Comparison geometry for substatic manifolds

Stefano Borghini (Università degli Studi di Trento)

Joint works with Mattia Fogagnolo (Università degli Studi di Padova)
and Andrea Pinamonti (Università degli Studi di Trento)

Recent advances in comparison geometry

Hangzhou, 26 February 2024

Substatic manifolds with horizon boundary

Riemannian manifolds (M, g) with a nonnegative smooth function f (substatic potential) such that

Substatic manifolds with horizon boundary

Riemannian manifolds (M, g) with a nonnegative smooth function f (substatic potential) such that

(i) $f\,\mathrm{Ric} - \nabla\nabla f + \Delta f\, g \geq 0$

Substatic manifolds with horizon boundary

Riemannian manifolds (M, g) with a nonnegative smooth function f (substatic potential) such that

(i) $f\,\mathrm{Ric} - \nabla\nabla f + \Delta f\, g \geq 0$

(ii) The boundary $\partial M = \{f = 0\}$ (horizon) is a *minimal* closed hypersurface and a regular level set for f.

Physical motivation: static spacetimes

Physical motivation: static spacetimes

A spacetime is a Lorentzian 4-manifold $(\mathcal{L}, \gamma)$ satisfying

$$\mathrm{Ric}_\gamma + \left(\Lambda - \frac{1}{2}\mathrm{R}_\gamma\right)\gamma \,=\, T\,,$$

where T is the *stress-energy tensor* and $\Lambda \in \mathbb{R}$ is the *cosmological constant*.

Physical motivation: static spacetimes

A spacetime is a Lorentzian 4-manifold $(\mathcal{L}, \gamma)$ satisfying

$$\mathrm{Ric}_\gamma + \left(\Lambda - \frac{1}{2}\mathrm{R}_\gamma\right)\gamma \,=\, T\,,$$

where T is the *stress-energy tensor* and $\Lambda \in \mathbb{R}$ is the *cosmological constant*.

$(\mathcal{L}, \gamma)$ is *static* if it splits as

$$\mathcal{L} = \mathbb{R} \times M\,, \quad \text{and } \gamma \,=\, -f^2\, dt \otimes dt + g\,,$$

with (M, g) Riemannian 3-manifold, $f : M \to \mathbb{R}_+$ smooth.

Physical motivation: static spacetimes

A spacetime is a Lorentzian 4-manifold $(\mathcal{L}, \gamma)$ satisfying

$$\mathrm{Ric}_\gamma + \left(\Lambda - \frac{1}{2}\mathrm{R}_\gamma\right)\gamma \,=\, T\,,$$

where T is the *stress-energy tensor* and $\Lambda \in \mathbb{R}$ is the *cosmological constant*.

$(\mathcal{L}, \gamma)$ is *static* if it splits as

$$\mathcal{L} = \mathbb{R} \times M\,, \quad \text{and } \gamma \,=\, -f^2\, dt \otimes dt + g\,,$$

with (M, g) Riemannian 3-manifold, $f : M \to \mathbb{R}_+$ smooth.

- If we assume vacuum ($T = 0$) we get $f\mathrm{Ric} - \nabla\nabla f + \Delta f\, g = 0$.
  (Other interesting cases: Einstein–Maxwell, perfect fluids, scalar fields, ...)

Physical motivation: static spacetimes

A spacetime is a Lorentzian 4-manifold $(\mathcal{L}, \gamma)$ satisfying

$$\mathrm{Ric}_\gamma + \left(\Lambda - \frac{1}{2}\mathrm{R}_\gamma\right)\gamma \,=\, T\,,$$

where T is the *stress-energy tensor* and $\Lambda \in \mathbb{R}$ is the *cosmological constant*.

$(\mathcal{L}, \gamma)$ is *static* if it splits as

$$\mathcal{L} = \mathbb{R} \times M\,, \quad \text{and } \gamma \,=\, -f^2\, dt \otimes dt + g\,,$$

with (M, g) Riemannian 3-manifold, $f : M \to \mathbb{R}_+$ smooth.

- If we assume vacuum ($T = 0$) we get $f\mathrm{Ric} - \nabla\nabla f + \Delta f\, g = 0$.
  (Other interesting cases: Einstein–Maxwell, perfect fluids, scalar fields, ...)
- If we assume Null Energy Condition ($T(X, X) \geq 0\ \forall\ X$ such that $\gamma(X, X) = 0$), we get (Wang–Wang–Zhang '17)

$$f\,\mathrm{Ric} - \nabla\nabla f + \Delta f\, g \,\geq\, 0$$

Examples

We will consider *noncompact* substatic manifolds. Main model solution:

$$M = [r_0, +\infty) \times \Sigma\,, \qquad g = \frac{dr \otimes dr}{f(r)^2} + r^2 g_\Sigma, \qquad r_0 > 0$$

Examples

We will consider *noncompact* substatic manifolds. Main model solution:

$$M = [r_0, +\infty) \times \Sigma , \qquad g = \frac{dr \otimes dr}{f(r)^2} + r^2 g_\Sigma , \qquad r_0 > 0$$

- $f = \sqrt{1 - 2mr^{2-n}} , \qquad \mathrm{Ric}_\Sigma \geq (n-2) g_\Sigma$

Examples

We will consider *noncompact* substatic manifolds. Main model solution:

$$M = [r_0, +\infty) \times \Sigma, \qquad g = \frac{dr \otimes dr}{f(r)^2} + r^2 g_\Sigma, \qquad r_0 > 0$$

- $f = \sqrt{1 - 2mr^{2-n}}, \qquad \mathrm{Ric}_\Sigma \geq (n-2) g_\Sigma$
  - If $\Sigma = \mathbb{S}^{n-1}$ and $m > 0$ this is Schwarzschild (black hole in vacuum).
  - If $\Sigma = \mathbb{S}^{n-1}$ and $m = 0$ this is $\mathbb{R}^n$.

Examples

We will consider *noncompact* substatic manifolds. Main model solution:

$$M = [r_0, +\infty) \times \Sigma, \qquad g = \frac{dr \otimes dr}{f(r)^2} + r^2 g_\Sigma, \qquad r_0 > 0$$

- $f = \sqrt{1 - 2mr^{2-n}}, \qquad \mathrm{Ric}_\Sigma \geq (n-2) g_\Sigma$
  - If $\Sigma = \mathbb{S}^{n-1}$ and $m > 0$ this is Schwarzschild (black hole in vacuum).
  - If $\Sigma = \mathbb{S}^{n-1}$ and $m = 0$ this is $\mathbb{R}^n$.
- $f = \sqrt{1 - 2mr^{2-n} + q^2 r^{4-2n}} \qquad \mathrm{Ric}_\Sigma \geq (n-2) g_\Sigma.$

Examples

We will consider *noncompact* substatic manifolds. Main model solution:

$$M = [r_0, +\infty) \times \Sigma\,, \qquad g = \frac{dr \otimes dr}{f(r)^2} + r^2 g_\Sigma, \qquad r_0 > 0$$

- $f = \sqrt{1 - 2mr^{2-n}}\,, \qquad \mathrm{Ric}_\Sigma \geq (n-2)g_\Sigma$
  - If $\Sigma = \mathbb{S}^{n-1}$ and $m > 0$ this is Schwarzschild (black hole in vacuum).
  - If $\Sigma = \mathbb{S}^{n-1}$ and $m = 0$ this is $\mathbb{R}^n$.
- $f = \sqrt{1 - 2mr^{2-n} + q^2 r^{4-2n}} \qquad \mathrm{Ric}_\Sigma \geq (n-2)g_\Sigma$.
  - If $\Sigma = \mathbb{S}^{n-1}$ and $m > |q|$ this is Reissner-Nordstrom (charged black hole).

