# Trading linearity for ellipticity

## A novel approach to global Lorentzian geometry

Nicola Gigli, SISSA

Joint w. Beran, Braun, Calisti, McCann, Ohanyan, Rott, Saemann

# A standard set of tools

On $\mathbb{R}^d$ and Riemannian manifolds, calculus is strongly based upon the concepts of:

- Sobolev function
- Elliptic operator
- Banach space

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On $\mathbb{R}^d$ and Riemannian manifolds, calculus is strongly based upon the concepts of:

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It is unclear what any of this is in Lorentzian signature, where:

$$g_{\mathrm{M}}(v,v) := |v_0|^2 - \sum |v_i|^2 \qquad \text{replaces} \qquad g_{\mathrm{E}}(v,v) := \sum |v_i|^2$$

$$\Box f = \partial_{00} f - \sum_i \partial_{ii} f \qquad \text{replaces} \qquad \Delta f := \sum_i \partial_{ii} f$$

# A motivation: going towards non-smooth geometry

$\sim$40 years ago Gromov proposed to study how curvature affects the shape of Riemannian manifolds (also) via metric geometry. The program has been a success.

More recently, a similar program has been started for Lorentzian geometry.

Clear indications that some non-trivial geometry is in place are:

- The non-smooth version of the Hawking singularity theorem (Cavalletti-Mondino '20)
- The non-smooth Lorentzian analogue of the Splitting theorem for Sectional$\geq 0$ (Beran-Ohanyan-Rott-Solis '22)

In the 'elliptic' case, lower Ricci bounds in the non-smooth setting are encoded via

- A "curvature-dimension condition" related to optimal transport (after Lott-Sturm-Villani '05)
- "Infinitesimal Hilbertianity" related to Sobolev functions (after G. '12)

# Content

- Introduction
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- Elliptic operators
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# The worlds

Riemannian /
Elliptic

Lorentzian /
Hyperbolic

# The flat case

$\mathbb{R}^d$ equipped with the Euclidean tensor $g_{\mathrm{E}}(v,v) := \sum v_i^2$

the induced norm $\|v\|_{\mathrm{E}} := \sqrt{g_{\mathrm{E}}(v,v)}$ satisfies:

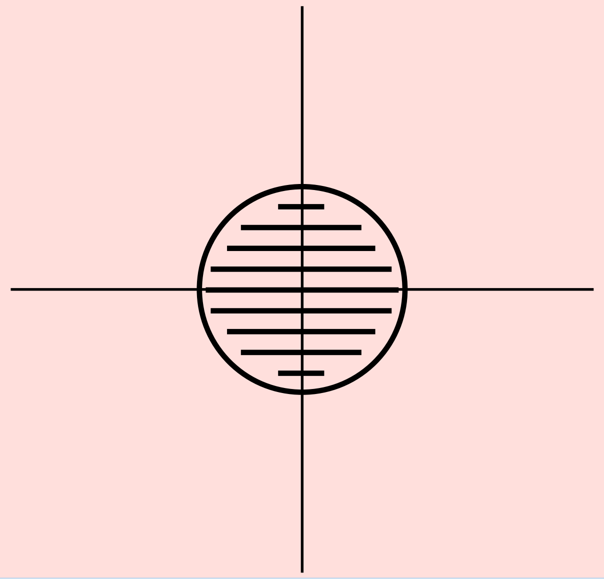
the triangle inequality $\|v+w\|_{\mathrm{E}} \leq \|v\|_{\mathrm{E}} + \|w\|_{\mathrm{E}}$

the Cauchy-Schwarz inequality $|g_{\mathrm{E}}(v,w)| \leq \|v\|_{\mathrm{E}}\|w\|_{\mathrm{E}}$

for any $v, w \in \mathbb{R}^d$

the unit ball is:

