Towards a further comprehension for mass inequalities

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This talk is based on the following works

- Positive mass theorem with arbitrary ends and its application [Z. '23] (Int. Math. Res. Not. IMRN)
- Riemannian Penrose inequality without horizon in dimension three [Z. '23+] (arXiv:2304.01769, to appear in Trans. Amer. Math. Soc.)

Outline

1. Asymptotically flat manifolds and previous mass inequalities
2. Recent developments and a conjecture on mass-systole inequality
3. Some progress on the mass-systole conjecture

1 Asymptotically flat manifolds and previous mass inequality

1.1 Asymptotically flat manifolds

- The interest on asymptotically flat manifolds comes from *general relativity*
- isolated gravity system
  - no substance gives flat space (Euclidean)
  - substance causes bending (curvature) with effect decay as distance increasing (model of space: asymptotically flat manifolds)
- Complete $(M^{n\geq 3}, g)$ is called asymptotically flat if
  - $M \setminus K$ has finitely many ends for compact K, each diffeomorphic to $\mathbb{R}^n \setminus B$
  - On each end E the metric g has expansion

$$g_{ij} = \delta_{ij} + O_2(r^{-\mu}) \text{ with } \mu > \frac{n-2}{2}$$

  - $R(g) \in L^1$

1.2 Geometric quantities of asymptotically flat manifolds

- **A**rnowitt-**D**eser-**M**isner (ADM) mass
  - Let (E, g) be one asymptotically flat end.

$$m_{ADM}(M, g, E) := \frac{1}{2(n-1)\omega_{n-1}} \int_{S_\infty} (\partial_j g_{ij} - \partial_i g_{jj})\nu^i_{\mathbb{E}} \, d\sigma_{\mathbb{E}},$$

    where

$$\partial_i \left(\partial_j g_{ij} - \partial_i g_{jj}\right) = R(g) + O(r^{-2-2\mu}).$$

  - Examples
    * Euclidean space (one end E) with $m_{ADM}(M, g, E) = 0$
    * Schwarzschild manifold (M, g)

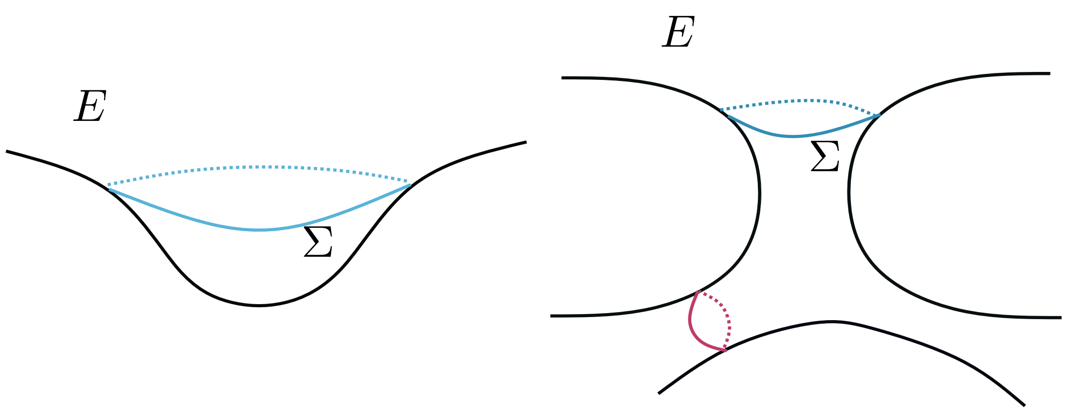
$$M = \mathbb{R}^n \setminus \{O\} \text{ and } g = \left(1 + \frac{m}{r^{n-2}}\right)^{\frac{4}{n-2}} g_{\mathbb{E}}$$

      has two ends E_1 and E_2 corresponding to O and ∞ respectively with $m(M, g, E_1) = m(M, g, E_2) = 2m$.