Examples

We will consider *noncompact* substatic manifolds. Main model solution:

$$M = [r_0, +\infty) \times \Sigma\,, \qquad g = \frac{dr \otimes dr}{f(r)^2} + r^2 g_\Sigma, \qquad r_0 > 0$$

- $f = \sqrt{1 - 2mr^{2-n}}\,, \qquad \mathrm{Ric}_\Sigma \geq (n-2) g_\Sigma$
  - If $\Sigma = \mathbb{S}^{n-1}$ and $m > 0$ this is Schwarzschild (black hole in vacuum).
  - If $\Sigma = \mathbb{S}^{n-1}$ and $m = 0$ this is $\mathbb{R}^n$.
- $f = \sqrt{1 - 2mr^{2-n} + q^2 r^{4-2n}} \qquad \mathrm{Ric}_\Sigma \geq (n-2) g_\Sigma.$
  - If $\Sigma = \mathbb{S}^{n-1}$ and $m > |q|$ this is Reissner-Nordstrom (charged black hole).
- $f = \sqrt{1 - 2mr^{2-n} + r^2} \qquad \mathrm{Ric}_\Sigma \geq (n-2) g_\Sigma.$

Examples

We will consider *noncompact* substatic manifolds. Main model solution:

$$M = [r_0, +\infty) \times \Sigma\,, \qquad g = \frac{dr \otimes dr}{f(r)^2} + r^2 g_\Sigma, \qquad r_0 > 0$$

- $f = \sqrt{1 - 2mr^{2-n}}\,, \qquad \mathrm{Ric}_\Sigma \geq (n-2) g_\Sigma$
  - If $\Sigma = \mathbb{S}^{n-1}$ and $m > 0$ this is Schwarzschild (black hole in vacuum).
  - If $\Sigma = \mathbb{S}^{n-1}$ and $m = 0$ this is $\mathbb{R}^n$.
- $f = \sqrt{1 - 2mr^{2-n} + q^2 r^{4-2n}} \qquad \mathrm{Ric}_\Sigma \geq (n-2) g_\Sigma.$
  - If $\Sigma = \mathbb{S}^{n-1}$ and $m > |q|$ this is Reissner-Nordstrom (charged black hole).
- $f = \sqrt{1 - 2mr^{2-n} + r^2} \qquad \mathrm{Ric}_\Sigma \geq (n-2) g_\Sigma.$
  - If $\Sigma = \mathbb{S}^{n-1}$ and $m > 0$ this is Anti de Sitter-Schwarzschild.
  - If $\Sigma = \mathbb{S}^{n-1}$ and $m = 0$ this is the hyperbolic space.

Natural connection with $\mathrm{CD}(0,1)$ condition

$(M, \tilde{g})$ satisfies the $CD(0, N)$ condition if there exists $\psi \in \mathscr{C}^2(M)$ such that the N-Bakry–Émery Ricci tensor is nonnegative

$$\mathrm{Ric}_{\tilde{g}} + \tilde{\nabla}\tilde{\nabla}\psi - \frac{d\psi \otimes d\psi}{N-n} \geq 0.$$

Natural connection with $\mathrm{CD}(0,1)$ condition

$(M, \tilde{g})$ satisfies the $CD(0, N)$ condition if there exists $\psi \in \mathscr{C}^2(M)$ such that the N-Bakry–Émery Ricci tensor is nonnegative

$$\mathrm{Ric}_{\tilde{g}} + \tilde{\nabla}\tilde{\nabla}\psi - \frac{d\psi \otimes d\psi}{N - n} \geq 0.$$

Let (M, g, f) be substatic. Consider (Brendle, Chrusciél, Woolgar, Reiris, ...)

$$\tilde{g} = \frac{g}{f^2}, \qquad \psi = -(n-1)\log f.$$

Natural connection with $\mathrm{CD}(0,1)$ condition

$(M,\tilde{g})$ satisfies the $CD(0,N)$ condition if there exists $\psi \in \mathscr{C}^2(M)$ such that the N-Bakry–Émery Ricci tensor is nonnegative

$$\mathrm{Ric}_{\tilde{g}} + \tilde{\nabla}\tilde{\nabla}\psi - \frac{d\psi \otimes d\psi}{N-n} \geq 0.$$

Let (M,g,f) be substatic. Consider (Brendle, Chrusciél, Woolgar, Reiris, ...)

$$\tilde{g} = \frac{g}{f^2}, \qquad \psi = -(n-1)\log f.$$

Then $\tilde{g}$ satisfies

$$\mathrm{Ric}_{\tilde{g}} + \tilde{\nabla}\tilde{\nabla}\psi + \frac{d\psi \otimes d\psi}{n-1} \geq 0,$$

i.e. 1-Bakry–Émery is nonnegative.

Natural connection with $\mathrm{CD}(0,1)$ condition

$(M, \tilde{g})$ satisfies the $CD(0, N)$ condition if there exists $\psi \in \mathscr{C}^2(M)$ such that the N-Bakry–Émery Ricci tensor is nonnegative

$$\mathrm{Ric}_{\tilde{g}} + \tilde{\nabla}\tilde{\nabla}\psi - \frac{d\psi \otimes d\psi}{N - n} \geq 0.$$

Let (M, g, f) be substatic. Consider (Brendle, Chrusciél, Woolgar, Reiris, ...)

$$\tilde{g} = \frac{g}{f^2}, \qquad \psi = -(n-1)\log f.$$

Then $\tilde{g}$ satisfies

$$\mathrm{Ric}_{\tilde{g}} + \tilde{\nabla}\tilde{\nabla}\psi + \frac{d\psi \otimes d\psi}{n - 1} \geq 0,$$

i.e. 1-Bakry–Émery is nonnegative.

Fundamental elements of comparison geometry in this setting have been recently studied (Wylie, Wylie–Yeroshkin, Ohta, Lu–Minguzzi–Ohta, Kuwae–Sakurai, Kuwae–Li, Sakurai, ...)!

Laplacian comparison

ρ distance from $x \in M$ with respect to $\tilde{g} = g/f^2$. We define the reparametrized distance η_x via

$$\begin{cases} \frac{\partial}{\partial\rho}\eta_x = f^2 & \text{in } M\,, \\ \eta_x(x) = 0 \end{cases}$$

Laplacian comparison

ρ distance from $x \in M$ with respect to $\tilde{g} = g/f^2$. We define the reparametrized distance η_x via

$$\begin{cases} \frac{\partial}{\partial\rho}\eta_x = f^2 & \text{in } M\,, \\ \eta_x(x) = 0 \end{cases}$$

This is a first order PDE $\Rightarrow$ it gives a smooth η_x *outside the cut locus*.

(η_x is the length of the radial $\tilde{g}$-geodesics with respect to $f^4\tilde{g} = f^2 g$.)

Laplacian comparison

ρ distance from $x \in M$ with respect to $\tilde{g} = g/f^2$. We define the reparametrized distance η_x via

$$\begin{cases} \frac{\partial}{\partial\rho}\eta_x = f^2 & \text{in } M\,, \\ \eta_x(x) = 0 & \end{cases}$$

This is a first order PDE $\Rightarrow$ it gives a smooth η_x *outside the cut locus*.
(η_x is the length of the radial $\tilde{g}$-geodesics with respect to $f^4\tilde{g} = f^2 g$.)

Riccati equation for the g-mean curvature H of the level sets of ρ:

$$\frac{\partial}{\partial\rho}\left(\frac{f}{\mathrm{H}} - \frac{\eta_x}{n-1}\right) \geq 0\,.$$

Laplacian comparison

ρ distance from $x \in M$ with respect to $\tilde{g} = g/f^2$. We define the reparametrized distance η_x via

$$\begin{cases} \frac{\partial}{\partial\rho}\eta_x = f^2 & \text{in } M\,, \\ \eta_x(x) = 0 & \end{cases}$$

This is a first order PDE $\Rightarrow$ it gives a smooth η_x *outside the cut locus*.
(η_x is the length of the radial $\tilde{g}$-geodesics with respect to $f^4\tilde{g} = f^2 g$.)

