On Bernstein type theorems for minimal graphs under Ricci lower bounds

joint works with G. Colombo, E.S. Gama, M. Magliaro and M. Rigoli

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MINIMAL GRAPHS ON MANIFOLDS

(M,σ) complete Riemannian manifold, dimension m

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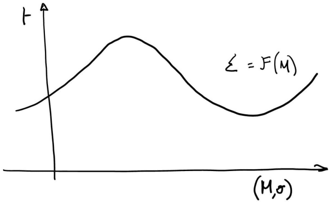
$u : M \to \mathbb{R}$

$F : M \to M \times \mathbb{R}$

$x \mapsto (x, u(x))$



endow $M \times \mathbb{R}$ with metric $\sigma + dt^2$

g induced metric on $\Sigma \quad \Longrightarrow \quad \Sigma = (M,g)$

$$\Sigma \text{ is minimal} \quad \Longleftrightarrow \quad \boxed{\operatorname{div}\left(\frac{Du}{\sqrt{1+|Du|^2}}\right) = 0} \quad \text{on } M \quad \text{(MS)}$$

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Notice: (MS) writes as $\boxed{\Delta_g u = 0}$

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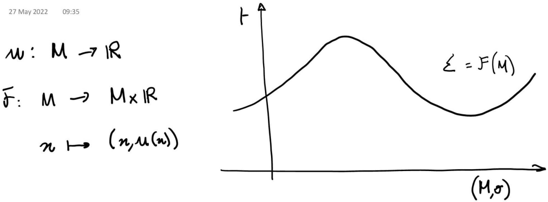

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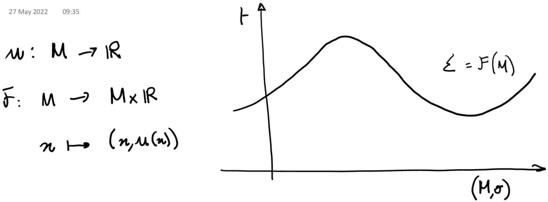

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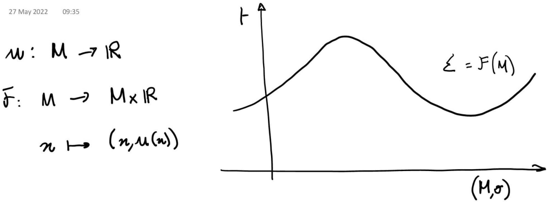

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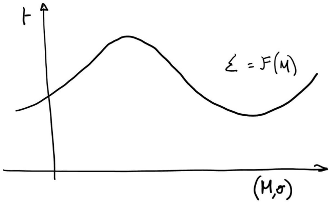

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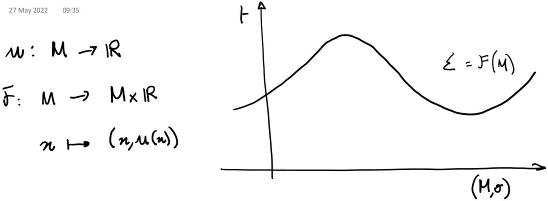

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$(\mathscr{B}1)$ all solutions to (MS) on $\mathbb{R}^m$ are affine

holds if and only if $m \leq 7$.

(Bernstein '15, De Giorgi '65, Almgren '66, Simons '68, Bombieri-De Giorgi-Giusti '69)

$(\mathscr{B}2)$ Solutions to (MS) on $\mathbb{R}^m$ with $u_-(x) = \mathcal{O}(|x|)$ are affine
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$(\mathscr{B}3)$ Positive solutions to (MS) on $\mathbb{R}^m$ are constant
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Notice: $(\mathscr{B}1) \Rightarrow (\mathscr{B}2) \Rightarrow (\mathscr{B}3)$.

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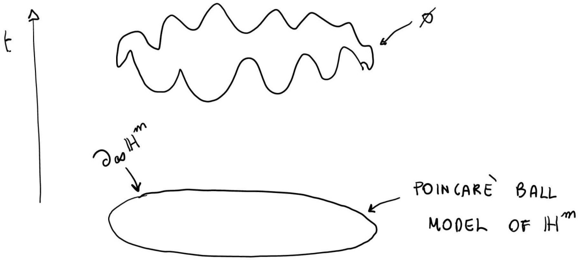
for which manifolds M properties $(\mathscr{B}1)$, $(\mathscr{B}2)$, $(\mathscr{B}3)$ hold?

