Recent Advances in Comparison Geometry

# Network flow: the charm of the (apparent) simplicity

Alessandra Pluda

j.w. J. Lira, R. Mazzeo & M. Saez, M. Pozzetta,

C. Mantegazza & M. Novaga, Y. Tonegawa

Dipartimento di Matematica

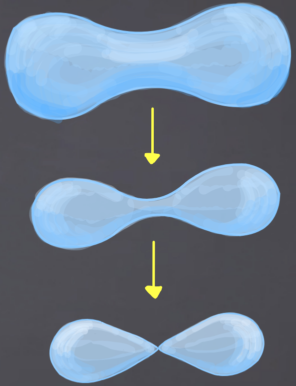
Università di Pisa

# Mean Curvature Flow

$\Sigma$ surface, $\varphi : [0,T) \times \Sigma \to \mathbb{R}^3$

$$\partial_t \varphi = H\nu$$

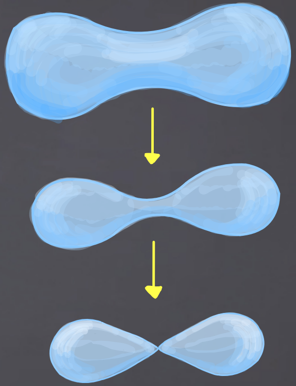
* Variational nature
* Geometric evolution equation
* System of second order PDE



# Mean Curvature Flow

$\Sigma$ surface, $\varphi : [0, T) \times \Sigma \to \mathbb{R}^3$

$$\partial_t \varphi = H\nu$$



- Variational nature
- Geometric evolution equation
- System of second order PDE

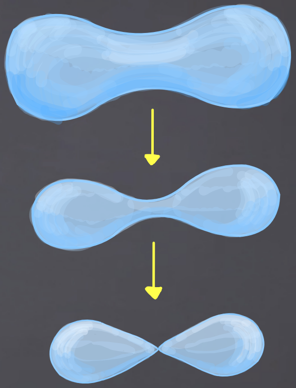
Surfaces

Huisken J.diff Geom. ‘84, Ecker-Huisken Ann. of Math ‘89, Invent. Math ‘91
Ilmanen Mem. Amer. Math. Soc. ‘94, Huisken-Sinestrari Acta Math ‘99
White JAMS ‘03 Ann. of Math ‘05,
Colding-Minicozzi Ann. of Math ‘12, Ann. of Math ‘15 ...

# Mean Curvature Flow

$\Sigma$ surface, $\varphi : [0,T) \times \Sigma \to \mathbb{R}^3$

$$\partial_t \varphi = H\nu$$



* Variational nature
* Geometric evolution equation
* System of second order PDE

Curves

Gage-Hamilton J.diff Geom. ‘86
Grayson JdG ‘87, Ann. of Math ‘89
Angenent Ann. of Math ‘90, JdG ‘91

Surfaces

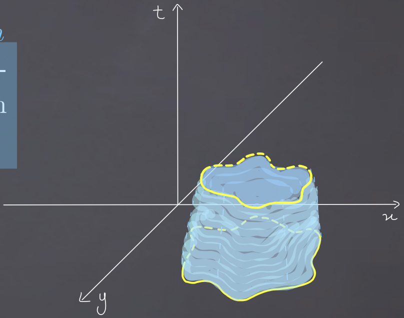
Huisken J.diff Geom. ‘84, Ecker-Huisken Ann. of Math ‘89, Invent. Math ‘91
Ilmanen Mem. Amer. Math. Soc. ‘94, Huisken-Sinestrari Acta Math ‘99
White JAMS ‘03 Ann. of Math ‘05,
Colding-Minicozzi Ann. of Math ‘12, Ann. of Math ‘15 ...

# Curve shortening flow

Curve shortening flow $\quad (\partial_t \gamma)^\perp = k\nu$

*Grayson's theorem*

A simple closed curve evolving by curvature becomes eventually convex and then shrinks to a round point in finite time.



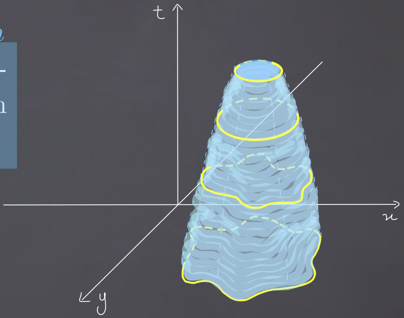

Gage-Hamilton J.diff Geom. '86, Grayson J.diff Geom. '87, Ann. of Math '89

# Curve shortening flow

Curve shortening flow $\quad (\partial_t \gamma)^\perp = k\nu$

*Grayson's theorem*

A simple closed curve evolving by curvature becomes eventually convex and then shrinks to a round point in finite time.



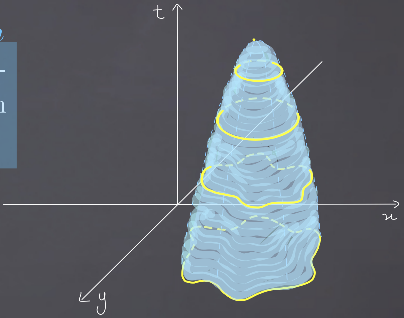

Gage-Hamilton J.diff Geom. '86, Grayson J.diff Geom. '87, Ann. of Math '89

# Curve shortening flow

Curve shortening flow $\qquad (\partial_t \gamma)^\perp = k\nu$

*Grayson's theorem*

A simple closed curve evolving by curvature becomes eventually convex and then shrinks to a round point in finite time.




$$\|k\|_2 \geq \frac{\int |k|\, ds}{(L(\gamma))^{1/2}} \geq \frac{2\pi}{(L(\gamma))^{1/2}}$$

$T_{\max} = \frac{A_0}{2\pi}$ with $A_0$ initial area

Gage-Hamilton J.diff Geom. '86, Grayson J.diff Geom. '87, Ann. of Math '89

# Network flow

A step further: singular surfaces

Network $\mathcal{N}$: connected set, composed of finitely many regular embedded curves $\gamma^i$ that meet at their endpoints in junctions.