Riccati equation for the g-mean curvature H of the level sets of ρ:

$$\frac{\partial}{\partial\rho}\left(\frac{f}{\mathrm{H}} - \frac{\eta_x}{n-1}\right) \geq 0\,.$$

Proposition (Wylie '16)

$$0 < \frac{\mathrm{H}}{f} = \Delta\rho + \frac{1}{f}\langle\nabla f \,|\, \nabla\rho\rangle \leq \frac{n-1}{\eta_x}$$

within the cut locus of x.

Horizon and ends

Let (M, g) be substatic.

With respect to $\tilde{g} = g/f^2$, the horizon ∂M becomes an end $(\rho \to +\infty)$. On the other hand, the reparametrized distance η has finite limit.

Horizon and ends

Let (M, g) be substatic.

With respect to $\tilde{g} = g/f^2$, the horizon ∂M becomes an end ($\rho \to +\infty$). On the other hand, the reparametrized distance η has finite limit.

We consider two kinds of ends for a substatic solution (M, g, f).

Horizon and ends

Let (M, g) be substatic.

With respect to $\tilde{g} = g/f^2$, the horizon ∂M becomes an end $(\rho \to +\infty)$. On the other hand, the reparametrized distance η has finite limit.

We consider two kinds of ends for a substatic solution (M, g, f).

(i) An end is *f-complete* if any ray γ has infinite $\tilde{g}$-length $(\rho \to +\infty)$ and

$$\int_0^{+\infty} f(\gamma(t))dt = +\infty \,. \quad (\Rightarrow \eta \to +\infty)$$

Main example: Schwarzschild (in particular $\mathbb{R}^n$).

Horizon and ends

Let (M, g) be substatic.

With respect to $\tilde{g} = g/f^2$, the horizon ∂M becomes an end ($\rho \to +\infty$). On the other hand, the reparametrized distance η has finite limit.

We consider two kinds of ends for a substatic solution (M, g, f).

(i) An end is *f-complete* if any ray γ has infinite $\tilde{g}$-length ($\rho \to +\infty$) and

$$\int_0^{+\infty} f(\gamma(t))dt = +\infty \,. \quad (\Rightarrow \eta \to +\infty)$$

Main example: Schwarzschild (in particular $\mathbb{R}^n$).

(ii) An end is *conformally compact* if any ray has finite $\tilde{g}$-length. The end becomes a boundary. We also require the metric to extend smoothly to the conformal boundary.

Main example: Anti de Sitter–Schwarzschild (in particular hyperbolic space).

Substatic Splitting Theorems

Theorem (Wylie '16, B.–Fogagnolo)

Let (M, g) be substatic.

(i) *If (M, g) has two f-complete ends, then*

$$M = \mathbb{R} \times L\,, \quad g = f^2 ds \otimes ds + g_L.$$

Substatic Splitting Theorems

Theorem (Wylie '16, B.–Fogagnolo)

Let (M, g) be substatic.

(i) *If (M, g) has two f-complete ends, then*

$$M = \mathbb{R} \times L\,, \quad g = f^2 ds \otimes ds + g_L.$$

Consequence: if $\partial M \neq \emptyset$ then $\partial M = \mathbb{R} \times \partial L$ noncompact, contradiction
$\Rightarrow$ if $\partial M \neq \emptyset$ there is only one f-complete end.

Substatic Splitting Theorems

Theorem (Wylie '16, B.–Fogagnolo)

Let (M, g) be substatic.

(i) *If (M, g) has two f-complete ends, then*

$$M = \mathbb{R} \times L\,, \quad g = f^2 ds \otimes ds + g_L.$$

Consequence: if $\partial M \neq \emptyset$ then $\partial M = \mathbb{R} \times \partial L$ noncompact, contradiction $\Rightarrow$ if $\partial M \neq \emptyset$ there is only one f-complete end.

(ii) *(M, g) cannot have two conformally compact ends, or an f-complete and a conformally compact one.*

Substatic Splitting Theorems

Theorem (Wylie '16, B.–Fogagnolo)

Let (M, g) be substatic.

(i) *If (M, g) has two f-complete ends, then*

$$M = \mathbb{R} \times L\,, \quad g = f^2 ds \otimes ds + g_L\,.$$

Consequence: if $\partial M \neq \emptyset$ then $\partial M = \mathbb{R} \times \partial L$ noncompact, contradiction $\Rightarrow$ if $\partial M \neq \emptyset$ there is only one f-complete end.

(ii) *(M, g) cannot have two conformally compact ends, or an f-complete and a conformally compact one.*

Proof.

Standard Cheeger–Gromoll/Kasue argument with the Busemann function/distance from the boundary □

Substatic Splitting Theorems

Theorem (Wylie '16, B.–Fogagnolo)

Let (M, g) be substatic.

(i) *If (M, g) has two f-complete ends, then*

$$M = \mathbb{R} \times L\,, \quad g = f^2 ds \otimes ds + g_L.$$

Consequence: if $\partial M \neq \emptyset$ then $\partial M = \mathbb{R} \times \partial L$ noncompact, contradiction $\Rightarrow$ if $\partial M \neq \emptyset$ there is only one f-complete end.

(ii) *(M, g) cannot have two conformally compact ends, or an f-complete and a conformally compact one.*

Proof.

Standard Cheeger–Gromoll/Kasue argument with the Busemann function/distance from the boundary □

(ii) generalizes a result of Chruściel–Simon for vacuum static metrics.

Substatic Bishop–Gromov monotonicity

Classical Bishop–Gromov Theorem for nonnegative Ricci: for every $x \in M$, $r = dist(x, \cdot)$, the functions

$$\frac{|\partial B(x,r)|}{r^{n-1}|\mathbb{S}^{n-1}|}, \qquad \frac{|B(x,r)|}{r^n|\mathbb{B}^n|}$$

are monotonically nonincreasing

Substatic Bishop–Gromov monotonicity

Classical Bishop–Gromov Theorem for nonnegative Ricci: for every $x \in M$, $r = dist(x, \cdot)$, the functions

$$\frac{|\partial B(x,r)|}{r^{n-1}|\mathbb{S}^{n-1}|}, \qquad \frac{|B(x,r)|}{r^n|\mathbb{B}^n|}$$

are monotonically nonincreasing

Theorem (B.–Fogagnolo)

The following functionals are monotonically nonincreasing:

$$A_x(t) = \frac{1}{|\mathbb{S}^{n-1}|}\int_{\{\rho=t\}} \frac{1}{\eta_x^{n-1}} d\sigma$$

Substatic Bishop–Gromov monotonicity

Classical Bishop–Gromov Theorem for nonnegative Ricci: for every $x \in M$, $r = dist(x, \cdot)$, the functions

$$\frac{|\partial B(x,r)|}{r^{n-1}|\mathbb{S}^{n-1}|}, \qquad \frac{|B(x,r)|}{r^n|\mathbb{B}^n|}$$

are monotonically nonincreasing

Theorem (B.–Fogagnolo)

The following functionals are monotonically nonincreasing:

$$A_x(t) = \frac{1}{|\mathbb{S}^{n-1}|}\int_{\{\rho=t\}} \frac{1}{\eta_x^{n-1}} d\sigma$$

$$V_x(t) = \frac{1}{t^n|\mathbb{B}^n|}\int_{\{\rho\leq t\}} \frac{1}{f}\left(\frac{\rho}{\eta_x}\right)^{n-1} d\sigma$$

Proof (monotonicity of $A(t)$).

The Laplacian comparison can be rephrased as

$$\mathrm{div}(X) \leq 0,$$

where

$$X = \frac{f}{\eta^{n-1}} \nabla \rho.$$

Proof (monotonicity of $A(t)$).

The Laplacian comparison can be rephrased as

$$\mathrm{div}(X) \leq 0,$$

where

$$X = \frac{f}{\eta^{n-1}} \nabla \rho.$$

Applying the Divergence Theorem in $\{t \leq \rho \leq T\}$ one gets

$$\int_{\{\rho = t\}} \left\langle X, \frac{\nabla \rho}{|\nabla \rho|} \right\rangle d\sigma \geq \int_{\{\rho = T\}} \left\langle X, \frac{\nabla \rho}{|\nabla \rho|} \right\rangle d\sigma.$$

Proof (monotonicity of $A(t)$).