If $M = \mathbb{H}^m$, *completely different picture*: Plateau's problem at infinity is always solvable!



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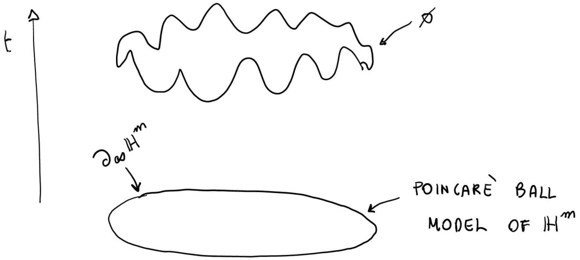

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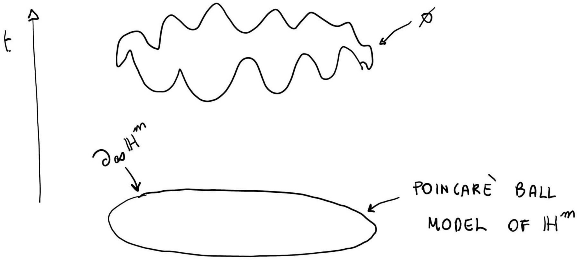

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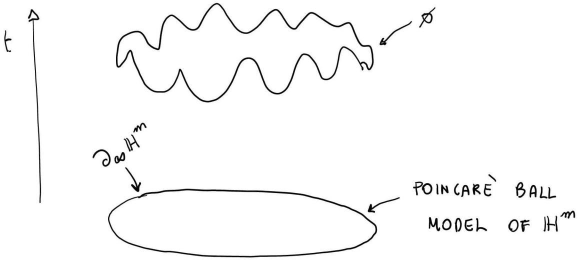

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$(\mathscr{B}1), (\mathscr{B}2), (\mathscr{B}3)$ might hold if Sec ≥ 0 or Ric ≥ 0:

1) Analogy with the theory of harmonic functions (recall: $\Delta_g u = 0$)

2) Cheeger-Colding's theory is available: if $o \in M$, $\lambda_j \to +\infty$, then

$$(M, \lambda_j^{-2}\sigma, o) \hookrightarrow (M_\infty, \mathrm{d}, o_\infty) \qquad \text{for some (nonsmooth) } M_\infty \text{ with Ric} \geq 0.$$

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PROPERTY $(\mathscr{B}1)$

Theorem

Let M^m be complete, $\mathrm{Ric} \geq 0$. Fix $o \in M$ and assume that

$$\int^{\infty} \frac{r}{|B_r(o)|} \mathrm{d}r = +\infty \tag{1}$$

Let u be a non-constant solution to (MS). *Then,*

- *$M = N \times \mathbb{R}$ with the product metric $\sigma_N + \mathrm{d}s^2$,*
- *In the variables $(y, s) \in N \times \mathbb{R}$ it holds $u(y, s) = as + b$ for some $a, b \in \mathbb{R}$*

Thus, $(\mathscr{B}1)$ holds.

In particular, it applies to surfaces with $\mathrm{Sec} \geq 0$.

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PROPERTIES $(\mathscr{B}2)$, $(\mathscr{B}3)$ IF $\mathrm{Sec} \geq 0$

Theorem (Colombo, Gama, M-, Rigoli 2022)

Let M be complete, $\mathrm{Sec} \geq 0$. Let u be a non-constant solution to (MS) and assume that

$$u_-(x) = \mathcal{O}\big(r(x)\big) \qquad \text{as } r(x) = \mathrm{dist}(x, o) \to \infty.$$

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- Borrowed from Cheeger-Colding-Minicozzi '95 and Moser '61.
- STEP 1: $\quad u_-(x) = \mathcal{O}(r(x)) \quad \Longrightarrow \quad |Du| \in L^\infty(M)$.
- STEP 2: $\quad$ blowdowns $(M, \lambda_j^{-2}\sigma, o) \to M_\infty$ split: $\quad M_\infty = N_\infty \times \mathbb{R}$.

- First:

$$|Du| \in L^\infty(M) \qquad \Longrightarrow \qquad u_j = \frac{u - u(o)}{\lambda_j} \to u_\infty \; : \; M_\infty \to \mathbb{R}.$$

- Key inequalities for balls $B_R \subset (M, \sigma)$ centered at o:

$$(i) \qquad \lim_{R\to\infty} \fint_{B_R} |Du|^2 = \sup_M |Du|^2$$

$$(ii) \qquad \lim_{R\to\infty} R^2 \fint_{B_R} |D^2 u|^2 = 0.$$

- Take limits: $\quad |Du_\infty| \neq 0, \quad |D^2 u_\infty| = 0 \quad$ on M_∞.