Network flow: gradient flow of the length

# Network flow

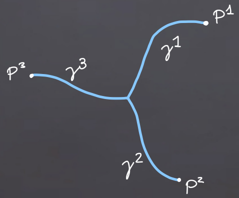
A step further: singular surfaces

Network $\mathcal{N}$: connected set, composed of finitely many regular embedded curves $\gamma^i$ that meet at their endpoints in junctions.

Network flow: gradient flow of the length

Consider a network $\mathcal{N} = \{\gamma^i\}_{i=1}^N$ and a variation $\widetilde{\mathcal{N}} = \{\widetilde{\gamma}^i\}_{i=1}^N = \{\gamma^i + t\varphi^i\}_{i=1}^N$

$$\frac{d}{dt} L(\widetilde{\mathcal{N}}(t)) = \frac{d}{dt}\left(\sum_i \int |\dot{\gamma}^i + t\dot{\varphi}^i| \,\mathrm{d}x\right) = \sum_i \int \left\langle -\vec{\kappa}^i, \varphi^i \right\rangle \mathrm{d}s + \langle \sum_i \tau^i, \varphi^i \rangle|_{\mathrm{bdry}}$$



# Network flow

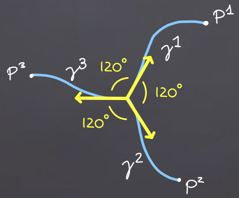
A step further: singular surfaces

Network $\mathcal{N}$: connected set, composed of finitely many regular embedded curves $\gamma^i$ that meet at their endpoints in junctions.

Network flow: gradient flow of the length

Consider a network $\mathcal{N} = \{\gamma^i\}_{i=1}^N$ and a variation $\widetilde{\mathcal{N}} = \{\widetilde{\gamma}^i\}_{i=1}^N = \{\gamma^i + t\varphi^i\}_{i=1}^N$

$$\frac{d}{dt}L(\widetilde{\mathcal{N}}(t)) = \frac{d}{dt}\left(\sum_i \int |\dot{\gamma}^i + t\dot{\varphi}^i|\,\mathrm{d}x\right) = \sum_i \int \left\langle -\vec{\kappa}^i, \varphi^i \right\rangle \mathrm{d}s + \langle \sum_i \tau^i, \varphi^i \rangle|_{\mathrm{bdry}}$$



# Network flow

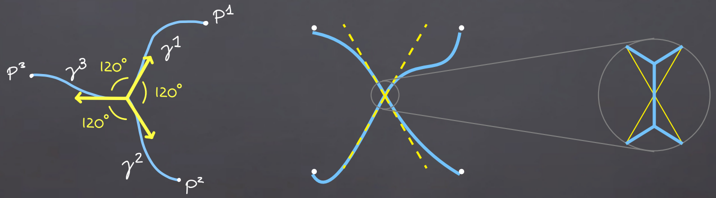
A step further: singular surfaces

Network $\mathcal{N}$: connected set, composed of finitely many regular embedded curves $\gamma^i$ that meet at their endpoints in junctions.

Network flow: gradient flow of the length

Consider a network $\mathcal{N} = \{\gamma^i\}_{i=1}^N$ and a variation $\widetilde{\mathcal{N}} = \{\widetilde{\gamma}^i\}_{i=1}^N = \{\gamma^i + t\varphi^i\}_{i=1}^N$

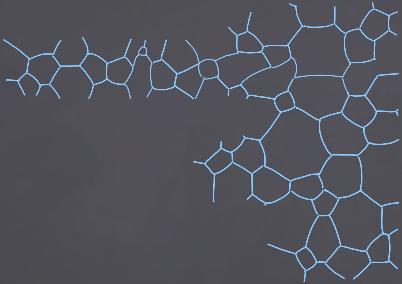
$$\frac{d}{dt} L(\widetilde{\mathcal{N}}(t)) = \frac{d}{dt}\left(\sum_i \int |\dot{\gamma}^i + t\dot{\varphi}^i| \, \mathrm{d}x\right) = \sum_i \int \left\langle -\vec{\kappa}^i, \varphi^i \right\rangle \mathrm{d}s + \langle \sum_i \tau^i, \varphi^i \rangle|_{\mathrm{bdry}}$$



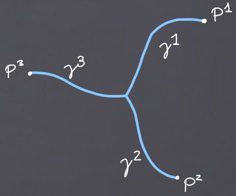

We expect to see networks with only triple junctions for almost all times.

# PDE formulation

Consider as initial datum a network with triple junctions.



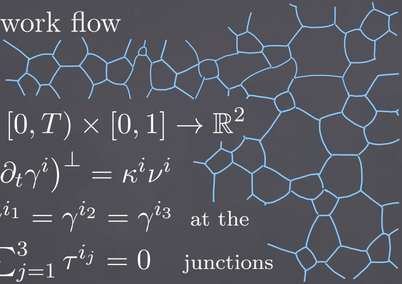
Simplest example: triod



# PDE formulation

Consider as initial datum a network with triple junctions.