The Laplacian comparison can be rephrased as

$$\mathrm{div}(X) \leq 0,$$

where

$$X = \frac{f}{\eta^{n-1}} \nabla \rho.$$

Applying the Divergence Theorem in $\{t \leq \rho \leq T\}$ one gets

$$\int_{\{\rho = t\}} \left\langle X, \frac{\nabla \rho}{|\nabla \rho|} \right\rangle d\sigma \geq \int_{\{\rho = T\}} \left\langle X, \frac{\nabla \rho}{|\nabla \rho|} \right\rangle d\sigma.$$

$$\int_{\{\rho = t\}} \frac{f}{\eta^{n-1}} |\nabla \rho| d\sigma \geq \int_{\{\rho = T\}} \frac{f}{\eta^{n-1}} |\nabla \rho| d\sigma.$$

Proof (monotonicity of $A(t)$).

The Laplacian comparison can be rephrased as

$$\mathrm{div}(X) \leq 0,$$

where

$$X = \frac{f}{\eta^{n-1}} \nabla \rho.$$

Applying the Divergence Theorem in $\{t \leq \rho \leq T\}$ one gets

$$\int_{\{\rho = t\}} \left\langle X, \frac{\nabla \rho}{|\nabla \rho|} \right\rangle d\sigma \geq \int_{\{\rho = T\}} \left\langle X, \frac{\nabla \rho}{|\nabla \rho|} \right\rangle d\sigma.$$

$$\int_{\{\rho = t\}} \frac{f}{\eta^{n-1}} |\nabla \rho| d\sigma \geq \int_{\{\rho = T\}} \frac{f}{\eta^{n-1}} |\nabla \rho| d\sigma.$$

$$|\nabla \rho| = 1/f \;\Rightarrow\; A(t) \geq A(T)$$

□

Proof (monotonicity of $V(t)$).

Coarea formula:

$$V(t) = \frac{1}{t^n|\mathbb{B}^n|}\int_{\{\rho\leq t\}}\frac{1}{f}\left(\frac{\rho}{\eta}\right)^{n-1}d\mu = \frac{n}{t^n|\mathbb{S}^{n-1}|}\int_0^t\int_{\{\rho=\tau\}}\left(\frac{\rho}{\eta}\right)^{n-1}d\sigma d\tau$$
$$= \frac{n}{t^n}\int_0^t\tau^{n-1}A(\tau)d\tau$$

Proof (monotonicity of $V(t)$).

Coarea formula:

$$V(t) = \frac{1}{t^n|\mathbb{B}^n|}\int_{\{\rho\leq t\}}\frac{1}{f}\left(\frac{\rho}{\eta}\right)^{n-1}d\mu = \frac{n}{t^n|\mathbb{S}^{n-1}|}\int_0^t\int_{\{\rho=\tau\}}\left(\frac{\rho}{\eta}\right)^{n-1}d\sigma d\tau$$
$$= \frac{n}{t^n}\int_0^t\tau^{n-1}A(\tau)d\tau$$

On the one hand, exploiting area monotonicity:

$$V(t) = \frac{n}{t^n}\int_0^t\tau^{n-1}A(\tau)d\tau \geq \frac{n}{t^n}A(t)\int_0^t\tau^{n-1}d\tau = A(t)$$

Proof (monotonicity of $V(t)$).

Coarea formula:

$$V(t) = \frac{1}{t^n|\mathbb{B}^n|}\int_{\{\rho\leq t\}}\frac{1}{f}\left(\frac{\rho}{\eta}\right)^{n-1}d\mu = \frac{n}{t^n|\mathbb{S}^{n-1}|}\int_0^t\int_{\{\rho=\tau\}}\left(\frac{\rho}{\eta}\right)^{n-1}d\sigma d\tau$$
$$= \frac{n}{t^n}\int_0^t \tau^{n-1}A(\tau)d\tau$$

On the one hand, exploiting area monotonicity:

$$V(t) = \frac{n}{t^n}\int_0^t \tau^{n-1}A(\tau)d\tau \geq \frac{n}{t^n}A(t)\int_0^t \tau^{n-1}d\tau = A(t)$$

On the other hand, differentiating:

$$V'(t) = \frac{n}{t^n}t^{n-1}A(t) - \frac{n^2}{t^{n+1}}\int_0^t \tau^{n-1}A(\tau)d\tau = \frac{n}{t}\left[A(t)-V(t)\right]\leq 0\,.$$

□

Distance from a hypersurface

Ω compact domain $\partial\Omega = \partial M \sqcup \Sigma$, where Σ hypersurface with

$$\mathrm{H}_\Sigma > 0 .$$

Distance from a hypersurface

Ω compact domain $\partial\Omega = \partial M \sqcup \Sigma$, where Σ hypersurface with

$$\mathrm{H}_\Sigma > 0 \,.$$

ρ distance from Σ with respect to $\tilde{g} = g/f^2$. Define reparametrized distance η_Σ by

$$\begin{cases} \frac{\partial}{\partial\rho}\eta_\Sigma = f^2 & \text{in } M \setminus \Omega \\ \eta_\Sigma = (n-1)\dfrac{f}{\mathrm{H}_\Sigma} & \text{on } \Sigma. \end{cases}$$

Distance from a hypersurface

Ω compact domain $\partial\Omega = \partial M \sqcup \Sigma$, where Σ hypersurface with

$$\mathrm{H}_\Sigma > 0\,.$$

ρ distance from Σ with respect to $\tilde{g} = g/f^2$. Define reparametrized distance η_Σ by

$$\begin{cases} \frac{\partial}{\partial\rho}\eta_\Sigma = f^2 & \text{in } M \setminus \Omega \\ \eta_\Sigma = (n-1)\dfrac{f}{\mathrm{H}_\Sigma} & \text{on } \Sigma. \end{cases}$$

As before, one proves:

$$\frac{\partial}{\partial\rho}\left(\frac{f}{\mathrm{H}} - \frac{\eta_\Sigma}{n-1}\right) \geq 0 \quad \Rightarrow \quad 0 < \frac{\mathrm{H}}{f} = \Delta\rho + \frac{1}{f}\langle\nabla f \,|\, \nabla\rho\rangle \leq \frac{n-1}{\eta_\Sigma}$$

Distance from a hypersurface

Ω compact domain $\partial\Omega = \partial M \sqcup \Sigma$, where Σ hypersurface with

$$\mathrm{H}_\Sigma \, > \, 0\,.$$

ρ distance from Σ with respect to $\tilde{g} = g/f^2$. Define reparametrized distance η_Σ by

$$\begin{cases} \frac{\partial}{\partial\rho}\eta_\Sigma = f^2 & \text{in } M \setminus \Omega \\ \eta_\Sigma = (n-1)\dfrac{f}{\mathrm{H}_\Sigma} & \text{on } \Sigma. \end{cases}$$

As before, one proves:

$$\frac{\partial}{\partial\rho}\left(\frac{f}{\mathrm{H}} - \frac{\eta_\Sigma}{n-1}\right) \geq 0 \quad \Rightarrow \quad 0 < \frac{\mathrm{H}}{f} = \Delta\rho + \frac{1}{f}\langle\nabla f \,|\, \nabla\rho\rangle \leq \frac{n-1}{\eta_\Sigma}$$

$$\rightsquigarrow \quad A_\Sigma(t) \, = \, \frac{1}{|\mathbb{S}^{n-1}|}\int_{\{\rho=t\}} \frac{1}{\eta_\Sigma^{n-1}}\, d\sigma \ \ \text{monotonically nonincreasing.}$$

Towards a Willmore-type inequality

$$\int_{\Sigma} \left[\frac{\mathrm{H}}{(n-1)f} \right]^{n-1} d\sigma \; = \; |\mathbb{S}^{n-1}| A_{\Sigma}(0) \geq |\mathbb{S}^{n-1}| \lim_{t \to +\infty} A_{\Sigma}(t)$$

Towards a Willmore-type inequality

$$\int_\Sigma \left[\frac{\mathrm{H}}{(n-1)f}\right]^{n-1} d\sigma \;=\; |\mathbb{S}^{n-1}| A_\Sigma(0) \geq |\mathbb{S}^{n-1}| \lim_{t\to+\infty} A_\Sigma(t)$$

Proposition

If, for any Σ_1, Σ_2,

$$\left|\frac{\eta_{\Sigma_1}}{\eta_{\Sigma_2}}(x) - 1\right| \to 0 \qquad (1)$$

uniformly as $\rho(x) \to +\infty$, then

$$\mathrm{AVR}_f(g) := \lim_{t\to+\infty} A_\Sigma(t) = \lim_{t\to+\infty} V_\Sigma(t)$$

does not depend on Σ.