  Flow of Du_∞ gives splitting.

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Theorem (Colombo, Magliaro, M-, Rigoli '21, Q. Ding '21)

A complete manifold M with $\mathrm{Ric} \geq 0$ satisfies $(\mathscr{B}3)$:

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Previously shown by Rosenberg, Schulze, Spruck '13 under the further condition $\mathrm{Sec} \geq -\kappa^2$, $\kappa \in \mathbb{R}^+$.

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Our main gradient estimate

Theorem

Let M^m complete with $\mathrm{Ric} \geq -(m-1)\kappa^2$*, for constant* $\kappa \geq 0$.

Let u be a positive solution to (MS) *on an open set* $\Omega \subset M$.

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(i) $\partial\Omega$ *locally Lipschitz and* $|\partial\Omega \cap B_R| \leq C_1 \exp\{C_2 R^2\}$, *or*

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Then

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$$\Delta_g W^{-1} + \left[\| \, \mathrm{II} \, \|^2 + \mathrm{Ric}\left(\frac{Du}{W}\right)\right] W^{-1} = 0 \qquad \text{on } \Sigma.$$

Korevaar's method: compute $\Delta_g(W\eta)$, for η a (carefully crafted) cutoff depending on u and r, the distance in (M,σ) from a fixed point.

Problem: need to evaluate

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The proof

Fix $C > \kappa\sqrt{m-1}$, $\quad z = We^{-Cu}$

- CLAIM: the following set is empty for every $\tau > 0$:

$$\Omega' := \left\{ x \in \Omega \ : \ z(x) > \max\left\{1, \limsup_{y \to \partial\Omega} \frac{W(y)}{e^{\kappa\sqrt{m-1}u(y)}}\right\} + \tau \right\}$$

Once the claim is shown, thesis follows by letting $\tau \to 0$, $C \downarrow \kappa\sqrt{m-1}$.

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$$\mathscr{L}_g\phi = W^2 \mathrm{div}_g\left(W^{-2}\nabla\phi\right) \qquad \text{on } \Sigma$$

Since $\|\nabla u\|^2 = \frac{W^2-1}{W^2}$,

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$$\mathscr{L}_g \phi = W^2 \mathrm{div}_g \left(W^{-2} \nabla \phi \right) \qquad \text{on } \Sigma$$

Since $\|\nabla u\|^2 = \frac{W^2-1}{W^2}$,

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The proof

Fix $C > \kappa\sqrt{m-1}$, $\quad z = We^{-Cu}$

- CLAIM: the following set is empty for every $\tau > 0$:

$$\Omega' := \left\{ x \in \Omega \ : \ z(x) > \max\left\{ 1, \limsup_{y \to \partial\Omega} \frac{W(y)}{e^{\kappa\sqrt{m-1}u(y)}} \right\} + \tau \right\}$$

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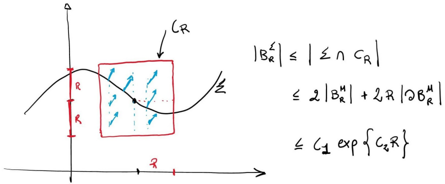

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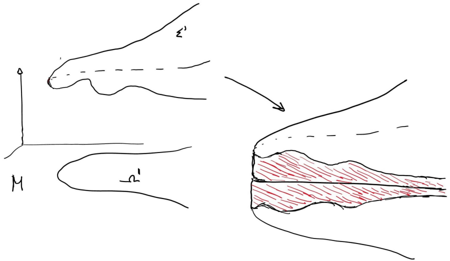

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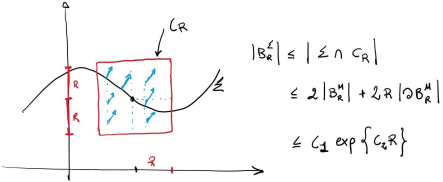

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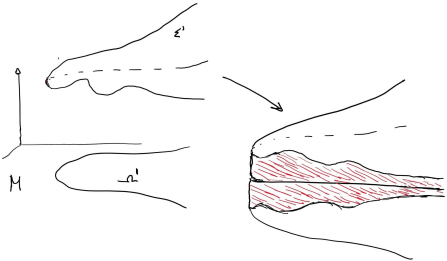