Network flow



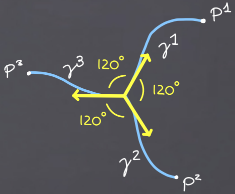
$\gamma^i : [0,T) \times [0,1] \to \mathbb{R}^2$

$$\begin{cases} \left(\partial_t \gamma^i\right)^\perp = \kappa^i \nu^i \\ \gamma^{i_1} = \gamma^{i_2} = \gamma^{i_3} \quad \text{at the} \\ \sum_{j=1}^{3} \tau^{i_j} = 0 \quad \text{junctions} \end{cases}$$

+ fixed endpoints

or periodic boundary conditions

Simplest example: triod




* Junctions are free to move
* Tangential motion

# Motivations

∗ Develop a theory of strong solutions to mean curvature flow for surfaces with mild singularities

∗ Enrich the list of generic singularities in mean curvature flow
↬ topological singularities with bounded curvature

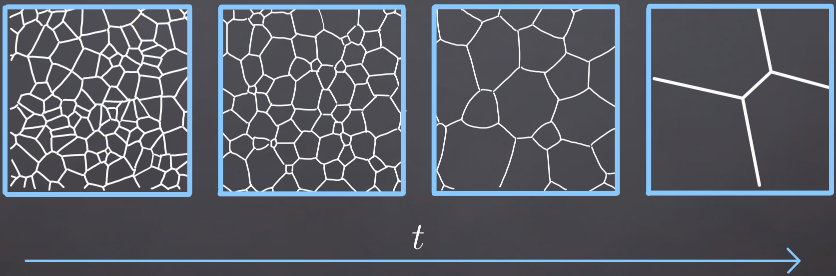
∗ Close the gap between simulation and theory
↬ capture the coarsening behavior of the network flow

# Expected evolution

Initial network with complex topology

Then, $\exists$ solution in the maximal time interval $[0, T)$
with singularities at times $t_1 < t_2 < \ldots < T$

* if $T < \infty$: everything vanishes (example: closed curve)
* if $T = \infty$: convergence to a network composed of straight segments with drastically simpler topology/structure



# Basic properties

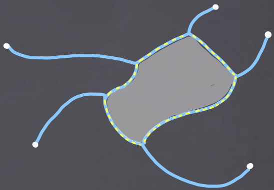
Consider a region bounded by a loop $\ell$ composed of $m$ curves.

By Gauss-Bonnet we have

$\partial_t A = (m/3 - 2)\pi$ Von Neumann law

and

$$|2 - m/3|\pi \leq \int_\ell |k| \, \mathrm{d}s \leq \|k\|_2 \sqrt{L(\ell)}$$



# Basic properties

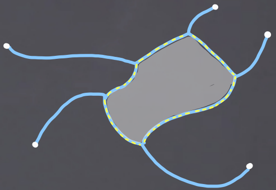
Consider a region bounded by a loop $\ell$ composed of $m$ curves.

By Gauss-Bonnet we have

$\partial_t A = (m/3 - 2)\pi$ Von Neumann law

and

$$|2 - m/3|\pi \leq \int_\ell |k|\, \mathrm{d}s \leq \|k\|_2 \sqrt{L(\ell)}$$



Suppose that $\widetilde{t}$ is a singular time

Then, as $t \nearrow \widetilde{t}$ at least one of the following happens:

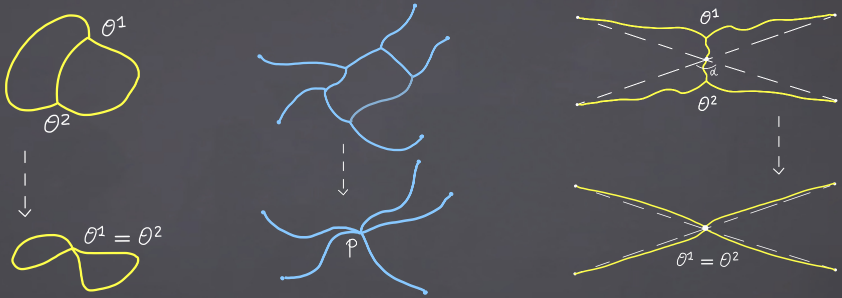
*i)* the inferior limit of the length of at least one curve is zero;

*ii)* the $L^2$–norm of the curvature becomes unbounded.

Mantegazza-Novaga-Pluda-Schulze Astérisque '2x

# Singularities

Example of singularities



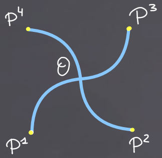

*Type-0 singularities*

When two triple junctions coalesce without the vanishing of a region the curvature remains bounded.

Mantegazza-Novaga-Pluda J. Reine Angew. Math. (Crelle's J.) ‘22

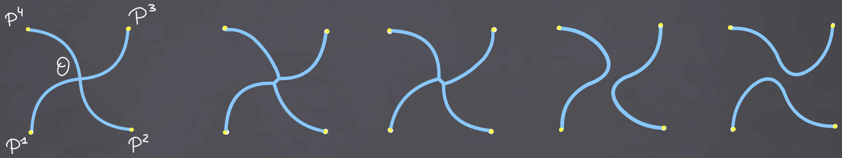
# Flow past singularities

Solution $\mathcal{N}(t)_{t\in[0,T)}$ with $0 \leq t_1 < \ldots < t_N \leq T$ singular times