Towards a Willmore-type inequality

$$\int_\Sigma \left[\frac{\mathrm{H}}{(n-1)f}\right]^{n-1} d\sigma \;=\; |\mathbb{S}^{n-1}| A_\Sigma(0) \geq |\mathbb{S}^{n-1}| \lim_{t\to+\infty} A_\Sigma(t)$$

Proposition

If, for any Σ_1, Σ_2,

$$\left|\frac{\eta_{\Sigma_1}}{\eta_{\Sigma_2}}(x) - 1\right| \to 0 \tag{1}$$

uniformly as $\rho(x) \to +\infty$, then

$$\mathrm{AVR}_f(g) \;:=\; \lim_{t\to+\infty} A_\Sigma(t) \;=\; \lim_{t\to+\infty} V_\Sigma(t)$$

does not depend on Σ.

When (1) holds we say that the end is uniform.

Towards a Willmore-type inequality

$$\int_\Sigma \left[\frac{\mathrm{H}}{(n-1)f}\right]^{n-1} d\sigma \;=\; |\mathbb{S}^{n-1}|A_\Sigma(0) \geq |\mathbb{S}^{n-1}| \lim_{t\to+\infty} A_\Sigma(t)$$

Proposition

If, for any Σ_1, Σ_2,

$$\left|\frac{\eta_{\Sigma_1}}{\eta_{\Sigma_2}}(x) - 1\right| \to 0 \tag{1}$$

uniformly as $\rho(x) \to +\infty$, then

$$\mathrm{AVR}_f(g) := \lim_{t\to+\infty} A_\Sigma(t) = \lim_{t\to+\infty} V_\Sigma(t)$$

does not depend on Σ.

When (1) holds we say that the end is uniform.
This is ensured e.g. if $f \to 1$ or $|\nabla \log f| \leq C\rho^{-1-\epsilon}$.

Towards a Willmore-type inequality

$$\int_\Sigma \left[\frac{\mathrm{H}}{(n-1)f}\right]^{n-1} d\sigma \;=\; |\mathbb{S}^{n-1}| A_\Sigma(0) \geq |\mathbb{S}^{n-1}| \lim_{t\to+\infty} A_\Sigma(t)$$

Proposition

If, for any Σ_1, Σ_2,

$$\left|\frac{\eta_{\Sigma_1}}{\eta_{\Sigma_2}}(x) - 1\right| \to 0 \qquad (1)$$

uniformly as $\rho(x) \to +\infty$, then

$$\mathrm{AVR}_f(g) := \lim_{t\to+\infty} A_\Sigma(t) = \lim_{t\to+\infty} V_\Sigma(t)$$

does not depend on Σ.

When (1) holds we say that the end is uniform.
This is ensured e.g. if $f \to 1$ or $|\nabla \log f| \leq C\rho^{-1-\epsilon}$.
Asymptotically flat ends are uniform f-complete and $\mathrm{AVR}_f(g) = 1$.

Substatic Willmore inequality

Theorem (B.–Fogagnolo)

Let (M, g) be substatic with a uniform f-complete end. Let Ω be a compact domain with $\partial\Omega = \partial M \sqcup \Sigma$, where Σ has strictly positive mean-curvature. Then

$$\int_\Sigma \left(\frac{\mathrm{H}}{f}\right)^{n-1} d\sigma \geq (n-1)^{n-1} |\mathbb{S}^{n-1}| \mathrm{AVR}_f(g) \,.$$

Substatic Willmore inequality

Theorem (B.–Fogagnolo)

Let (M, g) be substatic with a uniform f-complete end. Let Ω be a compact domain with $\partial\Omega = \partial M \sqcup \Sigma$, where Σ has strictly positive mean-curvature. Then

$$\int_\Sigma \left(\frac{\mathrm{H}}{f}\right)^{n-1} d\sigma \geq (n-1)^{n-1}|\mathbb{S}^{n-1}|\mathrm{AVR}_f(g)\,.$$

If equality holds, then in $M \setminus \Omega$ it holds

$$g \;=\; f^2 d\rho \otimes d\rho + \eta^2 g_0\,,$$

where g_0 is a metric on Σ.

Substatic Willmore inequality

Theorem (B.–Fogagnolo)

Let (M, g) be substatic with a uniform f-complete end. Let Ω be a compact domain with $\partial\Omega = \partial M \sqcup \Sigma$, where Σ has strictly positive mean-curvature. Then

$$\int_\Sigma \left(\frac{\mathrm{H}}{f}\right)^{n-1} d\sigma \geq (n-1)^{n-1}|\mathbb{S}^{n-1}|\mathrm{AVR}_f(g)\,.$$

If equality holds, then in $M \setminus \Omega$ it holds

$$g \;=\; f^2 d\rho \otimes d\rho + \eta^2 g_0\,,$$

where g_0 is a metric on Σ.

In non-negative Ricci curvature: proved by Agostiniani–Fogagnolo–Mazzieri. Proof in terms of distances and Riccati due to X. Wang.

Towards an isoperimetric inequality

We follow the approach of Fogagnolo–Mazzieri, that uses the Willmore inequality.

Towards an isoperimetric inequality

We follow the approach of Fogagnolo–Mazzieri, that uses the Willmore inequality.

The Willmore inequality being in terms of H/f, suggests to look at the problem

$$\inf\{P(E), \quad E \text{ homologous to } \partial M\}$$

with fixed

$$|\Omega_\Sigma|_f = \int_\Omega f \, d\mu = V,$$

where

$$\partial\Omega_\Sigma = \Sigma \sqcup \partial M,$$

Towards an isoperimetric inequality

We follow the approach of Fogagnolo–Mazzieri, that uses the Willmore inequality.

The Willmore inequality being in terms of H/f, suggests to look at the problem

$$\inf\{P(E), \quad E \text{ homologous to } \partial M\}$$

with fixed

$$|\Omega_\Sigma|_f = \int_\Omega f \, d\mu = V,$$

where

$$\partial\Omega_\Sigma = \Sigma \sqcup \partial M,$$

because these f-isoperimetric sets have

$$\frac{\mathrm{H}}{f} = \text{cnst}.$$

Substatic isoperimetric inequality

Theorem (B.–Fogagnolo)

Let (M, g) be substatic of dimension $n \leq 7$ with a uniform f-complete end. Assume there exists an exhaustion of outward minimizing hypersurfaces homologous to ∂M.