- Separation systole (used in mass-systole conjecture later)
  - Let E be an end of (M, g) and Σ be the boundary of a region Ω satisfying that $E\Delta\Omega$ is bounded. Let us call Σ a separation of E. Define

$$\mathrm{sys}(M, g, E) := \inf\{\mathrm{area}(\Sigma) : \ \Sigma \text{ is a separation of } E\}.$$

  - Separation of E



  - Examples
    * Euclidean space (manifolds with one end E) has $\mathrm{sys}(M, g, E) = 0$
    * Schwarzschild manifold (two ends) with a fixed end E satisfies

$$\mathrm{sys}(M, g, E) = \text{ area of the unique closed minimal hypersurface}$$

1.3 Previous mass inequalities

- Riemannian positive mass theorem

  (n_*: dimension where generic regularity from GMT holds, known for $n_* \leq 10$ [Smale '93, Chodosh-Mantoulidis-Schulze '23$^+$])

  – Theorem: Let (M^{n_*}, g) be an asymptotically flat manifold with nonnegative scalar curvature. Then for each end E it holds $m_{ADM}(M, g, E) \geq 0$, where equality holds for some end E if and only if (M, g) is isometric to the Euclidean space $(\mathbb{R}^{n_*}, g_{\mathbb{E}})$.

  – Related works:

    [Schoen-Yau '79 '81] (non-compact) minimal surface (3D)

    [Witten '81] spinor method (3D)

    [Schoen '89] non-compact dimension descent argument (n^*D)

    [Lohkamp '99] Lohkamp compactification (PMT⇒PSC obstruction)

    [Schoen-Yau '17$^+$/'21] a claim for all dimensions

- Riemannian-Penrose inequality
  - Theorem: Let $(M^{n\leq 7}, \partial M, g)$ be a complete Riemannian manifold with compact inner boundary ∂M and only one-end E, which is asymptotically flat. If (M, g) has nonnegative scalar curvature and ∂M is minimal and outer-minimizing, then we have
$$m_{ADM}(M, g, E) \geq \frac{1}{2}\left(\frac{|\partial M|_g}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}},$$
where equality holds if and only if (M, g) is isometric to the half Schwarzschild manifold.
  - Related works:
[Huisken-Ilmanen '01] IMCF (3D, ∂M connected, $H_2(M, \partial M) = 0$)
[Bray '01] conformal flow (and Gauss-Bonnet formula) (3D)
[Bray-Lee '09] conformal flow (nD, $n \leq 7$)
Also see recent nonlinear potential methods by [Agostiniani-Mantegazza-Mazzieri-Oronzio '22$^+$ '23, Hirsch-Miao-Tam '22$^+$]
  - Open for $n > 7$ due to singularity issue from GMT in Bray's conformal flow

2 Recent developments and a conjecture on mass-systole inequality

2.1 Recent developments

- Liouville theorem for locally conformally flat manifolds
  - Theorem: Let $(M^{n\geq 3}, g)$ be a complete locally conformally flat manifold with nonnegative scalar curvature, whose conformal structure is induced by a conformal map $\Phi : M \to \mathbb{S}^n$. Then Φ is injective and $\partial\Phi(M)$ has zero Newtonian capacity.
  - known cases before recent development
    [Schoen-Yau '94 , Lectures on Differential Geometry]
    a1. $d(M, [g]) < \frac{(n-2)^2}{n}$ when $n \geq 5$
    a2. $d(M, [g]) < \frac{(n-2)^2}{n}$ and $R(g) \leq C$ when $n = 3, 4$
    b. Riemannian positive mass theorem holds for some more general class of asymptotically flat manifolds (made clear in next slide)
  - known fact (at that time): $d(M, [g]) \leq \frac{n}{2}$ and so Liouville theorem reduced to generalized Riemannian positive mass theorem with $n \leq 6$.