- compact
- convex
- of finite measure



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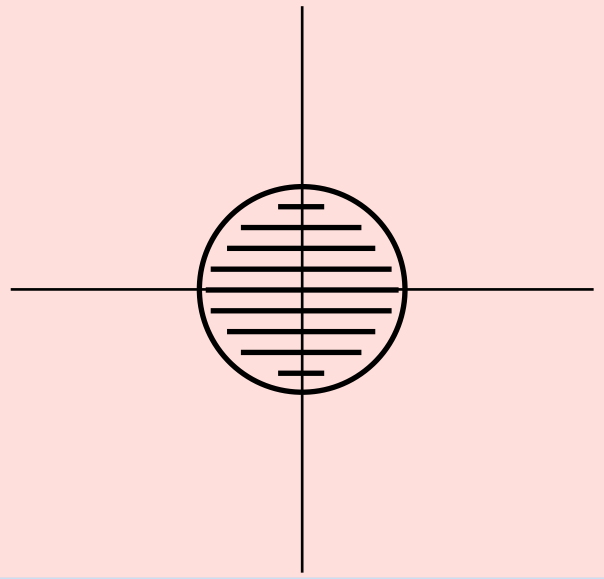

$\mathbb{R}^{1+d}$ equipped with the Minkowskian tensor $g_{\mathrm{M}}(v,v) := v_0^2 - \sum v_i^2$

the induced norm $\|v\|_{\mathrm{M}} := \sqrt{g_{\mathrm{M}}(v,v)}$ on $F := \{v : g_{\mathrm{M}}(v,v) \ge 0,\ v_0 \ge 0\}$ satisfies:

the unit ball is:

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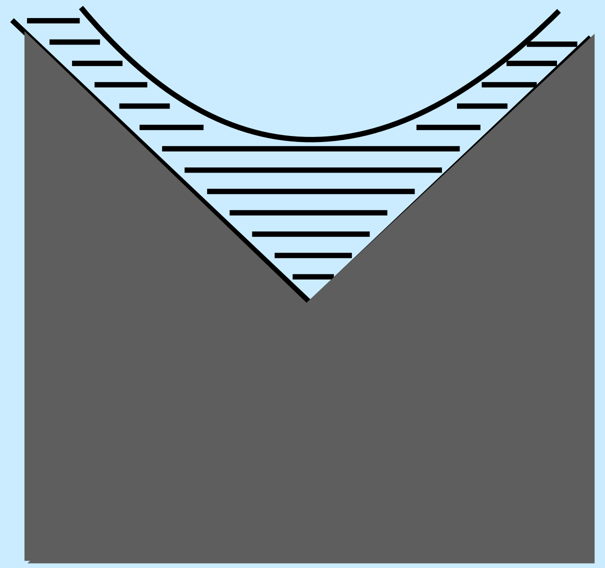

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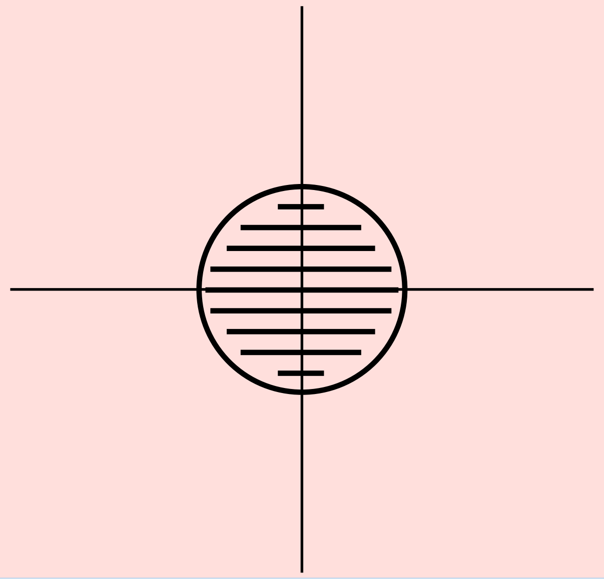