Substatic isoperimetric inequality

Theorem (B.–Fogagnolo)

Let (M, g) be substatic of dimension $n \leq 7$ with a uniform f-complete end. Assume there exists an exhaustion of outward minimizing hypersurfaces homologous to ∂M. Then, for any Σ homologous to ∂M, it holds

$$|\Sigma|^{\frac{n}{n-1}} - |\partial M|^{\frac{n}{n-1}} \geq n(\mathrm{AVR}_f(g)|\mathbb{S}^{n-1}|)^{\frac{1}{n-1}} |\Omega_\Sigma|_f \,.$$

Substatic isoperimetric inequality

Theorem (B.–Fogagnolo)

Let (M, g) be substatic of dimension $n \leq 7$ with a uniform f-complete end. Assume there exists an exhaustion of outward minimizing hypersurfaces homologous to ∂M. Then, for any Σ homologous to ∂M, it holds

$$|\Sigma|^{\frac{n}{n-1}} - |\partial M|^{\frac{n}{n-1}} \geq n(\mathrm{AVR}_f(g)|\mathbb{S}^{n-1}|)^{\frac{1}{n-1}}|\Omega_\Sigma|_f \,.$$

Equality holds if and only if g is a warped product

$$g = \frac{dr \otimes dr}{f(r)^2} + r^2 g_0$$

and Σ is a cross-section $\{r = \bar{r}\}$.

Substatic isoperimetric inequality

Theorem (B.–Fogagnolo)

Let (M, g) be substatic of dimension $n \leq 7$ with a uniform f-complete end. Assume there exists an exhaustion of outward minimizing hypersurfaces homologous to ∂M. Then, for any Σ homologous to ∂M, it holds

$$|\Sigma|^{\frac{n}{n-1}} - |\partial M|^{\frac{n}{n-1}} \geq n(\mathrm{AVR}_f(g)|\mathbb{S}^{n-1}|)^{\frac{1}{n-1}}|\Omega_\Sigma|_f \,.$$

Equality holds if and only if g is a warped product

$$g = \frac{dr \otimes dr}{f(r)^2} + r^2 g_0$$

and Σ is a cross-section $\{r = \bar{r}\}$.

Previous works in non-negative Ricci:
Inequality for $n \leq 7$ (Fogagnolo–Mazzieri).

Substatic isoperimetric inequality

Theorem (B.–Fogagnolo)

Let (M, g) be substatic of dimension $n \leq 7$ with a uniform f-complete end. Assume there exists an exhaustion of outward minimizing hypersurfaces homologous to ∂M. Then, for any Σ homologous to ∂M, it holds

$$|\Sigma|^{\frac{n}{n-1}} - |\partial M|^{\frac{n}{n-1}} \geq n(\mathrm{AVR}_f(g)|\mathbb{S}^{n-1}|)^{\frac{1}{n-1}} |\Omega_\Sigma|_f\,.$$

Equality holds if and only if g is a warped product

$$g = \frac{dr \otimes dr}{f(r)^2} + r^2 g_0$$

and Σ is a cross-section $\{r = \bar{r}\}$.

Previous works in non-negative Ricci:
Inequality for $n \leq 7$ (Fogagnolo–Mazzieri). Inequality in every dimension (Brendle).

Substatic isoperimetric inequality

Theorem (B.–Fogagnolo)

Let (M, g) be substatic of dimension $n \leq 7$ with a uniform f-complete end. Assume there exists an exhaustion of outward minimizing hypersurfaces homologous to ∂M. Then, for any Σ homologous to ∂M, it holds

$$|\Sigma|^{\frac{n}{n-1}} - |\partial M|^{\frac{n}{n-1}} \geq n(\mathrm{AVR}_f(g)|\mathbb{S}^{n-1}|)^{\frac{1}{n-1}}|\Omega_\Sigma|_f\,.$$

Equality holds if and only if g is a warped product

$$g = \frac{dr \otimes dr}{f(r)^2} + r^2 g_0$$

and Σ is a cross-section $\{r = \overline{r}\}$.

Previous works in non-negative Ricci:
Inequality for $n \leq 7$ (Fogagnolo–Mazzieri). Inequality in every dimension (Brendle).
Rigidity in every dimension (Antonelli–Pasqualetto–Pozzetta–Semola).

Proof (heuristic).

Let Σ_V be isoperimetric for the f-volume homologous to ∂M for any volume V. Then, H/f is constant.

Proof (heuristic).

Let Σ_V be isoperimetric for the f-volume homologous to ∂M for any volume V. Then, H/f is constant. We have by Willmore

$$\left(\frac{\mathrm{H}}{f}\right)^{n-1} |\Sigma_V| = \int_{\Sigma_V} \left(\frac{\mathrm{H}}{f}\right)^{n-1} d\sigma \geq (n-1)^{n-1} |\mathbb{S}^{n-1}| \mathrm{AVR}_f(g)$$

Proof (heuristic).

Let Σ_V be isoperimetric for the f-volume homologous to ∂M for any volume V. Then, H/f is constant. We have by Willmore

$$\left(\frac{\mathrm{H}}{f}\right)^{n-1} |\Sigma_V| = \int_{\Sigma_V} \left(\frac{\mathrm{H}}{f}\right)^{n-1} d\sigma \geq (n-1)^{n-1} |\mathbb{S}^{n-1}| \mathrm{AVR}_f(g)$$

In particular

$$I_f'(V) = \frac{\mathrm{H}}{f} \geq (n-1) \left(\frac{|\mathbb{S}^{n-1}| \mathrm{AVR}_f(g)}{I_f(V)}\right)^{\frac{1}{n-1}}$$

where I_f is the isoperimetric profile, that is $|\Sigma_V|$.

Proof (heuristic).

Let Σ_V be isoperimetric for the f-volume homologous to ∂M for any volume V. Then, H/f is constant. We have by Willmore

$$\left(\frac{\mathrm{H}}{f}\right)^{n-1} |\Sigma_V| = \int_{\Sigma_V} \left(\frac{\mathrm{H}}{f}\right)^{n-1} d\sigma \geq (n-1)^{n-1} |\mathbb{S}^{n-1}| \mathrm{AVR}_f(g)$$

In particular

$$I_f'(V) = \frac{\mathrm{H}}{f} \geq (n-1) \left(\frac{|\mathbb{S}^{n-1}| \mathrm{AVR}_f(g)}{I_f(V)}\right)^{\frac{1}{n-1}}$$

where I_f is the isoperimetric profile, that is $|\Sigma_V|$.
Integrating in V, and using

$$\lim_{V \to 0^+} I_f(V) = |\partial M|,$$

one gets

$$|\Sigma_V|^{\frac{n}{n-1}} - |\partial M|^{\frac{n}{n-1}} \geq n \left(\mathrm{AVR}_f(g) |\mathbb{S}^{n-1}|\right)^{\frac{1}{n-1}} V.$$

□

Comments on the proof: existence of isoperimetrics

Σ_V may not exist (the space is noncompact) $\rightsquigarrow$ consider the isoperimetric problem constrained in an outward minimising set B (idea due to Kleiner).
Here the problem has a solution.

Comments on the proof: existence of isoperimetrics

Σ_V may not exist (the space is noncompact) $\rightsquigarrow$ consider the isoperimetric problem constrained in an outward minimising set B (idea due to Kleiner).
Here the problem has a solution.

We prove that Σ_V does not touch the horizon (but it can touch ∂B).

Comments on the proof: existence of isoperimetrics

Σ_V may not exist (the space is noncompact) $\rightsquigarrow$ consider the isoperimetric problem constrained in an outward minimising set B (idea due to Kleiner).
Here the problem has a solution.

We prove that Σ_V does not touch the horizon (but it can touch ∂B).

To apply Willmore we need $\mathrm{H} > 0$:

Comments on the proof: existence of isoperimetrics

Σ_V may not exist (the space is noncompact) $\rightsquigarrow$ consider the isoperimetric problem constrained in an outward minimising set B (idea due to Kleiner).
Here the problem has a solution.

We prove that Σ_V does not touch the horizon (but it can touch ∂B).

To apply Willmore we need $\mathrm{H} > 0$:

- If Σ_V touches ∂B then

$$\left(\frac{\mathrm{H}}{f}\right)_{\partial\Sigma_V \setminus \partial B} \geq \left(\frac{\mathrm{H}}{f}\right)_{\partial\Sigma_V \cap \partial B} > 0$$

Comments on the proof: existence of isoperimetrics

Σ_V may not exist (the space is noncompact) $\rightsquigarrow$ consider the isoperimetric problem constrained in an outward minimising set B (idea due to Kleiner).
Here the problem has a solution.