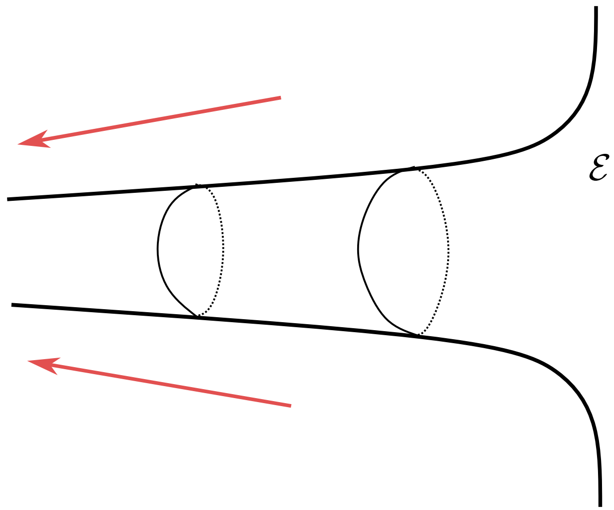
- asymptotically flat manifolds (with arbitrary ends)
  - complete Riemannian manifolds (M, g) with a distinguished asymptotically flat end $\mathcal{E}$
- Riemannian positive mass theorem (with arbitrary ends)
  - Theorem: Let $(M^{n_*}, g, \mathcal{E})$ be an asymptotically flat manifold with nonnegative scalar curvature. Then we have $m(M, g, \mathcal{E}) \geq 0$, where the equality holds if and only if (M, g) is the Euclidean space $(\mathbb{R}^{n_*}, g_{\mathbb{E}})$.
  - Related works: [Lesourd-Unger-Yau '21$^+$] asymptotically Schwarzwchild
    [Lee-Lesourd-Unger '23] general asymptotically flat (density theorem)
    [Z. '23] general asymptotically flat (through Geroch's compactification)
- Another approach to the Liouville theorem: generalized Geroch conjecture (raised in [Lesourd-Unger-Yau '20$^+$])
  - $T^{n_*} \sharp N$ (N can be non-compact) admits no complete metric with positive scalar curvature [Chodosh-Li '20$^+$]
  - [Wang-Zhang '22] proves generalized Geroch conjecture in all dimensions with extra spin assumption

2.2 A conjecture on mass-systole inequality

- Invalid case for Riemannian-Penrose inequality (no horizon)
  - extreme black hole (at infinity)
  - extreme Reissner-Nordström space

$$M = \mathbb{R}^3 \setminus \{O\} \text{ and } g = \left(1 + \frac{m}{r}\right)^2 g_{\mathbb{E}} \text{ with } m > 0$$

  - The intuition (mean-convex foliation)



- My mass-systole inequality conjecture
  - Conjecture: Let $(M^n, g, \mathcal{E})$ be an asymptotically flat manifold (with arbitrary ends) with nonnegative scalar curvature. Then we have

$$m_{ADM}(M, g, \mathcal{E}) \geq \frac{1}{2}\left(\frac{\mathrm{sys}(M, g, \mathcal{E})}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}},$$

    where equality holds if and only if the following happens:

    1. either $\mathrm{sys}(M, g, \mathcal{E}) = 0$ and (M, g) is the Euclidean space $(\mathbb{R}^n, g_{\mathbb{E}})$
    2. or $\mathrm{sys}(M, g, \mathcal{E}) > 0$ and there exists a separation sphere Σ of $\mathcal{E}$ with

$$\mathrm{area}(\Sigma) = \mathrm{sys}(M, g, \mathcal{E})$$

       enclosing the half Schwarzschild manifold (on the side of $\mathcal{E}$).

  - Some comments
    * Riemannian positive mass theorem (with arbitrary ends) is a special case
    * Riemannian-Penrose inequality follows from a doubling argument
    * This conjecture is open even in dimension three

3 Some progress on the conjecture

3.1 My recent results

- Riemannian positive mass theorem (with arbitrary ends) [Z. '23]
- Riemannian-Penrose inequality (with arbitrary ends) [Z. '23$^+$]
  - Theorem: Consider complete manifold $(M, g, \mathcal{E})$ with

    $$M = \mathbb{R}^3 \setminus \{O\},\ \mathcal{E} = \mathbb{R}^3 \setminus B_1 \text{ and } (\mathcal{E}, g) \text{ is asymptotically flat.}$$

    Then we have

    $$m_{ADM}(M, g, \mathcal{E}) \geq \sqrt{\frac{\operatorname{sys}(M, g, \mathcal{E})}{16\pi}},$$

    where equality holds if and only if $\operatorname{sys}(M, g, \mathcal{E}) > 0$ and there is a separation minimal 2-sphere Σ of $\mathcal{E}$ such that

    $$\operatorname{area}(\Sigma) = \operatorname{sys}(M, g, \mathcal{E})$$

    and the outside region is isometric to half Schwarzschild manifold.