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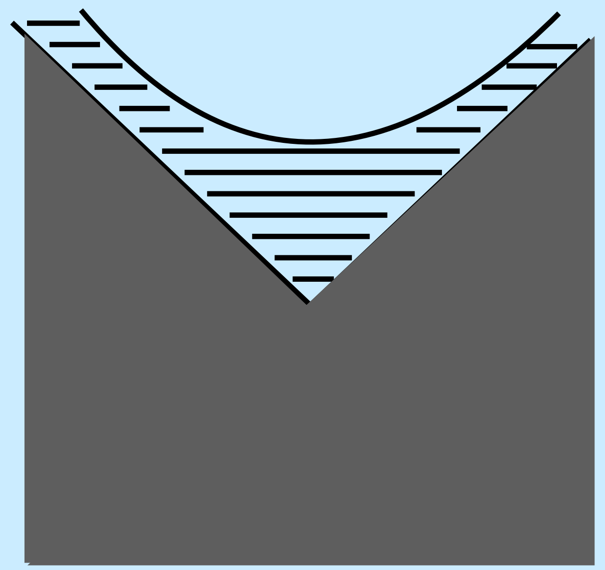

# The smooth curved case

A Riemannian manifold has a Euclidean scalar product on each tangent space

Geodesics $\gamma$ are local minimizers of $\int \|\dot\gamma_t\| \, \mathrm{d}t$

The formula

$$\frac{1}{q}\mathsf{d}^q(x,y) := \inf \frac{1}{q}\int_0^1 \|\dot\gamma_t\|^q \, \mathrm{d}t,$$

the inf being among curves from $x$ to $y$ defines a function $\mathsf{d} : M^2 \to \mathbb{R}^+$ independent on $q \geq 1$ that satisfies

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A Lorentzian manifold has a Minkowskian scalar product on each tangent space
*Causal* geodesics $\gamma$ are local maximizers of $\int \|\dot{\gamma}_t\| \, \mathrm{d}t$
The formula

$$\tfrac{1}{q}\ell^q(x,y) := \sup \tfrac{1}{q}\int_0^1 \|\dot{\gamma}_t\|^q \, \mathrm{d}t,$$

the sup being among curves from $x$ to $y$ defines a function $\ell : M^2 \to \mathbb{R}^+ \cup \{-\infty\}$
independent on $q \leq 1$ that satisfies

$$\ell(x,z) \geq \ell(x,y) + \ell(y,z)$$

We assume *Global hyperbolocity*, i.e.:
- time orientation
- no closed causal curves
- compactness of causal diamonds

# The non-smooth case

(Frechet 1906)

A metric space $(\mathrm{X}, \mathsf{d})$ is a set equipped with a symmetric function $\mathsf{d} : \mathrm{X}^2 \to \mathbb{R}^+$ satisfying

$$\mathsf{d}(x, x) = 0 \qquad \text{and} \qquad \mathsf{d}(x, z) \leq \mathsf{d}(x, y) + \mathsf{d}(y, z).$$

We assume $(\mathrm{X}, \mathsf{d})$ complete and separable
Balls $\{y : \mathsf{d}(x, y) < r\}$ generate a topology
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$\ell$ induces two partial orders via

$$x \leq y \quad \Leftrightarrow \quad \ell(x,y) \geq 0 \qquad \text{and} \qquad x < y \quad \Leftrightarrow \quad \ell(x,y) > 0$$

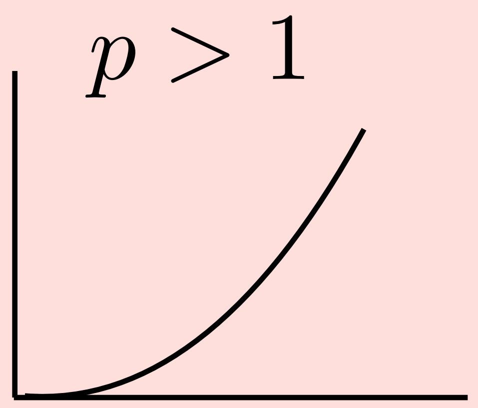
and the order $>$ induces a topology that we assume Polish

# Content

# Convex energies

$f \mapsto |\mathrm{d}f|$ is convex

For $p > 1$ let $u_p : \mathbb{R}^+ \to \mathbb{R}$ be defined as $u_p(z) := \frac{1}{p} z^p$.




