BIRS-IASM 24w5192: Partial Differential Equations: Deterministic and Probabilistic

Chenjie Fan (Chinese Academy of Sciences),
Weijun Xu (Peking University),
Ting Zhou (Zhejiang University)

18–23 August, 2024

1 Overview of the Field

The general area of PDEs with singular randomness has experienced vast breakthroughs and developments in recent years, notably parabolic PDEs with singular random noise, and (deterministic) dispersive PDEs with singular random initial data. A common feature of these two types of problems is that the randomness present is too singular for nonlinear operations in the equation to be well-defined classically. Hence, solutions are defined as limits of classical solutions with regularised noise/data after suitable *renormalisation*. In this procedure, randomness is used to obtain a well-defined limit.

In parabolic singular stochastic PDEs, recent advances of the theories of regularity structures ([27]) , paracontrolled distributions ([26]) and renormalisation group approaches ([28, 18]) make local well-posedness for sub-critical equations largely satisfactory.

On the other side, (deterministic) dispersive PDEs with singular random initial data started with Bourgain ([6]) and developed further by other authors (for example, [9]). The initial data in these settings are all below the critical threshold of the deterministic equations, and randomness are used to break such a barrier. Recent breakthroughs in this area include proofs of well-posedness of Schrödinger equations with Gaussian initial data all the way down to probabilistic criticality ([17, 16]), and the three dimensional cubic wave equation with Gibbs measure initial data ([7]). Long standing problems in the field are solved and new theories are emerging.

Important future questions in the above areas include understanding universalities, long-time existence without damping or knowledge of invariant measures, and well-posedness of the three dimensional cubic nonlinear Schrödinger equation with Gaussian free field or Gibbs measure initial data.

Recent developments in singular parabolic stochastic PDEs also experienced a shift of viewpoint – looking at solution theories that are maps with nonlinearities in the equation as inputs and solution maps for standard fixed equations as outputs (see [8, 10, 11]). This suggests potential connections with inverse problems arising from PDEs.

The PDE theory and analysis of *inverse problems* arise in various imaging and tomography methods. It amounts to a rigorous mathematical understanding of the dependence of the underlying PDE operator, linear or nonlinear, deterministic or stochastic, on partial knowledge of its solutions. In more general settings, these PDE based inverse problems have strong connections to some geometric inverse problems of boundary rigidity and inverse scattering theories.

The developments of the fields of PDE based inverse problems and inverse scattering theory have entered a new era due to the exploding amount of data available and advances in understanding more complicated physical models. For example, ever since the breakthrough in observing gravitational waves via LIGO, the inverse theory based on the nonlinear Einstein equation in general relativity (see [29]) has been established to reconstruct the space-time (i.e., the metric of a manifold describing the universe) structure from local measurements of gravitational waves or its light effects. The PDE and microlocal analysis tools developed there then extend to solve inverse problems for other nonlinear PDEs, including the Navier-Stokes-Maxwell system arising in magnetohydrodynamics, elastic equations in geophysics and the second-harmonic-generation Maxwell's equations in electromagnetism. Another example is related to the inverse problems for stochastic PDEs with highly oscillatory coefficients, arising in imaging and time-reversal algorithms in random media. When the heterogeneous medium has certain properties involving separation of scales, periodicity, or stationary ergodicity, homogenization theories have been developed and they provide macroscopic models for the heterogeneous equations. This paradigm also extends to the stability analysis in inverse theory, as demonstrated in [3, 4] for the kinetic theory.

2 Open Problems

Here are examples of some open questions in the field.

- Global well-posedness of singular stochastic PDEs or dispersive PDEs with random initial data, particularly without the existence of strong damping or invariant measures.
- Well-posedness of the three dimensional cubic nonlinear Schrödinger equation under Φ^4_3 initial data.
- Consider the validity and accuracy of the Navier-Stokes-Maxwell equations to model the magnetohydrodynamic system in order to solve an inverse problem in determining the constituents in the outer core of an extraterrestrial planet from orbital measurements of its magnetic field.
- Deeper understanding of homogenization theory and its role in analyzing the stability discrepancy in inverse problems for multi-scale physical models.
- Investigation of the light scattering by sub-wavelength resonant structures in some super-resolution imaging designs.

3 Presentation Highlights

The scientific programme consists of four research lecture series (three 50-minute lecture each) plus nine additional colloquium style research talks (50 minute each).

3.1 Lecture series

The lecture series are a novelty compared to usual research workshop programmes. Since our meeting consists of researchers from three different (though related) areas, we design four lecture series on four different topics, with the aim of introducing and motivating participants from diverse fields the main questions/problems and techniques as well as recent developments in the field. These arrangements turn out to be very successful — feedbacks from most participants said that they really enjoyed these series.