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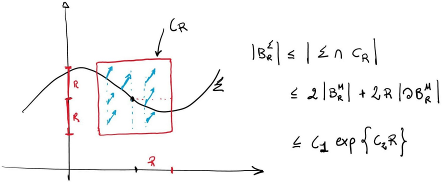

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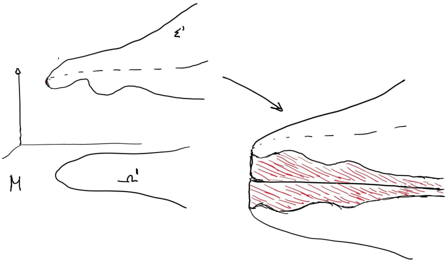

THEOREM (Grigoryan '99, Pigola-Rigoli-Setti '03):

If (N,h) is complete and

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then

$$(\star)\ :\ \boxed{\begin{cases} \Delta_h \omega \ge \omega & \text{on } \overline{U} \subset N, \\ \sup_U \omega < \infty \end{cases} \quad \Longrightarrow \quad \sup_U \omega \le \max\left\{0, \sup_{\partial U} \omega\right\}}$$

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(N,h) satisfies $(\star)$ *if and only if* there exists $v \in C^\infty(N)$ solving

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setting $\varrho = \log v \in C^\infty(N)$,

$$\begin{cases} \Delta_g \varrho + \|\nabla \varrho\|^2 \leq 1 \\ \varrho \geq 0, \qquad \varrho \text{ exhaustion on } N \end{cases}$$

Let $\delta, \varepsilon', \varepsilon$ be positive, small (specified later), and set

$$z_0 = W\left(e^{-Cu-\varepsilon\varrho} - \delta\right) < z$$

For ε, δ small enough, the upper level-set

$$\Omega_0' := \left\{ x \in \Omega \ : \ z_0(x) > \max\left\{1, \limsup_{y \to \partial\Omega} \frac{W(y)}{e^{\kappa\sqrt{m-1}u(y)}}\right\} + \tau \right\} \subset \Omega'.$$

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We compute on the graph Σ_0'

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\mathscr{L}_g z_0 &\geq \Big[\|C\nabla u + \varepsilon\nabla\varrho\|^2 - (m-1)\kappa^2\|\nabla u\|^2 - \varepsilon\Delta_g\varrho\Big] z_0 \\
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if ε' small enough and $\varepsilon << \varepsilon'$.

Using $\Delta\varrho + \|\nabla\varrho\|^2 \leq 1$ and $\varepsilon << 1$,

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$$|Du| \in L^\infty(M) \quad \Longrightarrow$$

$$L \doteq W\Delta_g = \frac{1}{\sqrt{\sigma}}\partial_i\big(W\sqrt{\sigma}g^{ij}\partial_j\big) \qquad \text{uniformly elliptic on } (M,\sigma).$$

$$(i) \qquad \lim_{R\to\infty} \fint_{B_R} |Du|^2 = \sup_M |Du|^2$$

$$(ii) \qquad \lim_{R\to\infty} R^2 \fint_{B_R} |D^2u|^2 = 0.$$

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- The function

$$f \doteq (\sup_M |Du|^2) - |Du|^2 = (\sup_M W^2) - W^2$$

is non-negative, bounded, and

$$Lf \leq -2\|\,\mathrm{II}\,\|^2 W^3 = -2|D^2u|^2 W \leq 0.$$

Want: $\forall\, 0 \leq f \in L^\infty(M)$ solving $Lf \leq 0$,

$$\fint_{B_R} f \to \inf_M f, \qquad R^2 \fint_{B_R} Lf \to 0.$$

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$$\begin{cases} \partial_t v = Lv & \text{on } M \times \mathbb{R}^+ \\ v(x,0^+) = f(x) & \forall x \in M. \end{cases} \quad \Longrightarrow \quad f(x,t) = \int_M h(x,y,t) f(y) \mathrm{d}y$$

Notice: $0 \leq v(x,t) \leq \|f\|_\infty$.

The L-heat kernel h satisfies (Saloff-Coste '92)

$$(i) \quad \left(1 + \frac{\mathrm{d}(x,y)}{\sqrt{t}}\right)^{-\frac{m}{2}} \frac{C_1}{|B_{\sqrt{t}}(x)|} e^{\frac{-\mathrm{d}^2(x,y)}{C_2 t}}$$

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Step 1: We show

$$\fint_{B_{\sqrt{t}}(x)} f \to 0 \qquad \text{as } t \to \infty.$$

(compare with Repnikov-Eidelman '66,'67). We follow P. Li '86.