# Flow past singularities

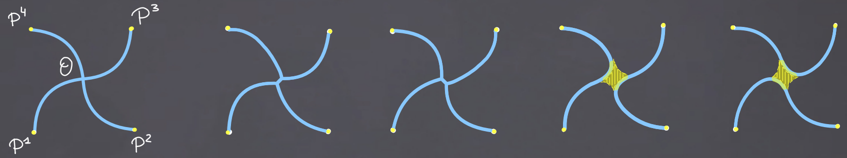
Solution $\mathcal{N}(t)_{t\in[0,T)}$ with $0 \leq t_1 < \ldots < t_N \leq T$ singular times




* issue 1: jump of topology from $t = t_i$ to $t > t_i$
* issue 2: non–uniqueness

# Flow past singularities

Solution $\mathcal{N}(t)_{t\in[0,T)}$ with $0 \le t_1 < \ldots < t_N \le T$ singular times



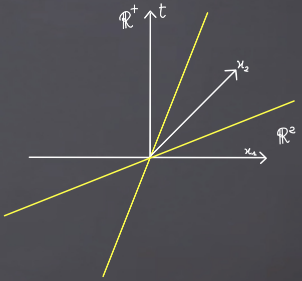

* issue 1: jump of topology from $t = t_i$ to $t > t_i$
* issue 2: non–uniqueness

* $\mathcal{N}(t)$ solves the network flow $\forall t \in (t_i, t_{i+1})$
* $\mathcal{N}(t)$, as a set, is continuous at $t_1, \ldots, t_N$

# A special case

$\mathcal{N}(t)$ at a singular time $\widetilde{t}$ is a fan $\mathcal{F}$ of half–lines $h_1, \ldots, h_\ell$

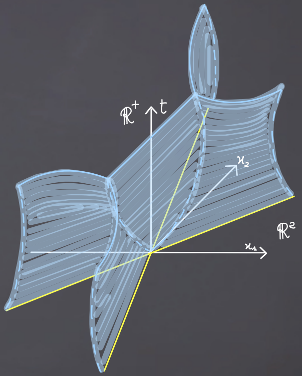
Then, all expanding solitons with non-compact branches $\gamma^1, \ldots, \gamma^\ell$ asymptotic at infinity to $h_1, \ldots, h_\ell$ are network flows out of $\mathcal{F}$. In particular, there exists a flow past singularity.



# A special case

$\mathcal{N}(t)$ at a singular time $\widetilde{t}$ is a fan $\mathcal{F}$ of half–lines $h_1, \ldots, h_\ell$

Then, all expanding solitons with non-compact branches $\gamma^1, \ldots, \gamma^\ell$ asymptotic at infinity to $h_1, \ldots, h_\ell$ are network flows out of $\mathcal{F}$. In particular, there exists a flow past singularity.




Expanding solitons evolve self-similarly by magnification.

$\gamma(t, x) = \lambda(t)\eta(x/\lambda(t)) \qquad k - \eta^\perp = 0$ and $\lambda = \sqrt{2t}$

Variational proof - expanding solitons are critical point of the length in $(\mathbb{R}^2, g)$ with $g = e^{|x|^2}|dx|$

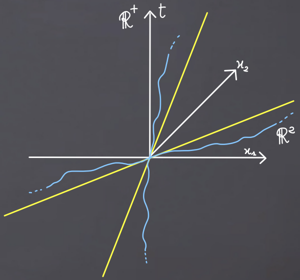
# General case - asymptotic expansion

Let $\mathcal{N}$ be a network of four curves meeting at the origin.

Suppose there exists a solution past singularity

Let $(t, x) \in [0, T) \times [0, 1] =: Q^i$.

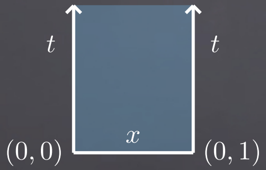
We interpret each curve $\gamma^i$ as a map
$(t, x) \to (t, \gamma^i(t, x)) \in \mathbb{R}^+_t \times \mathbb{R}^2_{x,y} =: Z$




$+$ a new curve $\gamma^5$ defined of $P := \{(t, x) \in \mathbb{R}^+_t \times \mathbb{R} \,|\, 0 \leq x \leq \sqrt{t}\}$

# Blown–up construction: domain

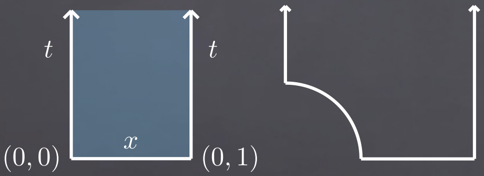
$Q^i_h :=$ blow-up of $Q^i$
obtained parabolically blowing up the singular point $(0,0)$



# Blown–up construction: domain

$Q^i_h :=$ blow-up of $Q^i$
obtained parabolically blowing up the singular point $(0,0)$
namely, by introducing parabolic polar coordinate near $(0,0)$

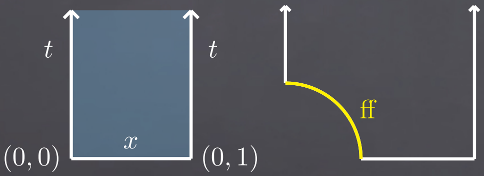
$$t = \rho\cos\omega, x = \rho^2\sin\omega \Leftrightarrow \rho = \sqrt{t + x^2} \geq 0,\ \omega = \arcsin(\frac{t}{\rho^2}) = \arccos(\frac{x}{\rho}) \in [0, \pi/2]$$



