Ricci flow smoothing and its application to scalar curvature rigidity

Man-Chun Lee

The Chinese University of Hong Kong

Feb, 2024
IASM-BIRS 5 day workshop

Motivation: Rigidity of torus and sphere

Rigidity related to scalar curvature:

Motivation: Rigidity of torus and sphere

Rigidity related to scalar curvature:

Theorem (Gromov-Lawson)

Riemannian metric g on n-torus $\mathbb{T}^n$ are flat.

Motivation: Rigidity of torus and sphere

Rigidity related to scalar curvature:

Theorem (Gromov-Lawson)

Riemannian metric g on n-torus $\mathbb{T}^n$ are flat.

Theorem (Llarull)

If (M,g) is closed spin manifold with $\mathcal{R} \geq n(n-1)$ and admits 1-lipschitz map $f : M \to \mathbb{S}^n$, then f is an isometry or of zero degree.

Motivation: Rigidity of torus and sphere

Rigidity related to scalar curvature:

Theorem (Gromov-Lawson)

Riemannian metric g on n-torus $\mathbb{T}^n$ are flat.

Theorem (Llarull)

If (M, g) is closed spin manifold with $\mathcal{R} \geq n(n-1)$ and admits 1-lipschitz map $f : M \to \mathbb{S}^n$, then f is an isometry or of zero degree.

Question: What if g (and f in case of sphere) is non-smooth?

Motivating (related) Questions:

- Gromov: What is the most reasonable compactness in scalar curvature geometry?
- Gromov: Define the notion of $\mathcal{R} \geq \kappa$ for non-smooth metrics

Motivating (related) Questions:

- Gromov: What is the most reasonable compactness in scalar curvature geometry?
- Gromov: Define the notion of $\mathcal{R} \geq \kappa$ for non-smooth metrics
- Schoen: If g is a L^∞ metric on $\mathbb{T}^n \setminus S$ outside some singularity S such that $\mathcal{R} \geq 0$ on R_{reg}, is S a removable singularity?

Motivating (related) Questions:

- Gromov: What is the most reasonable compactness in scalar curvature geometry?
- Gromov: Define the notion of $\mathcal{R} \geq \kappa$ for non-smooth metrics
- Schoen: If g is a L^∞ metric on $\mathbb{T}^n \setminus S$ outside some singularity S such that $\mathcal{R} \geq 0$ on R_{reg}, is S a removable singularity?
- Do we have positive Theorem for manifolds with "corners"?

notion of $\mathcal{R} \geq \kappa$ for C^0 metrics

Definition

Given a C^0 metric g on closed manifold M, we say that $\mathcal{R} \geq \kappa$ if there is a sequence of smooth metrics g_i on M such that $\mathcal{R}_i \geq \kappa - o(1)$ and $g_i \to g$ in C^0.

notion of $\mathcal{R} \geq \kappa$ for C^0 metrics

Definition

Given a C^0 metric g on closed manifold M, we say that $\mathcal{R} \geq \kappa$ if there is a sequence of smooth metrics g_i on M such that $\mathcal{R}_i \geq \kappa - o(1)$ and $g_i \to g$ in C^0. We might as well replace κ by a continuous function on M

notion of $\mathcal{R} \geq \kappa$ for C^0 metrics

Definition

Given a C^0 metric g on closed manifold M, we say that $\mathcal{R} \geq \kappa$ if there is a sequence of smooth metrics g_i on M such that $\mathcal{R}_i \geq \kappa - o(1)$ and $g_i \to g$ in C^0. We might as well replace κ by a continuous function on M

Que: How natural is this?

notion of $\mathcal{R} \geq \kappa$ for C^0 metrics

Definition

Given a C^0 metric g on closed manifold M, we say that $\mathcal{R} \geq \kappa$ if there is a sequence of smooth metrics g_i on M such that $\mathcal{R}_i \geq \kappa - o(1)$ and $g_i \to g$ in C^0. We might as well replace κ by a continuous function on M

Que: How natural is this?

e.g. If $\mathcal{R} \geq 0$ in this sense on $\mathbb{T}^n$, is g a flat torus?

Result of Gromov, Bamler

Theorem (Gromov, Bamler)

Suppose g_i is a metric on M with $\mathcal{R}_i \geq \kappa$ such that $g_i \to g_\infty$ in C^0 for some smooth metric g_∞, then $\mathcal{R}_\infty \geq \kappa$.