3.2 Proof of Riemannian-Penrose inequality (with arbitrary ends)

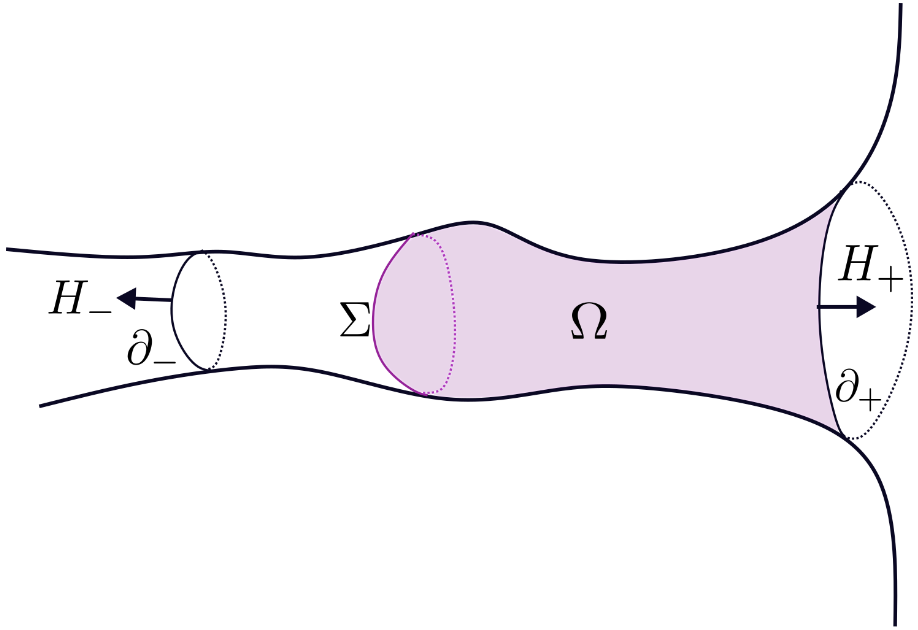
3.2.1 The inequality

- Techniques:
  - an approximation scheme of μ-bubbles
- Starting point
  - ADM-Hawking mass inequality [Huisken-Ilmanen '01]
    Let $(M^3, \partial M, g)$ be a complete Riemannian manifold with compact inner boundary ∂M and a unique end E, which is asymptotically flat. If M has nonnegative scalar curvature and satisfies $H_2(M, \partial M) = 0$, and the boundary ∂M is connected and outer-minimizing, then we have

$$m(M, g, E) \geq \sqrt{\frac{|\partial M|_g}{16\pi}}\left(1 - \frac{1}{16\pi}\int_{\partial M} H^2 \mathrm{d}\sigma_g\right).$$

  - Plan: find for each small $\epsilon > 0$ a 3-ball Ω_ϵ containing the origin O such that
    * $area(\partial\Omega_\epsilon) \leq A_0$ with A_0 independent of ϵ
    * $H(\partial\Omega_\epsilon) = \epsilon$

- The μ-bubble method



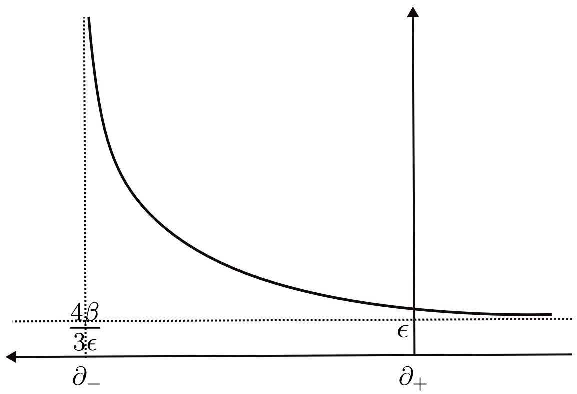
  - Let $(V, \partial_\pm)$ be a Riemannian band with a smooth function $h : V \to \mathbb{R}$.
  - Consider the functional

$$\mathcal{A}^h(\Sigma, \Omega) = \text{area}(\Sigma) + \int_\Omega h \, \mathrm{d}\mu_g.$$

  - existence of a smooth minimizer needs $h - H_+ < 0$ on ∂_+ and $h + H_- > 0$ on ∂_-

- The set-up of our μ-bubble problem
  - ∂_+ is taken to be some coordinate sphere with $H_+ \geq \epsilon_0 > 0$
  - ∂_- is taken to be $h = +\infty$ after h is determined
  - Take $h_{\epsilon,\beta}(t) = \epsilon \coth\left(-\frac{3}{4}\epsilon t + \beta\right)$ with $0 < \epsilon < \epsilon_0$ and β to be determined later, which satisfies