# Blown–up construction: domain

$Q^i_h :=$ blow-up of $Q^i$
obtained parabolically blowing up the singular point $(0,0)$
namely, by introducing parabolic polar coordinate near $(0,0)$

$$t = \rho\cos\omega, x = \rho^2\sin\omega \Leftrightarrow \rho = \sqrt{t + x^2} \geq 0,\ \omega = \arcsin(\frac{t}{\rho^2}) = \arccos(\frac{x}{\rho}) \in [0, \pi/2]$$

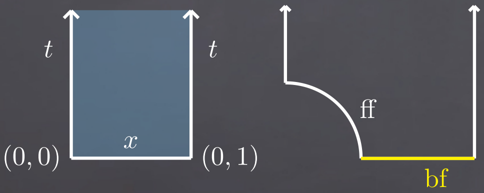



ff: $\varrho = 0,\ \omega \in [0, \pi/2]$

# Blown–up construction: domain

$Q^i_h :=$ blow-up of $Q^i$
obtained parabolically blowing up the singular point $(0,0)$
namely, by introducing parabolic polar coordinate near $(0,0)$

$$t = \rho \cos\omega, x = \rho^2 \sin\omega \Leftrightarrow \rho = \sqrt{t + x^2} \geq 0,\ \omega = \arcsin(\frac{t}{\rho^2}) = \arccos(\frac{x}{\rho}) \in [0, \pi/2]$$



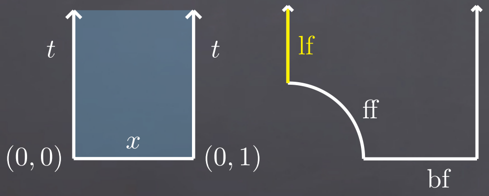

ff: $\varrho = 0,\ \omega \in [0, \pi/2]$

bf: $\omega = 0$ initial data

# Blown–up construction: domain

$Q^i_h :=$ blow-up of $Q^i$
obtained parabolically blowing up the singular point $(0,0)$
namely, by introducing parabolic polar coordinate near $(0,0)$

$$t = \rho\cos\omega, x = \rho^2 \sin\omega \Leftrightarrow \rho = \sqrt{t + x^2} \geq 0,\ \omega = \arcsin(\frac{t}{\rho^2}) = \arccos(\frac{x}{\rho}) \in [0, \pi/2]$$




ff: $\varrho = 0,\ \omega \in [0, \pi/2]$

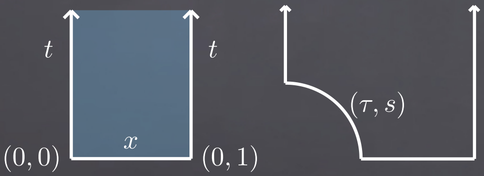
bf: $\omega = 0$ initial data

lf: $\omega = \pi/2$ Herring cond.

# Blown–up construction: domain

$Q^i_h :=$ blow-up of $Q^i$
obtained parabolically blowing up the singular point $(0,0)$
namely, by introducing parabolic polar coordinate near $(0,0)$

$$t = \rho\cos\omega, x = \rho^2\sin\omega \Leftrightarrow \rho = \sqrt{t+x^2} \geq 0,\ \omega = \arcsin(\frac{t}{\rho^2}) = \arccos(\frac{x}{\rho}) \in [0,\pi/2]$$




ff: $\varrho = 0,\ \omega \in [0,\pi/2]$

bf: $\omega = 0$ initial data

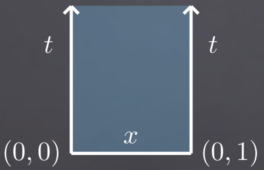
lf: $\omega = \pi/2$ Herring cond.

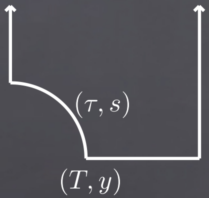
where $\tau = \sqrt{t},\ s = \frac{x}{\sqrt{t}}$

# Blown–up construction: domain

$Q^i_h$ := blow-up of $Q^i$
obtained parabolically blowing up the singular point $(0,0)$
namely, by introducing parabolic polar coordinate near $(0,0)$

$$t = \rho\cos\omega, x = \rho^2\sin\omega \Leftrightarrow \rho = \sqrt{t + x^2} \geq 0,\ \omega = \arcsin(\frac{t}{\rho^2}) = \arccos(\frac{x}{\rho}) \in [0, \pi/2]$$




ff: $\varrho = 0,\ \omega \in [0, \pi/2]$

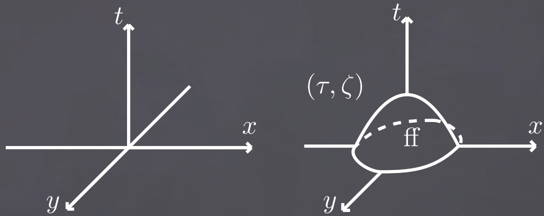
bf: $\omega = 0$ initial data

lf: $\omega = \pi/2$ Herring cond.

where $\tau = \sqrt{t}$, $s = \frac{x}{\sqrt{t}}$ and $T = \frac{t}{x^2}$, $y = x$

# Blown–up construction: range

$Z_h$:= blow-up of $Z$
obtained parabolically blowing up $x = y = t = 0$,




projective coordinates valid away from bf: $\tau = \sqrt{2t}$ and $\zeta = \frac{z}{\sqrt{2t}}$