Result of Gromov, Bamler

Theorem (Gromov, Bamler)

Suppose g_i is a metric on M with $\mathcal{R}_i \geq \kappa$ such that $g_i \to g_\infty$ in C^0 for some smooth metric g_∞, then $\mathcal{R}_\infty \geq \kappa$. (localizable, just lazy)

Result of Gromov, Bamler

Theorem (Gromov, Bamler)

Suppose g_i is a metric on M with $\mathcal{R}_i \geq \kappa$ such that $g_i \to g_\infty$ in C^0 for some smooth metric g_∞, then $\mathcal{R}_\infty \geq \kappa$. (localizable, just lazy)

Hence, definition of C^0 metric is a direct generalization of smooth setting!

- Gromov: formulate $\mathcal{R} \geq 0$ using non-existence of local cube C^0 data

Result of Gromov, Bamler

Theorem (Gromov, Bamler)

Suppose g_i is a metric on M with $\mathcal{R}_i \geq \kappa$ such that $g_i \to g_\infty$ in C^0 for some smooth metric g_∞, then $\mathcal{R}_\infty \geq \kappa$. (localizable, just lazy)

Hence, definition of C^0 metric is a direct generalization of smooth setting!

- Gromov: formulate $\mathcal{R} \geq 0$ using non-existence of local cube C^0 data
- Bamler: using Ricci flow smoothing (seems more flexible)

Ricci flow smoothing

Ricci-Deturck flow (diffeomorphic to Ricci flow in smooth case):

$$\partial_t g_{ij} = -R_{ij} + \nabla_i V_j + \nabla_j V_i; \quad V^k = g^{ij}(\Gamma_{ij}^k - \tilde{\Gamma}_{ij}^k) \tag{1}$$

where $\tilde{g}$ is a fixed chosen metrc on M.

Ricci flow smoothing

Ricci-Deturck flow (diffeomorphic to Ricci flow in smooth case):

$$\partial_t g_{ij} = -R_{ij} + \nabla_i V_j + \nabla_j V_i; \quad V^k = g^{ij}(\Gamma_{ij}^k - \tilde{\Gamma}_{ij}^k) \tag{1}$$

where $\tilde{g}$ is a fixed chosen metrc on M.

This is diffeomorphic to Ricci flow:

$$\partial_t g = -2\text{Ric}(g(t)).$$

Ricci flow smoothing

Ricci-Deturck flow (diffeomorphic to Ricci flow in smooth case):

$$\partial_t g_{ij} = -R_{ij} + \nabla_i V_j + \nabla_j V_i; \quad V^k = g^{ij}(\Gamma_{ij}^k - \tilde{\Gamma}_{ij}^k) \tag{1}$$

where $\tilde{g}$ is a fixed chosen metrc on M.

This is diffeomorphic to Ricci flow:

$$\partial_t g = -2\text{Ric}(g(t)).$$

Why Ricci flow is useful?

Ricci flow smoothing

Ricci-Deturck flow (diffeomorphic to Ricci flow in smooth case):

$$\partial_t g_{ij} = -R_{ij} + \nabla_i V_j + \nabla_j V_i; \quad V^k = g^{ij}(\Gamma_{ij}^k - \tilde{\Gamma}_{ij}^k) \tag{1}$$

where $\tilde{g}$ is a fixed chosen metrc on M.

This is diffeomorphic to Ricci flow:

$$\partial_t g = -2\text{Ric}(g(t)).$$

Why Ricci flow is useful? Because this is indeed the best second order variation to increase $\mathcal{R}$:

$$\left(\frac{\partial}{\partial t} - \Delta\right)\mathcal{R} = 2|\text{Ric}|^2 \geq \frac{2}{n}\mathcal{R}^2.$$

Ricci flow smoothing

Theorem (Simon, Koch-Lamn, Burkhardt-Guim)

Suppose g_0 is a continuous metric on closed manifold $(M, \tilde{g})$ such that

$$(1-\varepsilon_n)\tilde{g} \le g_0 \le (1+\varepsilon_n)\tilde{g}$$

on M, then there exists a short-time solution to the Ricci-Deturck flow $g(t)$ (with respect to $\tilde{g}$) such that $g(t) \to g_0$ as $t \to 0$ in C^0 and

$$|\tilde{\nabla}^k g(t)| \le o(1)t^{-k/2} \le C(n,k,\tilde{g})t^{-k/2}$$

for all $k > 0$. Moreover, the solution is unique within the class of solution achieving g_0 in C^0.