We prove that Σ_V does not touch the horizon (but it can touch ∂B).

To apply Willmore we need $\mathrm{H} > 0$:

- If Σ_V touches ∂B then

$$\left(\frac{\mathrm{H}}{f}\right)_{\partial\Sigma_V \setminus \partial B} \geq \left(\frac{\mathrm{H}}{f}\right)_{\partial\Sigma_V \cap \partial B} > 0$$

- H cannot be zero: there are no minimal hypersurfaces homologous to ∂M (B.–Fogagnolo, a combination of Riccati and MCF).

Comments on the proof: existence of isoperimetrics

Σ_V may not exist (the space is noncompact) $\rightsquigarrow$ consider the isoperimetric problem constrained in an outward minimising set B (idea due to Kleiner).
Here the problem has a solution.

We prove that Σ_V does not touch the horizon (but it can touch ∂B).

To apply Willmore we need $\mathrm{H} > 0$:

- If Σ_V touches ∂B then

$$\left(\frac{\mathrm{H}}{f}\right)_{\partial\Sigma_V \setminus \partial B} \geq \left(\frac{\mathrm{H}}{f}\right)_{\partial\Sigma_V \cap \partial B} > 0$$

- H cannot be zero: there are no minimal hypersurfaces homologous to ∂M (B.–Fogagnolo, a combination of Riccati and MCF).
- If H/f is negative, the outward minimizing hull is minimal (Fogagnolo–Mazzieri)$\Rightarrow$ impossible.

Further directions

- Improve the isoperimetric inequality:
  - remove the assumption on the existence of an exhaustion of outward minimizing hypersurfaces (IMCF?).
  - remove the dimensional threshold $n \leq 7$ (Brendle's strategy? RCD framework?).

Further directions

- Improve the isoperimetric inequality:
  - remove the assumption on the existence of an exhaustion of outward minimizing hypersurfaces (IMCF?).
  - remove the dimensional threshold $n \leq 7$ (Brendle's strategy? RCD framework?).
- Other results from comparison geometry? Minkowski inequality (through IMCF perhaps)? Compact case (Schwarzschild–de Sitter)?

Further directions

- Improve the isoperimetric inequality:
  - remove the assumption on the existence of an exhaustion of outward minimizing hypersurfaces (IMCF?).
  - remove the dimensional threshold $n \leq 7$ (Brendle's strategy? RCD framework?).
- Other results from comparison geometry? Minkowski inequality (through IMCF perhaps)? Compact case (Schwarzschild–de Sitter)?
- Study

$$f\,\mathrm{Ric} - \nabla^2 f + (\Delta f) g \geq -\mu g\,, \quad \mu \in \mathbb{R}$$

Equality given by V-static solutions (related to Besse conjecture).

Further directions

- Improve the isoperimetric inequality:
  - remove the assumption on the existence of an exhaustion of outward minimizing hypersurfaces (IMCF?).
  - remove the dimensional threshold $n \leq 7$ (Brendle's strategy? RCD framework?).
- Other results from comparison geometry? Minkowski inequality (through IMCF perhaps)? Compact case (Schwarzschild–de Sitter)?
- Study

$$f\,\mathrm{Ric} - \nabla^2 f + (\Delta f)g \geq -\mu g\,, \quad \mu \in \mathbb{R}$$

  Equality given by V-static solutions (related to Besse conjecture).
- Can we replace η with the $f^2 g$-distance? If so, we would get rid of the uniform assumption.

Substatic Heintze–Karcher inequality

Theorem (Li–Xia '17, Fogagnolo–Pinamonti '22)

Let (M, g) be substatic and assume

$$\frac{\nabla\nabla f}{f} \in C^{0,\alpha}(M \cup \partial M)$$

Let Ω be a compact domain with $\partial\Omega = \partial M \sqcup \Sigma$, where Σ is a connected, smooth strictly mean-convex hypersurface. Then

$$\int_\Sigma \frac{f}{\mathrm{H}}\, d\sigma \geq \frac{n}{n-1}\int_\Omega f\, d\mu + \left(\int_{\partial M} |\nabla f|\, d\sigma\right)^2 \left(\int_{\partial M} |\nabla f|^2 \frac{\mathrm{H}}{f}\, d\sigma\right)^{-1},$$

where Ω is the bounded set enclosed by Σ and ∂M.

Substatic Heintze–Karcher inequality

Theorem (B.–Fogagnolo–Pinamonti)

Let (M, g) be substatic and assume

$$\frac{\nabla\nabla f}{f} \in C^{0,\alpha}(M \cup \partial M)$$

Let Ω be a compact domain with $\partial\Omega = \partial M \sqcup \Sigma$, where Σ is a connected, smooth strictly mean-convex hypersurface. Then

$$\int_\Sigma \frac{f}{\mathrm{H}}\, d\sigma \geq \frac{n}{n-1}\int_\Omega f\, d\mu + \left(\int_{\partial M} |\nabla f|\, d\sigma\right)^2 \left(\int_{\partial M} |\nabla f|^2 \frac{\mathrm{H}}{f}\, d\sigma\right)^{-1},$$

where Ω is the bounded set enclosed by Σ and ∂M.
In case of equality, then Ω is isometric to a warped product

$$g = \frac{dr \otimes dr}{f(r)^2} + r^2 g_0 \,.$$

Substatic warped products

We now focus on substatic warped products

$$\left([r_0, \overline{r}] \times N \;,\; \frac{dr \otimes dr}{f(r)^2} + r^2 g_N\right) ,$$

where f is the substatic potential.

Substatic warped products

We now focus on substatic warped products

$$\left([r_0,\overline{r}]\times N\,,\ \frac{dr\otimes dr}{f(r)^2}+r^2g_N\right)\,,$$

where f is the substatic potential.

It always holds

$$\left(\mathrm{Ric}-\frac{\nabla\nabla f}{f}+\frac{\Delta f}{f}g\right)(\nabla r,\nabla r)\,=\,0\,.$$

∇f is constant on the boundary, the level sets have $\mathrm{H}/f=(n-1)/r$.
$\rightsquigarrow$ the Heintze–Karcher inequality rewrites as:

$$(n-1)\int_\Sigma \frac{f}{\mathrm{H}}\,d\sigma\geq n\int_\Omega f\,d\mu+r_0|\partial M|\,.$$

CMC hypersurfaces in substatic warped products

Brendle's contribution:

- Heintze–Karcher inequality for hypersurfaces Σ in a substatic warped product. If equality then Σ is umbilic. (weaker, but no hypothesis on $\nabla\nabla f/f$)

CMC hypersurfaces in substatic warped products

Brendle's contribution:

- Heintze–Karcher inequality for hypersurfaces Σ in a substatic warped product. If equality then Σ is umbilic. (weaker, but no hypothesis on $\nabla\nabla f/f$)
- CMC hypersurfaces saturate the Heintze–Karcher inequality ($\Rightarrow$ umbilic).

CMC hypersurfaces in substatic warped products

Brendle's contribution:

- Heintze–Karcher inequality for hypersurfaces Σ in a substatic warped product. If equality then Σ is umbilic. (weaker, but no hypothesis on $\nabla\nabla f/f$)
- CMC hypersurfaces saturate the Heintze–Karcher inequality ($\Rightarrow$ umbilic).
- Σ umbilic CMC. If $\mathrm{Ric}_{g_N} \geq (n-2)cg_N$ and

$$\text{(H4)} \qquad r\, f(r)f'(r) - f(r)^2 + c > 0\,,$$

  then Σ cross section.

CMC hypersurfaces in substatic warped products

Brendle's contribution:

- Heintze–Karcher inequality for hypersurfaces Σ in a substatic warped product. If equality then Σ is umbilic. (weaker, but no hypothesis on $\nabla\nabla f/f$)
- CMC hypersurfaces saturate the Heintze–Karcher inequality ($\Rightarrow$ umbilic).
- Σ umbilic CMC. If $\mathrm{Ric}_{g_N} \geq (n-2)cg_N$ and

$$\text{(H4)} \qquad r\, f(r)f'(r) - f(r)^2 + c > 0\,,$$

then Σ cross section.