$(t, \gamma^i(t,x))$ lifts to $(\frac{1}{2}\tau^2, \tau\eta^i(\tau, s))$

$\tau = 0$ is a defining function for ff

# The lifted equation

We lift each $\gamma^i$ from $Q_h^i$ to $Z_h$. This lifting is effected simply by writing

$$\partial_t \gamma = \frac{\partial_x^2 \gamma}{|\partial_x \gamma|^2}$$

using the coordinate systems $(\tau, s)$ on $Q_h^i$ and $(\tau, \zeta)$ on $Z_h$

# The lifted equation

We lift each $\gamma^i$ from $Q_h^i$ to $Z_h$. This lifting is effected simply by writing

$$\partial_t \gamma = \frac{\partial_x^2 \gamma}{|\partial_x \gamma|^2}$$

using the coordinate systems $(\tau, s)$ on $Q_h^i$ and $(\tau, \zeta)$ on $Z_h$

We get $\partial_t = \tau^{-2}(\tau\partial_\tau - s\partial_s)$ and $\partial_x = \tau^{-1}\partial_s$, hence

$$(\tau\partial_\tau + 1 - s\partial_s)\,\eta = \frac{\partial_s^2 \eta}{|\partial_s \eta|^2}$$

# The lifted equation

We lift each $\gamma^i$ from $Q_h^i$ to $Z_h$. This lifting is effected simply by writing

$$\partial_t \gamma = \frac{\partial_x^2 \gamma}{|\partial_x \gamma|^2}$$

using the coordinate systems $(\tau, s)$ on $Q_h^i$ and $(\tau, \zeta)$ on $Z_h$

We get $\partial_t = \tau^{-2}(\tau\partial_\tau - s\partial_s)$ and $\partial_x = \tau^{-1}\partial_s$, hence

$$(\tau\partial_\tau + 1 - s\partial_s)\,\eta = \frac{\partial_s^2 \eta}{|\partial_s \eta|^2}$$

Herring condition at $s = 0$

$\eta^1(\tau,0) = \eta^2(\tau,0) = \eta^5(\tau,0)$ and $\eta^3(\tau,0) = \eta^4(\tau,0) = \eta^5(\tau,1)$

$\frac{\partial_s \eta^1(\tau,0)}{|\partial_s \eta^1(\tau,0)|} = \frac{\partial_s \eta^2(\tau,0)}{|\partial_s \eta^2(\tau,0)|} = \frac{\partial_s \eta^5(\tau,0)}{|\partial_s \eta^5(\tau,0)|}$ and $\frac{\partial_s \eta^3(\tau,0)}{|\partial_s \eta^3(\tau,0)|} = \frac{\partial_s \eta^4(\tau,0)}{|\partial_s \eta^4(\tau,0)|} = -\frac{\partial_s \eta^5(\tau,1)}{|\partial_s \eta^5(\tau,1)|}$

# Initial datum

To write the system in the new coordinate we shall as well specify an initial condition at $\tau = 0$
Note that $\tau \partial_\tau \eta^i|_{\tau=0} = 0$, from we deduce that

$$\frac{\partial_s^2 \eta_0}{|\partial_s \eta_0|^2} + (s\partial_s - 1)\,\eta_0 = 0$$

# Initial datum

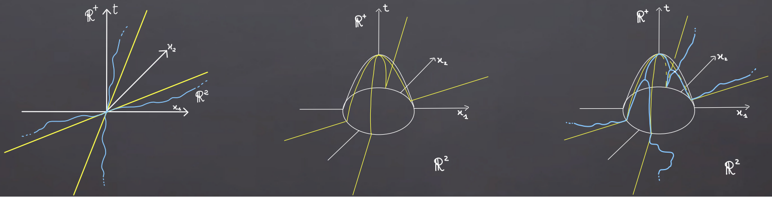
To write the system in the new coordinate we shall as well specify an initial condition at $\tau = 0$

Note that $\tau \partial_\tau \eta^i|_{\tau=0} = 0$, from we deduce that

$$\frac{\partial_s^2 \eta_0}{|\partial_s \eta_0|^2} + (s\partial_s - 1)\,\eta_0 = 0$$

that is nothing but the expander equation $k - \eta^\perp = 0$ in the new coordinate.

On the ff we have an expander!



# Existence of the flow past singularity

In the blown-up space the number of curves of the network at $\widetilde{t}$ is the same as the number of curves of the network at $t > \widetilde{t}$.

We can then say that $\mathcal{N}(t)$ is a solution of the flow past singularity if $t \searrow \widetilde{t}$ the maps converges in the blown-up space.

# Existence of the flow past singularity

In the blown-up space the number of curves of the network at $\widetilde{t}$ is the same as the number of curves of the network at $t > \widetilde{t}$.

We can then say that $\mathcal{N}(t)$ is a solution of the flow past singularity if $t \searrow \widetilde{t}$ the maps converges in the blown-up space.

There exists the network flow past singularity.

Moreover, the set of possible flow out is classified by the collection of (expanders) compatible with the irregular junction.

Ilmanen-Neves-Schulze J. diff. Geom ‘19, Lira-Mazzeo-Pluda-Sáez CPAM ‘23

# Topological complexity through a singularity

Flow-outs are classified by the collection of self–similar expanding solutions (expanders) compatible with the irregular junction

# Topological complexity through a singularity

Flow-outs are classified by the collection of self–similar expanding solutions (expanders) compatible with the irregular junction

If up to five curves concur at an irregular junction,
all the compatible expanders have no loops

# Topological complexity through a singularity

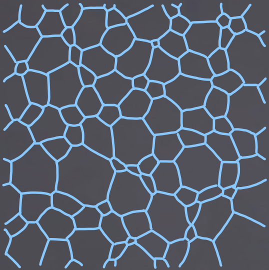
Flow-outs are classified by the collection of self–similar expanding solutions (expanders) compatible with the irregular junction

If up to five curves concur at an irregular junction, all the compatible expanders have no loops

If a region vanishes, then

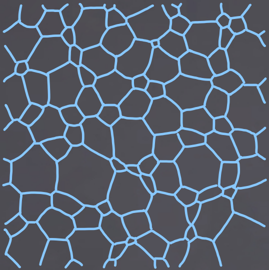
* the total number of curves decreases by at least three
* the total number of triple junctions decreases by at least two.