Ricci flow smoothing

Theorem (Simon, Koch-Lamn, Burkhardt-Guim)

Suppose g_0 is a continuous metric on closed manifold $(M, \tilde g)$ such that

$$(1-\varepsilon_n)\tilde g \le g_0 \le (1+\varepsilon_n)\tilde g$$

on M, then there exists a short-time solution to the Ricci-Deturck flow $g(t)$ (with respect to $\tilde g$) such that $g(t) \to g_0$ as $t \to 0$ in C^0 and

$$|\tilde\nabla^k g(t)| \le o(1) t^{-k/2} \le C(n,k,\tilde g) t^{-k/2}$$

for all $k > 0$. Moreover, the solution is unique within the class of solution achieving g_0 in C^0.

In particular, if $g(t), t \in (0, T]$ is a solution smooth for $t > 0$ which attains a smooth metric g_0 as $t \to 0$ in C^0, then $g(t)$ coincides with the standard solution.

Bamler's approach to Gromov Theorem

- g_i are all ε_n away from some fixed background metric $\tilde{g}$

Bamler's approach to Gromov Theorem

- g_i are all ε_n away from some fixed background metric $\tilde{g}$
- take $g_i(t)$, the geometric deformation of each g_i such that $\mathcal{R}(g_i(t)) \geq \kappa$ for $t > 0$.

Bamler's approach to Gromov Theorem

- g_i are all ε_n away from some fixed background metric $\tilde{g}$
- take $g_i(t)$, the geometric deformation of each g_i such that $\mathcal{R}(g_i(t)) \geq \kappa$ for $t > 0$.
- Parabolic smoothing implies $g_i(t) \to g_\infty(t)$ smoothly so that $\mathcal{R}(g_\infty(t)) \geq \kappa$.

Bamler's approach to Gromov Theorem

- g_i are all ε_n away from some fixed background metric $\tilde{g}$
- take $g_i(t)$, the geometric deformation of each g_i such that $\mathcal{R}(g_i(t)) \geq \kappa$ for $t > 0$.
- Parabolic smoothing implies $g_i(t) \to g_\infty(t)$ smoothly so that $\mathcal{R}(g_\infty(t)) \geq \kappa$.
- $g_\infty(t)$ coincides with the classicial smooth solution, result follows by letting $t \to 0$.

Bamler's approach to Gromov Theorem

- g_i are all ε_n away from some fixed background metric $\tilde{g}$
- take $g_i(t)$, the geometric deformation of each g_i such that $\mathcal{R}(g_i(t)) \geq \kappa$ for $t > 0$.
- Parabolic smoothing implies $g_i(t) \to g_\infty(t)$ smoothly so that $\mathcal{R}(g_\infty(t)) \geq \kappa$.
- $g_\infty(t)$ coincides with the classicial smooth solution, result follows by letting $t \to 0$.

- Huang-L. generalize Gromov's Theorem: if $g_i \to g$ in C^0 and $||\mathcal{R}_i - \kappa||_{L^{n/2}} \to 0$, then $\mathcal{R} \geq \kappa$ (indeed localizable).

Bamler's approach to Gromov Theorem

- g_i are all ε_n away from some fixed background metric $\tilde{g}$
- take $g_i(t)$, the geometric deformation of each g_i such that $\mathcal{R}(g_i(t)) \geq \kappa$ for $t > 0$.
- Parabolic smoothing implies $g_i(t) \to g_\infty(t)$ smoothly so that $\mathcal{R}(g_\infty(t)) \geq \kappa$.
- $g_\infty(t)$ coincides with the classicial smooth solution, result follows by letting $t \to 0$.

- Huang-L. generalize Gromov's Theorem: if $g_i \to g$ in C^0 and $||\mathcal{R}_i - \kappa||_{L^{n/2}} \to 0$, then $\mathcal{R} \geq \kappa$ (indeed localizable). This is related to Miao's PMT with corners.

Why care C^0 metrics?

This is related to positive mass Theorem with "singularity".

Why care C^0 metrics?

This is related to positive mass Theorem with "singularity".

Setting: g is smooth on $\mathbb{T}^n \setminus S$ (take M to be Asymptotically flat if you like PMT) with $\mathcal{R}(g) \geq 0$ outside S.

Why care C^0 metrics?

This is related to positive mass Theorem with "singularity".

Setting: g is smooth on $\mathbb{T}^n \setminus S$ (take M to be Asymptotically flat if you like PMT) with $\mathcal{R}(g) \geq 0$ outside S.

Question: under what conditions of S we can conclude g to be flat?