By the lower bound on h,

$$\begin{aligned} v(x,t) &= \int_M f(y) h(x,y,t) \mathrm{d}y \\ &\geq \frac{C_1}{|B_{\sqrt{t}}(x)|} \int_M f(y) \left(1 + \frac{\mathrm{d}(x,y)}{\sqrt{t}}\right)^{-\frac{m}{2}} e^{\frac{-\mathrm{d}^2(x,y)}{C_2 t}} \mathrm{d}y \\ &\geq \frac{C_1}{|B_{\sqrt{t}}(x)|} \int_{B_{\sqrt{t}}(x)} \cdots \geq \frac{C_3}{|B_{\sqrt{t}}(x)|} \int_{B_{\sqrt{t}}(x)} f \geq 0. \end{aligned}$$

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- (x,t) fixed, $\Omega_a = \{y : h(x,y,t) > a\}$.

$$\begin{aligned}\partial_t v(x,t) &= \int_M f\partial_t h = \lim_{a\to 0}\int_{\Omega_a} f\partial_t h \\ &= \lim_{a\to 0}\int_{\Omega_a} fL(h-a) \\ &= \lim_{a\to 0}\left\{\int_{\partial\Omega_a} f\partial_\nu(h-a) + \int_{\Omega_a}(h-a)Lf\right\} \le 0.\end{aligned}$$

- Thus,

$$v(x,t) \downarrow v_\infty(x) \qquad \text{as } t\to\infty,$$

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$$t \fint_{B_{\sqrt{t}}(x)} Lf \to 0 \qquad \text{as } t \to \infty.$$

We start from

$$\int_{\Omega_a} f \partial_t h \leq \int_{\Omega_a} (h-a) Lf$$

Key fact: there exists $\delta = \delta(C_j)$ and $k = k(C_j)$ such that if

$$a = \frac{\delta}{|B_{\sqrt{t}}(x)|}$$

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then

$$B_{\sqrt{t}}(x) \subset \Omega_{2a} \subset \Omega_a \subset B_{k\sqrt{t}}(x).$$

Step 2: We show

$$t \fint_{B_{\sqrt{t}}(x)} Lf \to 0 \qquad \text{as } t \to \infty.$$

We start from

$$\int_{\Omega_a} f \partial_t h \leq \int_{\Omega_a} (h-a) Lf$$

Key fact: there exists $\delta = \delta(C_j)$ and $k = k(C_j)$ such that if

$$a = \frac{\delta}{|B_{\sqrt{t}}(x)|}$$

then

$$B_{\sqrt{t}}(x) \subset \Omega_{2a} \subset \Omega_a \subset B_{k\sqrt{t}}(x).$$

$$B_{\sqrt{t}}(x) \subset \Omega_{2a} \subset \Omega_a \subset B_{k\sqrt{t}}(x), \qquad a = \frac{\delta}{|B_{\sqrt{t}}(x)|}$$

$$\begin{aligned}
0 &\geq \fint_{B_{\sqrt{t}}(x)} Lf = \frac{a}{\delta} \int_{B_{\sqrt{t}}(x)} Lf \\
&\geq \frac{1}{\delta} \int_{B_{\sqrt{t}}(x)} (h-a)Lf \geq \frac{1}{\delta} \int_{\Omega_a} (h-a)Lf \\
&\geq \frac{1}{\delta} \int_{\Omega_a} f \partial_t h \\
&\geq -\frac{1}{t\delta} \frac{C_5}{|B_{\sqrt{t}}(x)|} \int_{B_{k\sqrt{t}}(x)} f(y) \left(1 + \frac{\mathrm{d}(x,y)}{\sqrt{t}}\right)^{\frac{m}{2}} e^{\frac{-\mathrm{d}^2(x,y)}{C_6 t}} \, \mathrm{d}y \\
&\geq -\frac{C_7}{t} \frac{|B_{k\sqrt{t}}(x)|}{|B_{\sqrt{t}}(x)|} \fint_{B_{k\sqrt{t}}(x)} f \geq -\frac{C_8}{t} \fint_{B_{k\sqrt{t}}(x)} f
\end{aligned}$$

THANKS!