# Average growth of the area of the grains



$N^2$ grains, total length $L(\mathcal{N}) = \mathcal{O}(N)$
average area of a cell $= \mathcal{O}(1/N^2)$
average length of a loop $L = \mathcal{O}(1/N)$

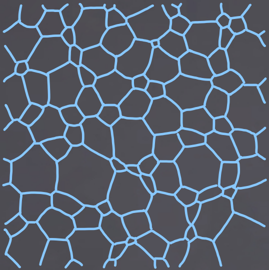
# Average growth of the area of the grains



$N^2$ grains, total length $L(\mathcal{N}) = \mathcal{O}(N)$
average area of a cell $= \mathcal{O}(1/N^2)$
average length of a loop $L = \mathcal{O}(1/N)$

Along each loop $\ell$ it holds $\int_\ell k^2 \, ds \geq \frac{C}{L(\ell)}$
with $C > 0$ for non-hexagonal cells.

# Average growth of the area of the grains

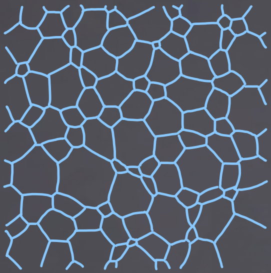


$N^2$ grains, total length $L(\mathcal{N}) = \mathcal{O}(N)$
average area of a cell $= \mathcal{O}(1/N^2)$
average length of a loop $L = \mathcal{O}(1/N)$

Along each loop $\ell$ it holds $\int_\ell k^2 \,\mathrm{d}s \geq \frac{C}{L(\ell)}$
with $C > 0$ for non-hexagonal cells.

Until $\sharp$ non–hexagonal $\sim N^2$
$\int_{\mathcal{N}} k^2 \,\mathrm{d}s \geq \frac{C}{L(\ell)} \sharp\, \text{cells} = CN^3$

# Average growth of the area of the grains



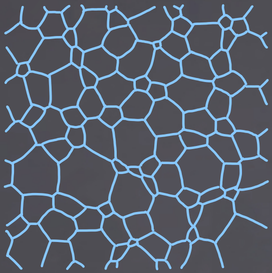
$N^2$ grains, total length $L(\mathcal{N}) = \mathcal{O}(N)$
average area of a cell $= \mathcal{O}(1/N^2)$
average length of a loop $L = \mathcal{O}(1/N)$

Along each loop $\ell$ it holds $\int_\ell k^2\,\mathrm{d}s \geq \frac{C}{L(\ell)}$
with $C > 0$ for non-hexagonal cells.

Until $\sharp$ non–hexagonal $\sim N^2$
$\int_\mathcal{N} k^2\,\mathrm{d}s \geq \frac{C}{L(\ell)} \sharp\,\text{cells} = CN^3$

From the gradient flow structure $\frac{d}{dt}L = -\int_\mathcal{N} k^2\,\mathrm{d}s$

# Average growth of the area of the grains



$N^2$ grains, total length $L(\mathcal{N}) = \mathcal{O}(N)$
average area of a cell $= \mathcal{O}(1/N^2)$
average length of a loop $L = \mathcal{O}(1/N)$

Along each loop $\ell$ it holds $\int_\ell k^2 \, \mathrm{d}s \geq \frac{C}{L(\ell)}$
with $C > 0$ for non-hexagonal cells.

Until $\sharp$ non–hexagonal $\sim N^2$
$\int_{\mathcal{N}} k^2 \, \mathrm{d}s \geq \frac{C}{L(\ell)} \sharp \, \text{cells} = CN^3$

From the gradient flow structure $\frac{d}{dt} L = - \int_{\mathcal{N}} k^2 \, \mathrm{d}s$

$$\frac{d}{dt} N(t) \lesssim CN^3(t) \Rightarrow \frac{1}{N(0)^2} - \frac{1}{N(t)^2} \lesssim -2Ct \qquad \frac{1}{N(0)^2} \to 0$$

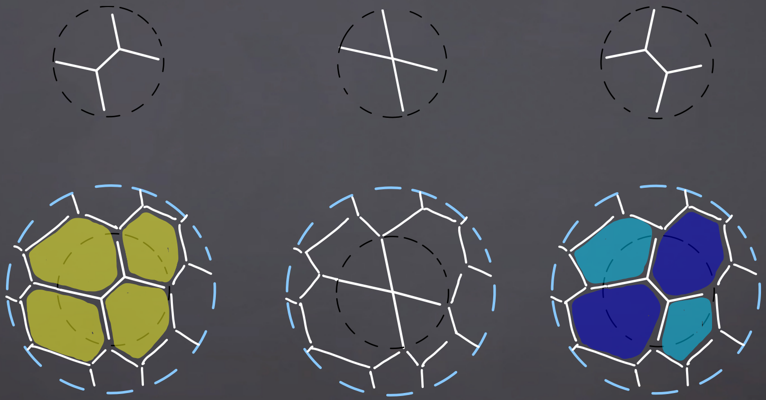
The average area grows linearly: $\frac{1}{N(t)^2} \geq 2Ct$