3.1.1 Topics on random data theory for PDEs

Yu Deng (University of Southern California) and **Haitian Yue (Shanghai Tech. University)** jointly gave a lecture series on PDEs with (singular) random initial data, starting with a motivative introduction of the classic work by Bourgain ([6]), to more recent series works on dispersive PDEs with singular random initial data as well as justification of wave kinetic equation in long time scales.

The first two lectures were given by **Haitian Yue**. In his first lecture, Haitian introduced the problem of deterministic dispersive PDEs starting with (singular) Gibbs measure initial data. He described the classical result of Bourgain ([6]), and then moved to his joint work with Deng and Nahmod ([17]) which developed the analytic framework of random averaging operators and pushed the 2D NLS problem to arbitrary power nonlinearity.

In his second lecture, Haitian continued on this line of investigation, this time discussed his recent work with Bringmann, Deng and Nahmod ([7]), which established the invariance of the Φ_3^4 initial data under the three dimensional cubic wave dynamics. This resolves a long-standing challenging problem in the field. The resolution relies on a deep para-controlled ansatz and a miraculous cancellation between two diverging stochastic objects.

The last lecture of this series was given by **Yu Deng**, focusing on his recent series of works with Hani on rigorous derivation of wave kinetic equation from the cubic nonlinear Schrödinger equation at the kinetic timescale ([13, 14]). These works solve a main conjecture in the theory of wave turbulence. The techniques and methods again involve deep and keen observation of the structures of various stochastic objects. These observations and methods turn out to be robust enough to be applicable to the problem of the long time derivation of the Boltzmann equation from hard sphere dynamics, which Yu also discussed his recent exciting work with Hani and Ma ([15]).

3.1.2 Stochastic quantisation

Hao Shen (University of Wisconsin-Madison) and **Weijun Xu (Peking University)** jointly gave a lecture series on various aspects of stochastic quantisation. The idea of stochastic quantisation dates back to Parisi and Wu, who proposed to construct a probability measure on the space of functions/distributions as an invariant measure of its Langevin dynamics (a stochastic PDE). This is an infinite dimensional analogue of sampling a distribution in $\mathbf{R}^d$ as the invariant measure of its associated Langevin SDE. However, in infinite dimensional situations, the singularity of the stochastic forcing makes nonlinear terms in the SPDE classically ill-defined, and recent advances including the theory of regularity structures ([27], para-controlled distributions ([26]) and renormalisation group approaches ([28, 18]) are developed in order to establish local-in-time well-posedness of these equations. Long time existence (and hence existence and convergence to invariant measures) are still largely challenging. This lecture series gave an overview of recent developments in these directions as well as future questions.

The series started with an introductory lecture by **Hao Shen**, motivating the set up of stochastic quantisations, giving concrete examples (dynamical Φ_d^4, sine-Gordon, Yang-Mills, etc.), and then moved into more detailed discussions on the simple example of dynamical Φ_2^4. He showed how to use the dynamical Φ_2^4 equation to construct and exploit properties of the Φ_2^4 measure.

The second lecture of this series was given by **Weijun Xu**, focusing on the recent variational method developed by Barashkov and Gubinelli ([5]). Although relatively recent, this method has already found applications in various situations involving constructing quantum field theories, and many very recent and ongoing works make essential use of it. Weijun illustrated the core of this method in the simpler context of Φ_2^4, and then briefly discussed the subtleties in dimension three and how to overcome them.

In the last lecture, **Hao Shen** gave a colloquium-style talk on the recent developments on stochastic quantisation of the Yang-Mills measures in two and three dimensions (the measure in the latter is still conjectural). This involves a series of works on local well-posedness of the Langevin dynamics in dimension two ([10]) and three ([11]), and the confirmation of the invariance of the 2D Yang-Mills measure under the Langevin dynamic ([12]).

3.1.3 Quantitative homogenization of interacting particle systems

In this series of mini-course, **Jean-Christophe Mourrat (ENS Lyon and CNRS)**, **Chenlin Gu (Tsinghua University)**, and **Tadahisa Funaki (Beijing Institute of Mathematical Sciences and Applications)** introduced the recent progress in interacting particle systems via the homogenization method. The course includes the essential ingredients of the renormalization approach in quantitative homogenization summarized in [2, 1], and its extension and application in several infinite interacting particle systems [24, 25, 22, 23]. This new combination allows to obtain an estimate of convergence rate in non-gradient exclusion system, which is missing for longtime. This lecture series is closely related to the research talk by Benjamin Fehrman.

Jean-Christophe Mourrat presented the basis of the renormalization approach in quantitative homogenization for elliptic equation in $\mathbf{R}^d$. The key idea is to use subadditive quantity and its dual problem to quantify the convergence rate of the effective conductivity. This idea is then generalized in the work [24], where an interacting particle system in continuous space was studied. In this model, the variational problem is described as the minimization of Dirichlet energy using the intrinsic derivative on configuration space, and such diffusion process was constructed before in [31].