Then $f \mapsto u_p(|\mathrm{d}f|)$ is convex and thus

$$f \qquad \mapsto \qquad \mathsf{E}_p(f) := \int u_p(|\mathrm{d}f|)\,\mathrm{d}\mathfrak{m} \qquad \text{is convex (and lsc)}$$

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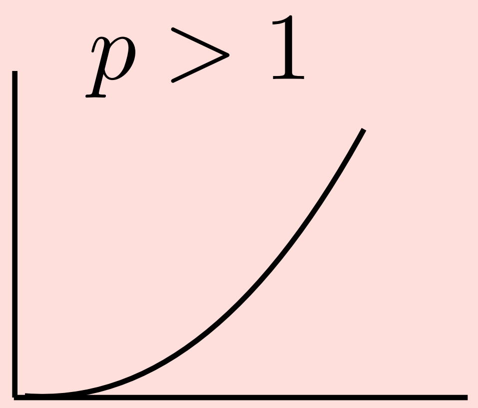


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$f : M \to \mathbb{R}$ is a *time* function if it is monotone, i.e. $x \leq y \quad \Rightarrow \quad f(x) \leq f(y)$.

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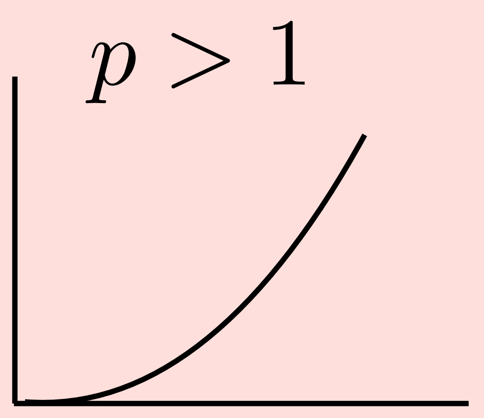


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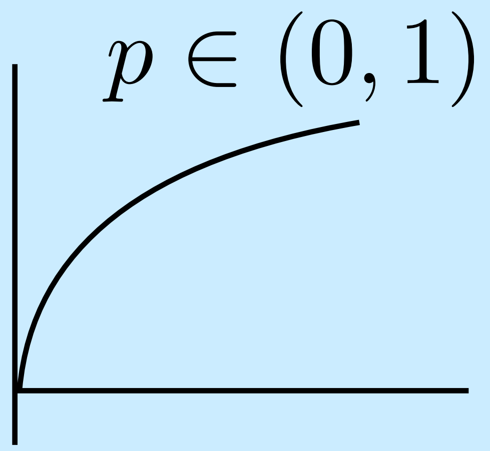


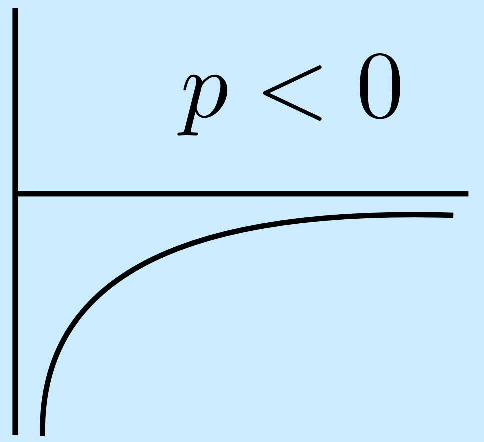

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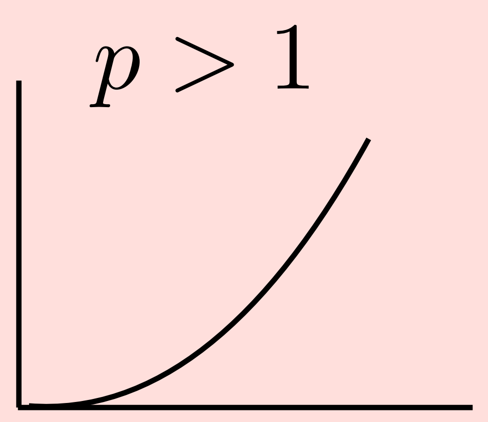


