant and $\int_M \sigma_2 > \int_M |W|^2$ is diffeomorphic to $\mathbb{S}^4$ or RP^4.

- (Ge-W.-Lin JDG (2009)) (M^3, g) with positive Yamabe constant and $\int_M \sigma_2 > 0$ is diffeomorphic to $\mathbb{S}^3$.

Thank you very much!