Chenlin Gu presented a recent research [22] collaborated with Funaki and Wang. In this work, the quantitative homogenization of diffusion matrix was obtained in a non-gradient exclusion process. Compared to the previous work [24], the main challenge comes from the constrain of particle number, which was discussed in details and requires a new lifting technique to extend the system to larger space of free particles. The result was then integrated to the classical framework in [23] of relative entropy method, and yielded a convergence rate for the hydrodynamic limit.

Tadahisa Funaki presented the further application of quantitative homogenization in non-gradient Glauber–Kawasaki dynamics [21]. In this dynamic, the particles evolve with speed-change rate, and there is also creation and annihilation. The result in [22] also allows to obtain the quantitative hydrodynamic limit of this non-gradient Glauber–Kawasaki dynamics, and it shows the phase-separating interface evolves according to the anisotropic curvature flow. Some perspectives about non-linear fluctuation in this model were mentioned in the end.

3.1.4 The Calderón inverse problem and inverse problems for hyperbolic equations

Gunther Uhlmann (University of Washington and HKUST) and **Yiran Wang (Emory University)** jointly gave a lecture series on the classical Calderón inverse problems and its recent developments for hyperbolic equations.

In his two lectures, **Gunther Uhlmann** introduced the classical Calderón problem motivated by applications such as EIT, geophysical prospection and so on. The topic was generalized to other inverse problems for both elliptic and hyperbolic PDEs. The results on Calderón type inverse problems are rich yet several important and difficult questions remain to be solved, such as the partial data problems for more general shaped boundaries and the anisotropic problem. The latter was closely related to the inverse problems on Riemannian manifolds. Then he showed recent results on inverse problems for fractional PDEs, which sheds light on both applications involving abnormal diffusive physics and how to solve the anisotropic Calderón problem for local operators by solving the corresponding question for the anisotropic nonlocal operator and then passing to the local one.

In **Yiran Wang**'s lecture on Inverse problem for hyperbolic equations on the theme of geometric optics, he motivated the inverse problem via applications of imaging using various progressing waves. Formulated on Lorentzian manifolds, these problems typically ask to reconstruct metric information from the measurements of waves. The general approach is to apply geometrical optics type of solutions that propagate along geodesics to eventually transform the inverse problem to an integral geometry problem: how to determine the metric (or a conformal factor of sound speed) from the geodesic ray transform. This integral geometry inverse problem is also the linearized problem for the important boundary rigidity problem. Yiran Wang introduced the recent seminal work by Vasy and Uhlmann in solving this problem based on a local approach. Then he also discussed the inverse problem for nonlinear hyperbolic equations, where microlocal tools are used to analyze the nonlinear interaction of waves generating new microlocal singularities that results in new waves to help solve the inverse problem of recovering the space-time metric and nonlinear coefficients in a cosmology setup.

3.2 Research talks

In addition to the four lecture series, we also scheduled nine colloquium-style research talks, given by some of the participants on their recent and ongoing works.

Jacky Chong (Peking University) talks about derivation of the compressible Euler equations from the dynamics of interacting Bose gas in the hard-core limit. One origin of the nonlinear dispersive PDEs is from many body quantum system. Roughly speaking, many body-Schrödinger can be used to derive (semi-classical) Gross-Pitevskii, which can be further used to derive Euler by considering semiclassical limit. Jacky's talk explains how can one directly derive Euler from many body Schrodinger. It combines idea from dispersive PDEs and mathematical physics.

Wenkui Du (University of Toronto) introduced, in his talk Singularity Models in Geometric Variational Problems, the geometric variational problems of mean curvature flow and the minimal surfaces. From a PDE point of view, he consider the Allen-Cahn equations solutions whose level sets converge to minimal surfaces and geodesics can be used to study regularity of minimal surfaces and geodesics and the multiplicity one problem. He talked about classification of ancient solutions of mean curvature flow, and surveyed recent progress on the regularity and singularity structure of limiting minimal interfaces generated by Allen Cahn equations on manifolds with boundary.

Ben Fehrman (Louisiana State University) gave a talk on *Non-equilibrium fluctuations, conservative stochastic PDE, and parabolic-hyperbolic PDE with irregular drift*. Far-from-equilibrium behaviour is widespread in physical systems. A statistical description of these events is provided by macroscopic fluctuation theory, a framework for non-equilibrium statistical mechanics that postulates a formula for the probability of a space-time fluctuation based on the constitutive equations of the system. This formula is formally obtained via a zero noise large deviations principle for the associated fluctuating hydrodynamics, which postulates a conservative, singular stochastic PDE to describe the system out-of equilibrium.