Why care C^0 metrics?

This is related to positive mass Theorem with "singularity".

Setting: g is smooth on $\mathbb{T}^n \setminus S$ (take M to be Asymptotically flat if you like PMT) with $\mathcal{R}(g) \geq 0$ outside S.

Question: under what conditions of S we can conclude g to be flat?

- Miao: need assumption on mean curvature across the co-dim 1 singularity
- Gromov, Li-Mantoulidis: some condition on angle for co-dim 2
- Schoen: there should be no requirement on high codimension singularity (even for L^∞ metrics ?)

C^0 metrics

- **Question:** does C^0 metric g satisfy $\mathcal{R}(g) \geq 0$ in the sense of approximation

C^0 metrics

- **Question:** does C^0 metric g satisfy $\mathcal{R}(g) \geq 0$ in the sense of approximation
- If it does, then we can find using Ricci flow approach $g(t)$ from g such that $\mathrm{Rm}(g(t)) \equiv 0$ and $g(t) \to g$ in C^0.

C^0 metrics

- **Question:** does C^0 metric g satisfy $\mathcal{R}(g) \geq 0$ in the sense of approximation
- If it does, then we can find using Ricci flow approach $g(t)$ from g such that $\mathrm{Rm}(g(t)) \equiv 0$ and $g(t) \to g$ in C^0.
- In particular, using Ricci flow perspective implies $g = \Phi^* g_{flat}$ outside S for some bi-Lipschitz homeomorphism $\Phi \in C^\infty(M \setminus S)$.

(isolated) high codimension is invisible

Theorem (L.-Tam)

Given $M = \mathbb{T}^n$. *Suppose* $g \in C^0(M) \cap C^\infty_{loc}(M \setminus S)$ *such that co-dim of* S *is at least* 3 *and* $\mathcal{R}(g) \geq 0$ *outside* S, *then* $\mathcal{R}(g) \geq 0$ *in the sense of approximation. In particular, there exists a bi-lipschitz homeomorphism* $\Phi \in C^\infty(M \setminus S)$ *such that* Φ *is an isometry from* (M, g) *to* (M, g_{flat}) *and* $g = \Phi^* g_{flat}$ *outside* S.

(isolated) high codimension is invisible

Theorem (L.-Tam)

Given $M = \mathbb{T}^n$. Suppose $g \in C^0(M) \cap C^\infty_{loc}(M \setminus S)$ such that co-dim of S is at least 3 and $\mathcal{R}(g) \geq 0$ outside S, then $\mathcal{R}(g) \geq 0$ in the sense of approximation. In particular, there exists a bi-lipschitz homeomorphism $\Phi \in C^\infty(M \setminus S)$ such that Φ is an isometry from (M, g) to (M, g_{flat}) and $g = \Phi^* g_{flat}$ outside S.

- Proved by Li-Mantoulidis in $3D$ under only L^∞ across S using minimal surface;

(isolated) high codimension is invisible

Theorem (L.-Tam)

Given $M = \mathbb{T}^n$. Suppose $g \in C^0(M) \cap C^\infty_{loc}(M \setminus S)$ such that co-dim of S is at least 3 and $\mathcal{R}(g) \geq 0$ outside S, then $\mathcal{R}(g) \geq 0$ in the sense of approximation. In particular, there exists a bi-lipschitz homeomorphism $\Phi \in C^\infty(M \setminus S)$ such that Φ is an isometry from (M, g) to (M, g_{flat}) and $g = \Phi^* g_{flat}$ outside S.

- Proved by Li-Mantoulidis in $3D$ under only L^∞ across S using minimal surface;
- Ricci flow method also work for manifold with non-positive Yamabe invariant $\sigma(M) \leq 0$ and critical scalar lower bound.

Maximum principle with singular initial data

Proposition (L.-Tam)

There is $\delta > 0$ such that if $g(t)$ is Ricci-Deturck flow on $M \times (0, T]$ such that

1. *$g(t)$ is bi-Lip to h;*
2. $|\nabla^h g(t)|^2 + |\nabla^{h,2} g(t)| \leq \delta t^{-1}$;
3. $g_0 \in C^\infty_{loc}(M \setminus \Sigma)$ *for* $\mathrm{co}-\dim(\Sigma) \geq 3$;

If $\mathcal{R}(g_0) \geq \sigma_0 \leq 0$ on Σ^c, then on $M \times (0, T]$,

$$\mathcal{R}(g(t)) \geq \sigma_0 \left(1 - \frac{2}{n}\sigma_0 t\right)^{-1}.$$

Maximum principle with singular initial data

Proposition (L.-Tam)