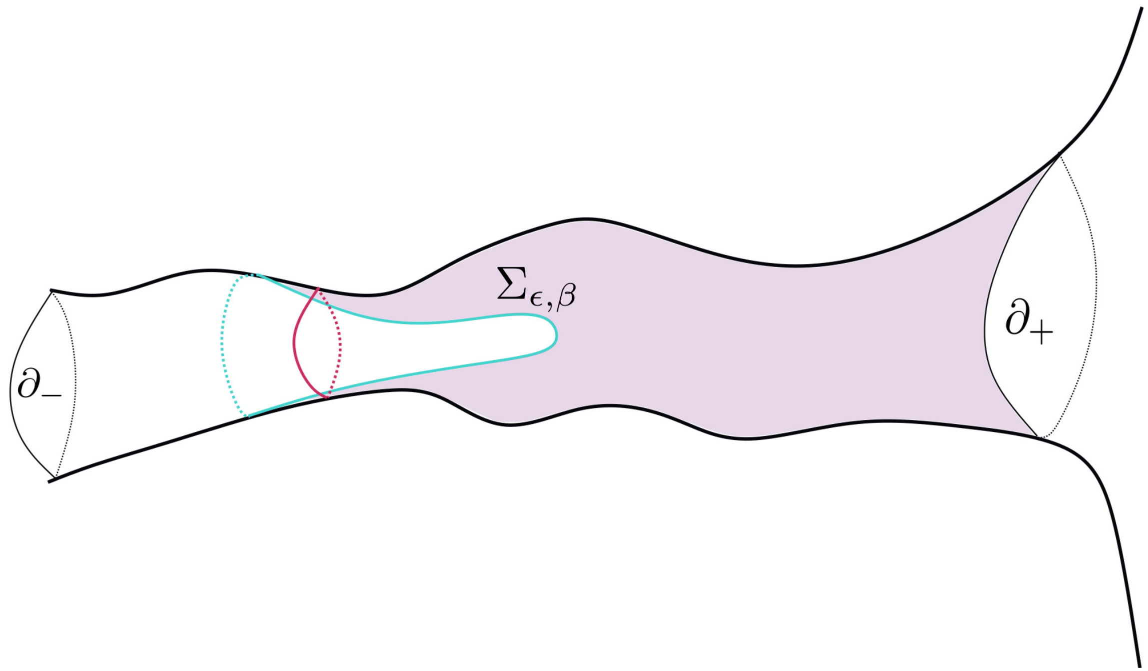
$$-2h'_{\epsilon,\beta} + \frac{3}{2}h^2_{\epsilon,\beta} = \frac{3}{2}\epsilon^2.$$



  - Let ρ be the distance function to ∂_+ (only $|\mathrm{d}\rho| \leq 1$ used so assuming smooth) and $h = h_{\epsilon,\beta} \circ \rho$ satisfying

$$-2|\mathrm{d}h| + \frac{3}{2}h^2 \geq \frac{3}{2}\epsilon^2.$$

- fixing phenomenon when $\text{sys}(M, g, \mathcal{E}) > 0$



we have

$$\int_{\Omega_{\epsilon,\beta}} h \, \mathrm{d}\mu_g + \text{area}(\Sigma_{\epsilon,\beta}) \leq \text{area}(\partial_+) =: A_0$$

and

$$\int_{\Omega_{\epsilon,\beta}} h \, \mathrm{d}\mu_g \geq \epsilon \cdot \text{sys}(M, g, \mathcal{E}) \cdot \text{dist}(\partial_+, \Sigma_{\epsilon,\beta}).$$

- topology and intrinsic diameter bound
  - 2nd variation formula

$$\int_{\Sigma_{\epsilon,\beta}} |\nabla\phi|^2 - (\mathrm{Ric}(\nu,\nu) + |A|^2 - \partial_\nu h)\phi^2 \geq 0$$