Theorem (Brendle '13)

In a substatic warped product satisfying (H4), the cross sections are the only CMC hypersurfaces.

Heintze–Karcher inequality in substatic warped products

Theorem (B.–Fogagnolo–Pinamonti)

Let (M, g) be a substatic warped product. Let Ω be a compact domain with $\partial\Omega = \partial M \sqcup \Sigma$, where Σ is a connected, smooth strictly mean-convex hypersurface. Then

$$(n-1)\int_\Sigma \frac{f}{\mathrm{H}}\, d\sigma \geq n\int_\Omega f\, d\mu + r_0|\partial M|\,,$$

where Ω is the bounded set enclosed by Σ and ∂M.
In case of equality, then Σ is a cross section.

Heintze–Karcher inequality in substatic warped products

Theorem (B.–Fogagnolo–Pinamonti)

Let (M, g) be a substatic warped product. Let Ω be a compact domain with $\partial\Omega = \partial M \sqcup \Sigma$, where Σ is a connected, smooth strictly mean-convex hypersurface. Then

$$(n-1)\int_\Sigma \frac{f}{\mathrm{H}}\, d\sigma \geq n \int_\Omega f\, d\mu + r_0|\partial M|\,,$$

where Ω is the bounded set enclosed by Σ and ∂M.
In case of equality, then Σ is a cross section.

Following Brendle, we then have:

Corollary

In a substatic warped product, the cross sections are the only CMC hypersurfaces.

Proof.

It is enough to show that $\nabla\nabla f/f \in C^{0,\alpha}(M \cup \partial M)$.

Proof.

It is enough to show that $\nabla\nabla f/f \in C^{0,\alpha}(M \cup \partial M)$.

- Following Brendle, let ρ be the $\tilde{g}$-distance from Σ and let $Q(t) = \int_{\{\rho=t\}} \frac{f}{\mathrm{H}} d\sigma$.

$$Q'(t) = -\frac{n}{n-1} \int\limits_{\{\rho=t\}} f \, d\sigma - \int\limits_{\{\rho=t\}} \left(\frac{f}{\mathrm{H}}\right)^2 \left[|\mathring{\mathrm{h}}|^2 + \left(\mathrm{Ric} - \frac{\nabla\nabla f}{f} + \frac{\Delta f}{f} g\right)(\nu,\nu)\right] d\sigma$$

Proof.

It is enough to show that $\nabla\nabla f/f \in C^{0,\alpha}(M \cup \partial M)$.

- Following Brendle, let ρ be the $\tilde{g}$-distance from Σ and let $Q(t) = \int_{\{\rho=t\}} \frac{f}{\mathrm{H}} d\sigma$.

$$Q'(t) = -\frac{n}{n-1}\int\limits_{\{\rho=t\}} f\, d\sigma - \int\limits_{\{\rho=t\}} \left(\frac{f}{\mathrm{H}}\right)^2 \left[|\mathring{\mathrm{h}}|^2 + \left(\mathrm{Ric} - \frac{\nabla\nabla f}{f} + \frac{\Delta f}{f} g\right)(\nu,\nu)\right] d\sigma$$

$$\Rightarrow \quad Q(0) - Q(t) \geq -\frac{n}{n-1}\int\limits_{\{0\leq\rho\leq t\}} f\, d\mu \qquad (\Rightarrow \text{H.–K. when } t \to +\infty)$$

Proof.

It is enough to show that $\nabla\nabla f/f \in C^{0,\alpha}(M \cup \partial M)$.

- Following Brendle, let ρ be the $\tilde{g}$-distance from Σ and let $Q(t) = \int_{\{\rho=t\}} \frac{f}{\mathrm{H}} d\sigma$.

$$Q'(t) = -\frac{n}{n-1}\int\limits_{\{\rho=t\}} f\, d\sigma - \int\limits_{\{\rho=t\}} \left(\frac{f}{\mathrm{H}}\right)^2 \left[|\mathring{\mathrm{h}}|^2 + \left(\mathrm{Ric} - \frac{\nabla\nabla f}{f} + \frac{\Delta f}{f} g\right)(\nu,\nu)\right] d\sigma$$

$$\Rightarrow \quad Q(0) - Q(t) \geq -\frac{n}{n-1}\int\limits_{\{0\leq\rho\leq t\}} f\, d\mu \qquad (\Rightarrow \text{H.–K. when } t \to +\infty)$$

- Σ saturates Heintze–Karcher $\Rightarrow$ Q constant $\Rightarrow$ $\Sigma_t = \{\rho = t\}$ umbilic and

$$\left(\mathrm{Ric} - \frac{\nabla\nabla f}{f} + \frac{\Delta f}{f} g\right)(\nu,\nu) = 0$$

on Σ_t $\forall$ t.

Proof.

- On the other hand, we also have

$$\left(\mathrm{Ric} - \frac{\nabla\nabla f}{f} + \frac{\Delta f}{f} g\right)(\nabla r, \nabla r) = 0$$

Proof.

- On the other hand, we also have

$$\left(\mathrm{Ric} - \frac{\nabla\nabla f}{f} + \frac{\Delta f}{f} g\right)(\nabla r, \nabla r) = 0$$

- Argument by Montiel: if Σ is not a cross section, ∇r and ν are a.e. linearly independent $\rightsquigarrow$ there exists $X \perp \nabla r$ such that

$$\left(\mathrm{Ric} - \frac{\nabla\nabla f}{f} + \frac{\Delta f}{f} g\right)(X, X) = 0 \qquad (1)$$

Proof.

- On the other hand, we also have

$$\left(\mathrm{Ric} - \frac{\nabla\nabla f}{f} + \frac{\Delta f}{f} g\right)(\nabla r, \nabla r) = 0$$

- Argument by Montiel: if Σ is not a cross section, ∇r and ν are a.e. linearly independent $\rightsquigarrow$ there exists $X \perp \nabla r$ such that

$$\left(\mathrm{Ric} - \frac{\nabla\nabla f}{f} + \frac{\Delta f}{f} g\right)(X, X) = 0 \qquad (1)$$

- B.–Fogagnolo–Pinamonti: substatic warped products satisfying (1) have the form:

$$g = \frac{dr \otimes dr}{f(r)^2} + r^2 g_N \,, \quad \mathrm{Ric}_{g_N} \geq (n-2)c g_N \,, \quad f = \sqrt{c - \lambda r^2 - \frac{2m}{r^{n-2}}}$$

where $c, \lambda, m \in \mathbb{R}$.

Proof.

- On the other hand, we also have

$$\left(\mathrm{Ric} - \frac{\nabla\nabla f}{f} + \frac{\Delta f}{f} g\right)(\nabla r, \nabla r) = 0$$

- Argument by Montiel: if Σ is not a cross section, ∇r and ν are a.e. linearly independent $\rightsquigarrow$ there exists $X \perp \nabla r$ such that

$$\left(\mathrm{Ric} - \frac{\nabla\nabla f}{f} + \frac{\Delta f}{f} g\right)(X, X) = 0 \qquad (1)$$

- B.–Fogagnolo–Pinamonti: substatic warped products satisfying (1) have the form:

$$g = \frac{dr \otimes dr}{f(r)^2} + r^2 g_N\,, \quad \mathrm{Ric}_{g_N} \geq (n-2) c g_N\,, \quad f = \sqrt{c - \lambda\, r^2 - \frac{2m}{r^{n-2}}}$$

where $c, \lambda, m \in \mathbb{R}$. These warped products satisfy $\nabla\nabla f / f \in C^{0,\alpha}(M \cup \partial M)$!
$\rightsquigarrow$ our rigidity statement triggers $\rightsquigarrow$ contradiction.

□

Thank you!