There is $\delta > 0$ such that if $g(t)$ is Ricci-Deturck flow on $M \times (0, T]$ such that

1. *$g(t)$ is bi-Lip to h;*
2. $|\nabla^h g(t)|^2 + |\nabla^{h,2} g(t)| \leq \delta t^{-1}$;
3. $g_0 \in C^\infty_{loc}(M \setminus \Sigma)$ *for* $\mathrm{co}-\dim(\Sigma) \geq 3$;

If $\mathcal{R}(g_0) \geq \sigma_0 \leq 0$ on Σ^c, then on $M \times (0, T]$,

$$\mathcal{R}(g(t)) \geq \sigma_0 \left(1 - \frac{2}{n}\sigma_0 t\right)^{-1}.$$

Remark: coincide with smooth case

Applications to PMT with high co-dim singularity

Same approach also implies

Theorem (L.-Tam, Chu-L.-Zhu)

Suppose $(M^n, g), n \leq 7$ is AF manifold such that $g \in C^\infty_{loc}(M \setminus \Sigma)$ for some compact Σ of co-dim ≥ 3 and is locally continuous. If $\mathcal{R} \geq 0$ outside Σ, then $m_{ADM}(E) \geq 0$ for any end E of M. Moreover, if $m_{ADM}(E') = 0$ for some end E', then (M, g) is isometric to Euclidean space as a metric space.

- In smooth case: ADM mass is preserved under Ricci flow

Applications to PMT with high co-dim singularity

Same approach also implies

Theorem (L.-Tam, Chu-L.-Zhu)

Suppose $(M^n, g), n \leq 7$ *is AF manifold such that* $g \in C^\infty_{loc}(M \setminus \Sigma)$ *for some compact* Σ *of co-dim* ≥ 3 *and is locally continuous. If* $\mathcal{R} \geq 0$ *outside* Σ*, then* $m_{ADM}(E) \geq 0$ *for any end* E *of* M*. Moreover, if* $m_{ADM}(E') = 0$ *for some end* E'*, then* (M, g) *is isometric to Euclidean space as a metric space.*

- In smooth case: ADM mass is preserved under Ricci flow
- In non-smooth case, ADM mass is non-decreasing under smoothing. (Mcferon-Szekelyhidi)

Another Existence result

Theorem (Chu-L.)

Suppose (M, h) *is a compact manifold and* g_0 *is a* $L^\infty \cap W^{1,n}$ *metric on* M*, then there is a Ricci-Deturck* h*-flow on* $M \times (0, S]$ *such that*

Another Existence result

Theorem (Chu-L.)

Suppose (M, h) is a compact manifold and g_0 is a $L^\infty \cap W^{1,n}$ metric on M, then there is a Ricci-Deturck h-flow on $M \times (0, S]$ such that

1. *$g(t)$ is smooth and uniformly bi-Lipschitz for all $t \in (0, S]$;*
2. $\sup_M t^{k/2}|\nabla^{k,h} g(t)| = o(1)$ *as* $t \to 0$ *for all* $k \in \mathbb{N}$;
3. *$g(t) \to g_0$ in $W^{1,n}$ as $t \to 0$;*
4. *If $g_0 \in C^\infty_{loc}(\Omega)$ for some $\Omega \Subset M$, then $g(t) \to g_0$ in $C^\infty_{loc}(\Omega)$ as $t \to 0$.*

Another Existence result

Theorem (Chu-L.)

Suppose (M, h) is a compact manifold and g_0 is a $L^\infty \cap W^{1,n}$ metric on M, then there is a Ricci-Deturck h-flow on $M \times (0, S]$ such that

1. *$g(t)$ is smooth and uniformly bi-Lipschitz for all $t \in (0, S]$;*
2. $\sup_M t^{k/2}|\nabla^{k,h} g(t)| = o(1)$ *as $t \to 0$ for all $k \in \mathbb{N}$;*
3. *$g(t) \to g_0$ in $W^{1,n}$ as $t \to 0$;*
4. *If $g_0 \in C^\infty_{loc}(\Omega)$ for some $\Omega \Subset M$, then $g(t) \to g_0$ in $C^\infty_{loc}(\Omega)$ as $t \to 0$.*

Remark: $o(1)$ in the asymptotic measures the asymptotic flatness in weak sense

Applications

Corollary (Chu-L.)