  - Schoen-Yau's rearrangement

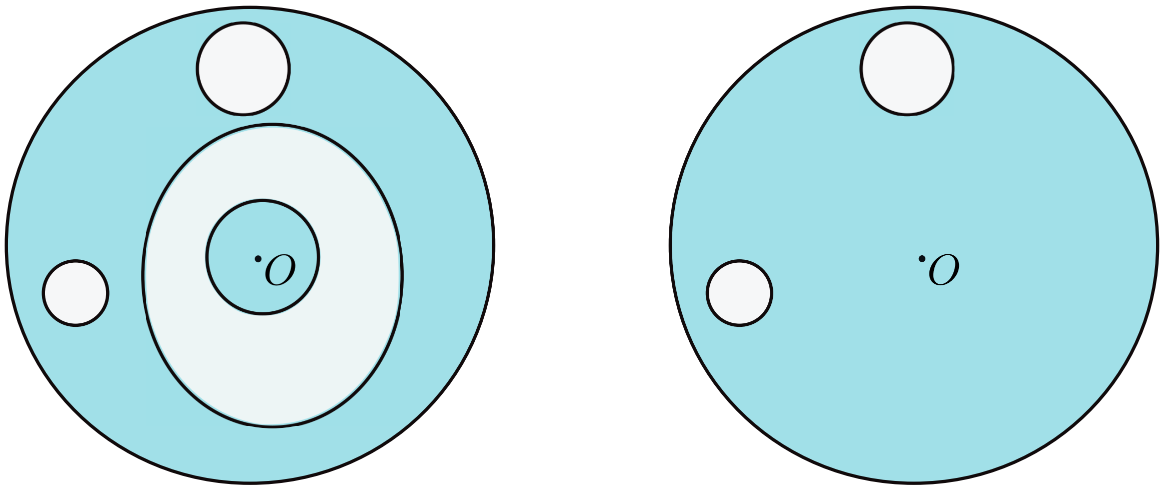
$$\int_{\Sigma} |\nabla\phi|^2 - \frac{1}{2}(R_M - R_\Sigma + \frac{3}{2}H^2 + |\mathring{A}|^2 + 2\partial_\nu h)\phi^2 \geq 0$$

    From the construction of h we see

$$\lambda_1(-\Delta + K_\Sigma) \geq \frac{3}{4}\epsilon^2$$

  - properties of $\Sigma_{\epsilon,\beta}$
    * (topology) $\Sigma_{\epsilon,\beta}$ is a sphere
    * (geometry) $\mathrm{diam}(\Sigma) \leq D_0$ (D_0 depending only on λ_1) [Schoen-Yau '83, Gromov '18]

- Take Σ_ϵ to be the limit of $\Sigma_{\epsilon,\beta}$. Then Σ_ϵ is the desired surface in the sense that
  - Σ_ϵ has uniform area bound A_0 (independent of ϵ)
  - Σ_ϵ has constant mean curvature ϵ
  - Σ_ϵ is embedded 2-sphere
- some technical modification
  - need to do component-picking: there is a unique component Σ_ϵ^o enclosing the origin O. We illustrate by the following figures (where the blue part is Ω_ϵ^c)



(a) Multiple enclosing spheres (b) a single enclosing sphere

3.2.2 The rigidity

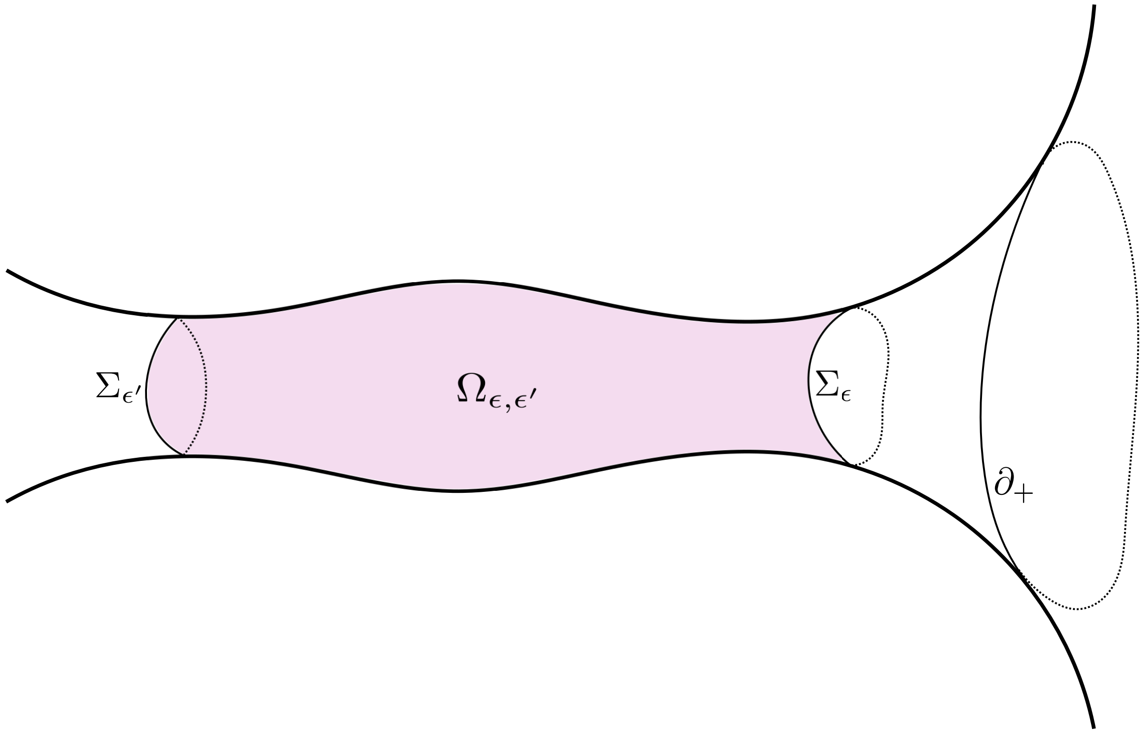
- Strategy: find the separation sphere attaining the separation systole
- Start from the equality