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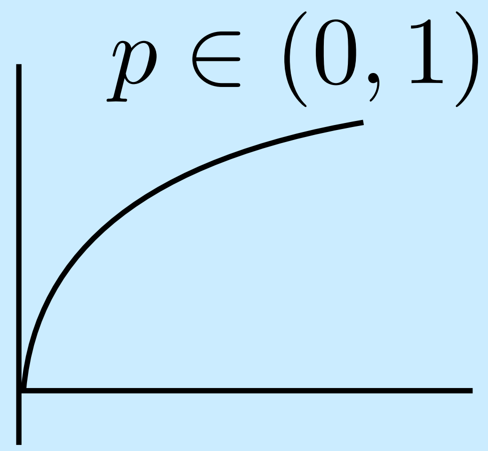


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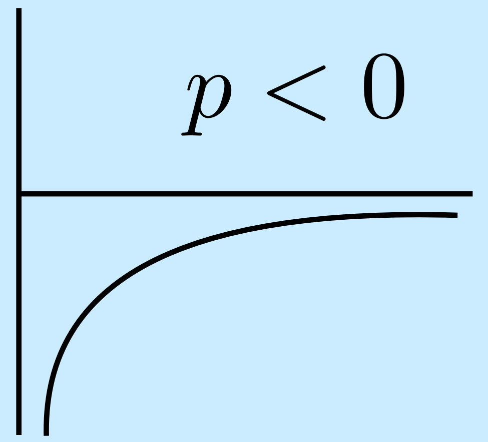


$$f \qquad \mapsto \qquad \mathsf{E}_p(f) := -\int u_p(|\mathrm{d}f|)\, \mathrm{d}\mathfrak{m} \qquad \text{is convex (and lsc) on time functions}$$

# Subdifferential of convex energies

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$$-\int g\Delta_p f := \lim_{\varepsilon\downarrow 0} \frac{\mathsf{E}_p(f+\varepsilon g) - \mathsf{E}_p(f)}{\varepsilon}$$

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These definitions only require $|\mathrm{d}f|$ and ‘thus’ can be performed in metric measure spaces

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# Sobolev functions & lower Ricci bounds

**Theorem** (G. '12 - $p = q = 2$)
Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be $\mathsf{CD}_q(0, N)$, $\bar{x} \in \mathrm{X}$ and $f := \frac{1}{q}\mathsf{d}^q(\bar{x}, \cdot)$.
Then

$$-\lim_{\varepsilon \downarrow 0} \frac{\mathsf{E}_p(f + \varepsilon g) - \mathsf{E}_p(f)}{\varepsilon} \leq N \int \rho \, \mathrm{d}\mathfrak{m}$$

for any $\rho \geq 0$ Lipschitz with bounded support.

Interpreted as: $\Delta_p f \leq N$.

$\mathsf{CD}$ condition introduced by Lott-Sturm-Villani
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**Theorem** (BBCGMcCORS '23)

Let $(\mathrm{X}, \ell, \mathfrak{m})$ be $\mathsf{TCD}_q(0, N)$, $\bar{x} \in \mathrm{X}$ and $f := \frac{1}{q}\ell^q(x, \cdot)$.
Then

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for any $\rho \geq 0$ such that $f + \rho$ is a time function.