Suppose M is a compact manifold with $\sigma(M) \leq 0$. If g is $L^\infty \cap W^{1,n}$ metric on M such that $g \in C^\infty_{loc}$ outside a co-dimension 3 singularity Σ, then $\mathrm{Ric}(g) \equiv 0$ outside Σ.

Applications

Corollary (Chu-L.)

Suppose M *is a compact manifold with* $\sigma(M) \leq 0$*. If* g *is* $L^\infty \cap W^{1,n}$ *metric on* M *such that* $g \in C^\infty_{loc}$ *outside a co-dimension* 3 *singularity* Σ*, then* $\mathrm{Ric}(g) \equiv 0$ *outside* Σ*.*

Corollary (Chu-L.)

Suppose (M, g) *is AF manifold such that* $g \in L^\infty \cap W^{1,n} \cap C^\infty_{loc}(M \setminus \Sigma)$ *for* Σ *of co-dim* ≥ 3*. If* $\mathcal{R} \geq 0$ *outside* Σ*, then* $m_{ADM}(E) \geq 0$ *for any end* E *of* M*. Moreover, if* $m_{ADM}(E') = 0$ *for some end* E'*, then* M *is diffeomorphic to* $\mathbb{R}^n$ *and* g *is flat outside* Σ*.*

Application to sphere's rigidity?

Recall Llarull Theorem: rigidity of sphere under **spin, area non-increasing, scalar curvature**.

Application to sphere's rigidity?

Recall Llarull Theorem: rigidity of sphere under **spin, area non-increasing, scalar curvature**.

Question (Gromov)

Is regualrlity making difference in sphere rigidity?

Application to sphere's rigidity?

Recall Llarull Theorem: rigidity of sphere under **spin, area non-increasing, scalar curvature**.

Question (Gromov)

Is reguarlity making difference in sphere rigidity? For example: if $f : M \to \mathbb{S}^n$ *is a distance non-increasing continuous map with non-zero degree, then is* f *a distance isometry under the same set of conditions* $(\mathcal{R} \geq n(n-1), spin)$*?*

Application to sphere's rigidity?

Recall Llarull Theorem: rigidity of sphere under **spin, area non-increasing, scalar curvature**.

Question (Gromov)

Is reguarlity making difference in sphere rigidity? For example: if $f : M \to \mathbb{S}^n$ *is a distance non-increasing continuous map with non-zero degree, then is* f *a distance isometry under the same set of conditions* $(\mathcal{R} \geq n(n-1), spin)$*?Or more generally,* g *is continuous metric with weak sense of* $\mathcal{R} \geq n(n-1)$*.*

Why care? Motivation from boundary rigidity

Question

Suppose g is a metric on $\mathbb{S}^n_+$ such that $g \geq g_{sph}$, $\mathcal{R} \geq n(n-1)$ and $H(g) \geq H(g_{sph})$ (Miao's condition) on $\partial\mathbb{S}^n_+$, then is $g = g_{sph}$?

Why care? Motivation from boundary rigidity

Question

Suppose g is a metric on $\mathbb{S}^n_+$ such that $g \geq g_{sph}$, $\mathcal{R} \geq n(n-1)$ and $H(g) \geq H(g_{sph})$ (Miao's condition) on $\partial \mathbb{S}^n_+$, then is $g = g_{sph}$?

Why related to singular version of Llarull?

Why care? Motivation from boundary rigidity

Question

Suppose g is a metric on $\mathbb{S}^n_+$ such that $g \geq g_{sph}$, $\mathcal{R} \geq n(n-1)$ and $H(g) \geq H(g_{sph})$ (Miao's condition) on $\partial\mathbb{S}^n_+$, then is $g = g_{sph}$?

Why related to singular version of Llarull?

- Taking double of g yield $(\mathbb{S}^n, \tilde{g})$ which is Lipschitz, $\mathcal{R} \geq n(n-1)$ in distribution sense (Lee) (and hence our weak sense) and $\tilde{g} \geq g_{sph}$ on $\mathbb{S}^n$.

Why care? Motivation from boundary rigidity

Question

Suppose g is a metric on $\mathbb{S}^n_+$ such that $g \geq g_{sph}$, $\mathcal{R} \geq n(n-1)$ and $H(g) \geq H(g_{sph})$ (Miao's condition) on $\partial\mathbb{S}^n_+$, then is $g = g_{sph}$?

Why related to singular version of Llarull?