$$m = \sqrt{\frac{\mathrm{sys}(M, g, \mathcal{E})}{16\pi}}.$$

- Improved area control for approximating μ-bubble
  - former estimate $\mathrm{area}(\Sigma_\epsilon) \leq A_0$
  - improved one $\mathrm{area}(\Sigma_\epsilon) \leq \mathrm{sys}(M, g, \mathcal{E}) + C\epsilon^2$ with C independent of ϵ

$$\sqrt{\frac{|\Sigma_\epsilon|}{16\pi}}\left(1 - \frac{\epsilon^2}{16\pi}|\Sigma_\epsilon|\right) \leq m = \sqrt{\frac{\mathrm{sys}(M, g, \mathcal{E})}{16\pi}}.$$

- Iterated approximating μ-bubbles and new fixing phenomenon



  - With Σ_ϵ as one boundary for $0 < \epsilon' < \epsilon$ we can construct approximation μ-bubble $\Sigma_{\epsilon'}$ (with respect to Σ_ϵ)
  - we have volume bound for jumped region $\mathrm{vol}(\Omega_{\epsilon,\epsilon'}) \leq C\epsilon^2/\epsilon'$

$$\mathrm{sys}(M, g, \mathcal{E}) \leq |\Sigma_{\epsilon'}| \leq |\Sigma_\epsilon| - \epsilon' \,\mathrm{vol}(\Omega_{\epsilon,\epsilon'}) \leq \mathrm{sys}(M, g, \mathcal{E}) + C\epsilon^2 - \epsilon' \,\mathrm{vol}(\Omega_{\epsilon,\epsilon'}).$$

- Bootstraping
  - Fix $\gamma \in (1,2)$. Find iterated approximating μ-bubbles

$$\Sigma_\epsilon, \Sigma_{\epsilon^\gamma}, \Sigma_{\epsilon^{\gamma^2}}, \ldots$$

  - uniform volume bound for jumped regions

$$\mathrm{vol}(\Sigma_\epsilon, \Sigma_{\epsilon^{\gamma^k}}) \leq C \sum_k \left(\epsilon^{\gamma^k}\right)^{2-\gamma} \leq C \Rightarrow \mathrm{dist}(\Sigma_{\epsilon^{\gamma^k}}, \Sigma_\epsilon) \leq \frac{C}{\mathrm{sys}(M, g, \mathcal{E})}$$

  - $\Sigma_{\epsilon^{\gamma^k}}$ pointed converges to a stable minimal surface Σ_0

- *compactness criterion* [Gromov-Lawson '83]

  If Σ is a stable minimal surface in 3-manifold (M, g) with $R(g) \geq 0$, then

$$\Sigma \text{ is non-compact} \Leftrightarrow |\Sigma| = +\infty.$$

- $|\Sigma_0| \leq A_g \Rightarrow \Sigma_0$ is a closed minimal surface and this returns to the classical case.

Thank you for your attention!