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TCD condition introduced by Cavalletti-Mondino
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It arises as variation of the energy

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Notice that $\Box \ell(\bar{x}, \cdot) = \Box_p \ell(\bar{x}, \cdot)$

# A case for ~~Riemannian~~ ~~Lorentzian~~ Hamiltonian geometry

After Agrachev '97, Ohta '13

$$H(\mathrm{p}) := \frac{1}{p}|\mathrm{p}|_{\mathrm{E}}^{p}$$

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$$H(\mathrm{p}) := \frac{1}{p}|\mathrm{p}|_{\mathrm{E}}^{p}$$

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Ricci curvature along every/future direction can be read in terms of a suitably defined $\mathrm{Ric}^H$

Such $\mathrm{Ric}^H$ satisfies the non-linear Bochner-Ohta identity:

$$-\partial_t(\Delta^H f_t(\gamma_t))|_{t=0} = \|\mathrm{Hess}^H f\|^2_{\mathsf{HS}(H)} + \mathrm{Ric}^H(\mathrm{d}f, \mathrm{d}f)$$

where $\quad \partial_t f_t + H(\mathrm{d}f_t) = 0$ and $\gamma_t' = \nabla^H f_t(\gamma_t)$.

Here $\|\mathrm{Hess}^H f\|_{\mathsf{HS}(H)} \geq 0$ and it is 0 iff $f$ is affine along the Hamiltonian flow.

# An example: the splitting theorem (statement)

**Theorem** (Cheeger-Gromoll '71)

Let $M$ be with $\mathrm{Ric}_M \geq 0$ and containing a line, i.e. a curve $\gamma : \mathbb{R} \to M$ with

$$\mathsf{d}(\gamma_t, \gamma_s) = |s - t| \qquad \forall t, s \in \mathbb{R}.$$

Then $M \sim \mathbb{R} \times_{\mathrm{E}} N$ for some Riemannian manifold $N$ with $\mathrm{Ric}_N \geq 0$.

# An example: the splitting theorem (statement)

**Theorem** (Cheeger-Gromoll '71)

Let $M$ be with $\mathrm{Ric}_M \geq 0$ and containing a line, i.e. a curve $\gamma : \mathbb{R} \to M$ with

$$\mathsf{d}(\gamma_t, \gamma_s) = |s - t| \qquad \forall t, s \in \mathbb{R}.$$

Then $M \sim \mathbb{R} \times_{\mathrm{E}} N$ for some Riemannian manifold $N$ with $\mathrm{Ric}_N \geq 0$.

**Theorem** (Galloway '84 - Eschenburg '88 - Newman '90)

Let $M$ be with $\mathrm{Ric}_M \geq 0$ in the timelike directions and containing a timelike line, i.e. a curve $\gamma : \mathbb{R} \to M$ with

$$\ell(\gamma_t, \gamma_s) = s - t \qquad \forall t < s, \; t, s \in \mathbb{R}.$$

Then $M \sim \mathbb{R} \times_{\mathrm{M}} N$ for some Riemannian manifold $N$ with $\mathrm{Ric}_N \geq 0$.

# The splitting theorem (basic considerations)

$M \sim \mathbb{R} \times_{\mathrm{E}} N$ iff there is $b : M \to \mathbb{R}$ non-constant with null Hessian

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Let $\begin{cases} b^+ := \lim_{t\uparrow\infty} \mathsf{d}(\cdot, \gamma_t) - t \\ b^- := \lim_{t\uparrow\infty} t - \mathsf{d}(\cdot, \gamma_{-t}) \end{cases}$ Then $\begin{cases} b^+ \geq b^- & \text{on } M \\ b^+ = b^- & \text{along } \gamma \end{cases}$

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# The splitting theorem (formal proof)

$$\mathrm{Ric} \geq 0 \quad \overset{\text{Lapl.comp.}}{\Rightarrow} \quad \begin{cases} \Delta b^+ \leq 0 \\ \Delta b^- \geq 0 \end{cases} \quad \overset{\text{strong max.pr.}}{\Rightarrow} \quad b^+ = b^-$$