- Taking double of g yield $(\mathbb{S}^n, \tilde{g})$ which is Lipschitz, $\mathcal{R} \geq n(n-1)$ in distribution sense (Lee) (and hence our weak sense) and $\tilde{g} \geq g_{sph}$ on $\mathbb{S}^n$.
- Then $\tilde{g} = g_{sph}$ on $\mathbb{S}^n$ and hence g! (Great :D)

Why care? Motivation from boundary rigidity

Question

Suppose g is a metric on $\mathbb{S}^n_+$ such that $g \geq g_{sph}$, $\mathcal{R} \geq n(n-1)$ and $H(g) \geq H(g_{sph})$ (Miao's condition) on $\partial\mathbb{S}^n_+$, then is $g = g_{sph}$?

Why related to singular version of Llarull?

- Taking double of g yield $(\mathbb{S}^n, \tilde{g})$ which is Lipschitz, $\mathcal{R} \geq n(n-1)$ in distribution sense (Lee) (and hence our weak sense) and $\tilde{g} \geq g_{sph}$ on $\mathbb{S}^n$.
- Then $\tilde{g} = g_{sph}$ on $\mathbb{S}^n$ and hence g! (Great :D)
- Or more generally, replace $\mathbb{S}^n_+$ with any $\Omega \subset \mathbb{S}^n$ with additional assumption $g = g_{sph}$ on boundary.

Main result

Theorem (L.-Tam, Cecchini-Hanke-Schick)

Suppose M is a closed spin manifold and g is a continuous metric on M with $\mathcal{R} \geq n(n-1)$ in the weak sense. Suppose $f : M \to \mathbb{S}^n$ is a distance non-increasing map with non-zero degree, then f is a distance isometry.

Main result

Theorem (L.-Tam, Cecchini-Hanke-Schick)

Suppose M is a closed spin manifold and g is a continuous metric on M with $\mathcal{R} \geq n(n-1)$ in the weak sense. Suppose $f : M \to \mathbb{S}^n$ is a distance non-increasing map with non-zero degree, then f is a distance isometry.

- Cecchini-Hanke-Schick: based on developing singular Dirac operator
- L.-Tam: based on parabolic smoothing

parabolic method: Reduction to smooth case

Theorem (smooth case)

Suppose $g(t)$ *is a solution to the Ricci flow on* $M \times [0, T]$ *and* $F : M \times [0, T] \to \mathbb{S}^n$ *such that* $\partial_t F = \Delta_{g(t),h(t)} F$, *if* $F_0^* h(0) \le g(0)$, *then*

$$F_t^* h(t) \le g(t)$$

on $M \times [0, T]$. *Here* $h(t)$ *is the standard shrinking sphere.*

parabolic method: Reduction to smooth case

Theorem (smooth case)

Suppose $g(t)$ is a solution to the Ricci flow on $M \times [0, T]$ and $F : M \times [0, T] \to \mathbb{S}^n$ such that $\partial_t F = \Delta_{g(t),h(t)} F$, if $F_0^* h(0) \le g(0)$, then

$$F_t^* h(t) \le g(t)$$

on $M \times [0, T]$. Here $h(t)$ is the standard shrinking sphere.

General case: $F_0 = f$ is non-smooth Lipschitz map

- approximate f by smooth map f_i using method of Greene-Wu;

parabolic method: Reduction to smooth case

Theorem (smooth case)

Suppose $g(t)$ is a solution to the Ricci flow on $M \times [0,T]$ and $F : M \times [0,T] \to \mathbb{S}^n$ such that $\partial_t F = \Delta_{g(t),h(t)} F$, if $F_0^* h(0) \leq g(0)$, then

$$F_t^* h(t) \leq g(t)$$

on $M \times [0,T]$. Here $h(t)$ is the standard shrinking sphere.

General case: $F_0 = f$ is non-smooth Lipschitz map

- approximate f by smooth map f_i using method of Greene-Wu;
- evolve f_i using harmonic map heat flow $F_i(t)$ coupled with Ricci flow;

parabolic method: Reduction to smooth case

Theorem (smooth case)

Suppose $g(t)$ is a solution to the Ricci flow on $M \times [0, T]$ and $F : M \times [0, T] \to \mathbb{S}^n$ such that $\partial_t F = \Delta_{g(t),h(t)} F$, if $F_0^* h(0) \le g(0)$, then

$$F_t^* h(t) \le g(t)$$

on $M \times [0, T]$. Here $h(t)$ is the standard shrinking sphere.