# From local to global?

Example: standard transition - topological complexity is preserved

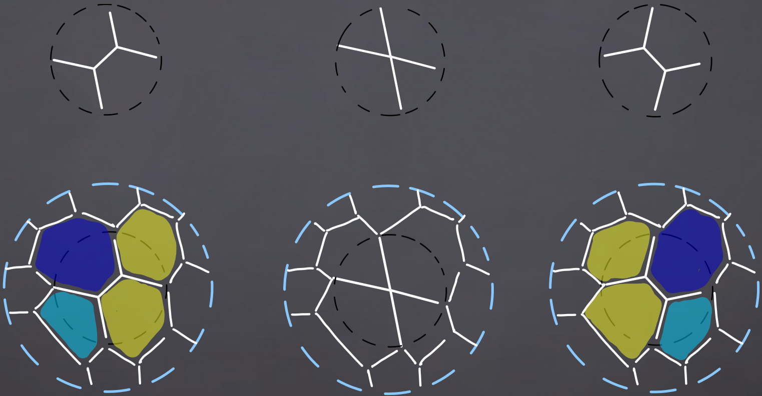
# From local to global?

Example: standard transition - topological complexity is preserved



# From local to global?

Example: standard transition - topological complexity is preserved



## Stability

Networks with triple junctions and straight segments are steady (each grain is a hexagon). Are they also attractors?

Pluda-Pozzetta Math. Ann. '23

# Stability

Networks with triple junctions and straight segments are steady (each grain is a hexagon). Are they also attractors?

Let $\mathcal{N}_*$ be a network with triple junctions and straight segments. Then,

$\exists \varepsilon = \varepsilon(\mathcal{N}_*)$ such that the network flow starting from any regular network $\mathcal{N}_0$ with

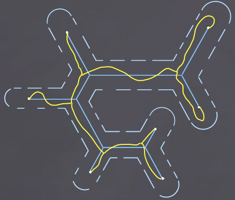
$$\|\mathcal{N}_* - \mathcal{N}_0\|_{H^2} < \varepsilon$$

exists for all times and converges to $\mathcal{N}_\infty$ with $L(\mathcal{N}_\infty) = L(\mathcal{N}_*)$.

Pluda-Pozzetta Math. Ann. '23

# Basin of attraction of critical points

Let $\mathcal{N}_*$ be a networks with triple junctions and straight segments and $d$ be the length of its shortest edge.

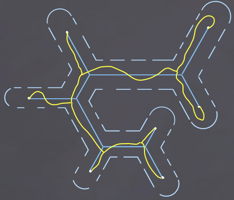


Then, $\mathrm{L}(\mathcal{N}) \geq \mathrm{L}(\mathcal{N}_*)$
for every $\mathcal{N}$ with
$\|\mathcal{N}_* - \mathcal{N}\|_{C^0} < \delta = \frac{\sqrt{3}}{8} d.$

Pluda-Pozzetta Bull. Lond. Math. Soc. '23

# Basin of attraction of critical points

Let $\mathcal{N}_*$ be a networks with triple junctions and straight segments and $d$ be the length of its shortest edge.



Then, $\mathrm{L}(\mathcal{N}) \geq \mathrm{L}(\mathcal{N}_*)$
for every $\mathcal{N}$ with
$\|\mathcal{N}_* - \mathcal{N}\|_{C^0} < \delta = \frac{\sqrt{3}}{8} d$.

The more complex the network, the smaller the $\delta$.

Indication that critical points with higher complexity should have a smaller basin of attraction.

Pluda-Pozzetta Bull. Lond. Math. Soc. '23

## Supporting arguments

* Topological complexity is non-increasing through singularities.

* Grains bound by less than six curves disappear during the evolution. The average area of the (surviving) grains grows linearly.

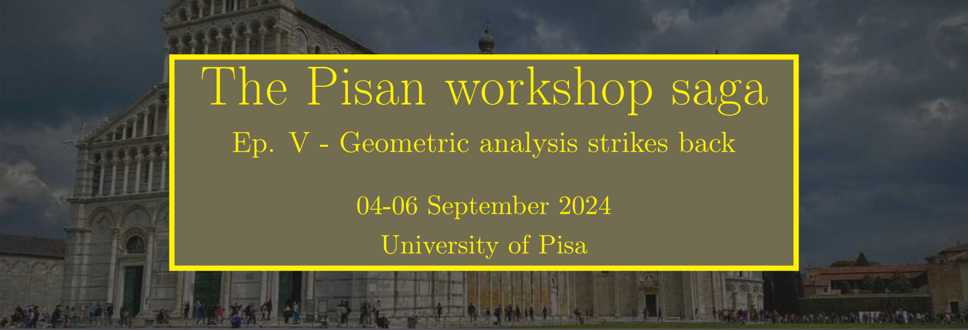
* The volume of the basin of attraction of all the many critical points of the length functional is small in the space of networks.

# Rigorous description of coarserning?

Chose randomly $n$ points in $\mathbb{R}^2$.

As initial datum for the network flow take the Voronoi partition associated with the given $n$ points.

Then, $\exists\, \varphi(n)$ negligible with respect to $n$ such that the probability that the limit network (as $t \to \infty$) has more than $\varphi(n)$ cells goes to zero as $n \to \infty$.



# The Pisan workshop saga

## Ep. V - Geometric analysis strikes back

04-06 September 2024

University of Pisa

https://sites.google.com/view/thepisanworkshopssaga/home