Use the Bochner identity $\frac{1}{2}\Delta|\mathrm{d}f|^2 - \langle \mathrm{d}f, \mathrm{d}\Delta f\rangle = \|\mathrm{Hess}\, f\|^2_{\mathsf{HS}} + \mathrm{Ric}(\nabla f, \nabla f) \geq 0$ to conclude that $\mathrm{Hess}\, b^+ = 0$

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Use the Bochner-Ohta identity $-\partial_t(\Delta^H f_t(\gamma_t))|_{t=0} = \|\mathrm{Hess}^H f\|^2_{\mathsf{HS}(H)} + \mathrm{Ric}^H(\mathrm{d}f, \mathrm{d}f) \geq 0$ to conclude that $\|\mathrm{Hess}^H b^+\|_{\mathsf{HS}(H)} = 0$

# Content

- Introduction
- Sobolev functions
- Elliptic operators
- Banach spaces

A *Banach space is*:

- a vector space $B$ together with
- a norm, i.e. a map $\|\cdot\| : B \to \mathbb{R}^+$ such that

$$\|\alpha v + \beta w\| \leq |\alpha|\|v\| + |\beta|\|w\| \qquad \text{for any } \alpha, \beta \in \mathbb{R},\ v, w \in B.$$

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**Definition** (G. '23)

An *Hyperbolic Banach space is*:

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# Cauchy vs Dedekind

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# Examples

$L^p(\mathrm{X}, \mathfrak{m})$ spaces, $p \geq 1$:

functions $f : \mathrm{X} \to \mathbb{R}$ with

$$\|f\|_p := \Big( \int |f|^p \, \mathrm{d}\mathfrak{m} \Big)^{\frac{1}{p}} < \infty$$

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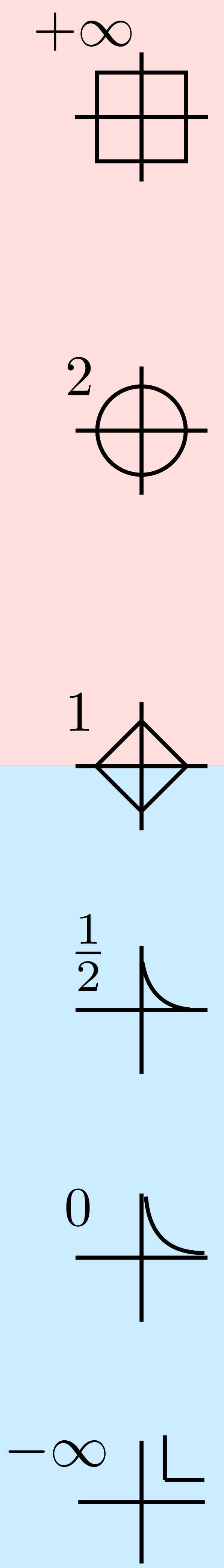

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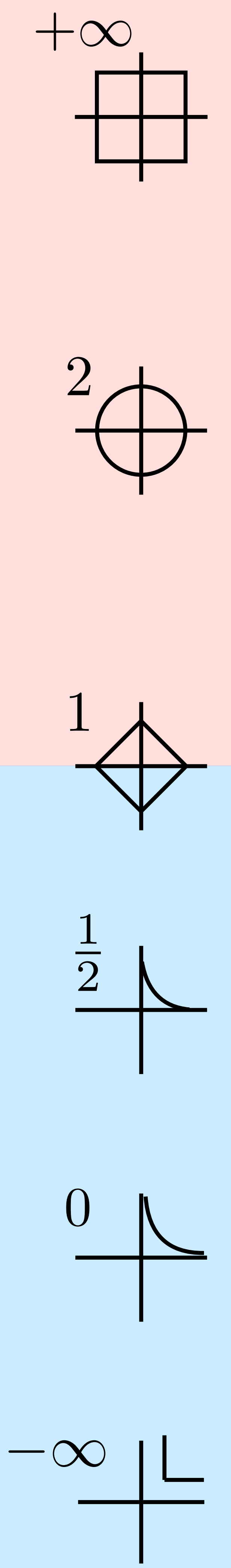

Thank you for the attention