General case: $F_0 = f$ is non-smooth Lipschitz map

- approximate f by smooth map f_i using method of Greene-Wu;
- evolve f_i using harmonic map heat flow $F_i(t)$ coupled with Ricci flow;
- Pass F_i to limiting map $F : M \to \mathbb{S}^n$ with $F(0) = f$ as C^0 initial data

Main smoothing result

Theorem (L.-Tam)

Under the assumption in main Theorem, there exists $F : (M, g(t)) \times (0, T] \to (\mathbb{S}^n, h(t))$ *such that* $\mathrm{Lip}(F) \leq 1$ *and*

$$d_h(F_t(x), f(x)) \leq C\sqrt{t}$$

for all $x \in M$ *and* $|\nabla^k dF| \leq Ct^{-k/2}$ *for all* k*. In particular,* F *is a distance isometry for all* $t > 0$*.*

Main smoothing result

Theorem (L.-Tam)

Under the assumption in main Theorem, there exists $F : (M, g(t)) \times (0, T] \to (\mathbb{S}^n, h(t))$ *such that* $\mathrm{Lip}(F) \le 1$ *and*

$$d_h(F_t(x), f(x)) \le C\sqrt{t}$$

for all $x \in M$ *and* $|\nabla^k dF| \le Ct^{-k/2}$ *for all* k. *In particular,* F *is a distance isometry for all* $t > 0$.

- Taking $t \to 0$, we conclude that f is a distance isometry.

Main smoothing result

Theorem (L.-Tam)

Under the assumption in main Theorem, there exists $F : (M, g(t)) \times (0, T] \to (\mathbb{S}^n, h(t))$ *such that* $\mathrm{Lip}(F) \le 1$ *and*

$$d_h(F_t(x), f(x)) \le C\sqrt{t}$$

for all $x \in M$ *and* $|\nabla^k dF| \le Ct^{-k/2}$ *for all* k*. In particular,* F *is a distance isometry for all* $t > 0$*.*

- Taking $t \to 0$, we conclude that f is a distance isometry.
- smoothing is independent of spin structure!

Further "application"

Question (Gromov?)

In the smooth case, if $f : M \to \mathbb{S}^n$ is a distance non-increasing map with non-zero degree and $\mathcal{R}(M) \geq n(n-1)$, then is f is isometry without a-priori spin condition??

Further "application"

Question (Gromov?)

In the smooth case, if $f : M \to \mathbb{S}^n$ *is a distance non-increasing map with non-zero degree and* $\mathcal{R}(M) \geq n(n-1)$, *then is* f *is isometry without a-priori spin condition??*

- Our method: this reduces the problem to non-existence of such map under $\mathcal{R} > n(n-1)$ (similar to Schoen-Yau's torus rigidity)

Further "application"

Question (Gromov?)

In the smooth case, if $f : M \to \mathbb{S}^n$ is a distance non-increasing map with non-zero degree and $\mathcal{R}(M) \geq n(n-1)$, then is f is isometry without a-priori spin condition??

- Our method: this reduces the problem to non-existence of such map under $\mathcal{R} > n(n-1)$ (similar to Schoen-Yau's torus rigidity)
- This is because if g is non-Einstein, Ricci flow smoothing yield strict inequality.

Further "application"

Question (Gromov?)

In the smooth case, if $f : M \to \mathbb{S}^n$ is a distance non-increasing map with non-zero degree and $\mathcal{R}(M) \geq n(n-1)$, then is f is isometry without a-priori spin condition??

- Our method: this reduces the problem to non-existence of such map under $\mathcal{R} > n(n-1)$ (similar to Schoen-Yau's torus rigidity)
- This is because if g is non-Einstein, Ricci flow smoothing yield strict inequality.
- Einstein case: trivial using comparison geometry (or standard harmonic map heat flow or graphical mean curvature flow)

Further "application"

Question (Gromov?)

In the smooth case, if $f : M \to \mathbb{S}^n$ is a distance non-increasing map with non-zero degree and $\mathcal{R}(M) \geq n(n-1)$, then is f is isometry without a-priori spin condition??

- Our method: this reduces the problem to non-existence of such map under $\mathcal{R} > n(n-1)$ (similar to Schoen-Yau's torus rigidity)
- This is because if g is non-Einstein, Ricci flow smoothing yield strict inequality.
- Einstein case: trivial using comparison geometry (or standard harmonic map heat flow or graphical mean curvature flow)
- This was done in 4D by Cecchini-Wang-Xie-Zhu.

THANK YOU!!