In his talk, Ben started with introduction to macroscopic fluctuation theory and fluctuating hydrodynamics, focusing particularly on the fluctuations of certain interacting particle processes about their hydrodynamic limits. Later, he showed how the associated MFT and fluctuating hydrodynamics lead to a class of conservative SPDEs with irregular coefficients, and how the study of large deviations principles for the particles processes and SPDEs leads to the analysis of parabolic-hyperbolic PDEs in energy critical spaces. The analysis makes rigorous the connection between MFT and fluctuating hydrodynamics in this setting, and provides a positive answer to a long-standing open problem for the large deviations of the zero range process ([19, 20]).

The scientific content of this talk is closely related to the lecture series by Funaki-Gu-Mourrat.

Soonsik Kwon (Korea Advanced Institute of Science and Technology) talks about the dynamic of Calogero-Moser Derivative NLS. It is a model which has been popularised by the recent work of Gerard and Lenzmann. This model shares a lot of common feature with mass critical NLS, but this model is complete integrable. The talk explores (constructive) finite time blow up and soliton resolution of the model, both being central topics of dispersive PDEs. The approach does not directly use the complete integrability but explores the connection of this model to Chern-Simons-Schrödinger.

In her talk on Inverse random scattering problems for stochastic wave equations, **Xu Wang (AMSS, Chinese Academy of Sciences)** introduced Inverse random scattering problems with a random source or potential for time-harmonic stochastic wave equations. These problems arise in applications where either the medium or the data involves randomness due to factors such as environmental noise, incomplete knowledge of the medium, fine-scale fluctuations in measurement. The unknown random source or potential is assumed to be a generalized fractional Gaussian random field. With information of the data observed in a bounded domain, the strength of the random source or potential is shown to be uniquely determined by a single realization of the magnitude of the wave field averaged over the frequency band almost surely.

In talk by **Yiran Wang (Emory University)** (different from his lecture series talk), with title *Inverse source problems for the Boltzmann equation in cosmology*, he considered an inverse problem of general relativity, particularly originated from the Cosmic Microwave Background (CMB). Based on the kinetic theory, the inverse problem is to recover the source term from the linearized photon distribution function that solves the Boltzmann equation. The problem then reduces to study the normal operator of the weighted light ray transform, which turns out to be difficult to handle than other space like geodesic ray transforms because of the limited angels. The microlocal structure of the operator was analyzed and stability results can be shown by proving the ellipticity of the normal operator. Furthermore, Yiran Wang showed some numerical simulation based on the singularity recovery.

Yakun Xi (Zhejiang University) gives a talk entitled *Can you hear where a drum is struck?*, which comes from the classical seminar question by Kac, *Can one hear the shape of a drum*, and goes all the way back to Weyl's classical result about determining the (asymptotic) number of eigen values by the geometry. Yakun's talk explores spectral analysis, lots of analysis are from the view point of waves (classical and qutuam) and intersect well with inverse problems. The main point is the Weyl's pointwise counting functions well decides the geometry of the ambient manifold.

Shengquan Xiang (Peking University) gave a talk on *Exponential mixing for random nonlinear wave equations: weak dissipation and localized control*. Ergodicity for nonlinear wave equations with degenerate forcing has been challenging questions for long time, mainly due to the lack of smoothing effect with respect to initial data for wave equations and the degeneracy of the noise in the non-parabolic setting. In an impressive joint work with colleagues from PKU ([30]), they established a new criterion for exponential mixing of random dynamical systems. This criterion is applicable to a wide range of systems, including in particular dispersive equations. Its verification is naturally related to several topics, i.e., asymptotic compactness in dynamical systems, global stability of evolution equations, and localized control problems. As an initial application, they exploit the exponential mixing of random nonlinear wave equations with degenerate damping, critical nonlinearity, and physically localised noise. Shengquan gave an inspiring lecture on this piece of work.

Jian Zhai (Fudan University) talked about inverse boundary value problems for nonlinear wave equations, with applications of imaging using acoustic and elastic waves. Based on the higher order linearization and the microlocal analysis of the singularity generated from nonlocal interaction of waves, the inverse problem of nonlinear wave equations formulated on a Lorentzian manifold can be solved. As a comparison, the theory for its counter linear model is still open due to the lack of unique continuation and the available approach of boundary control method. In particular, Zhai introduced his recent results on several inverse problems of recovering the nonlinear coefficients from the boundary Cauchy data of the solutions.

4 Scientific Progress Made

One aim of the meeting is to bridge various developments within PDEs with singular randomness, as well as their potential applications to inverse problems. Some progress out of the meeting include (but not restricted to) the following:

- People from diverse backgrounds (at least three different areas) have the chance to meet each other and discuss freely, opening a possibility for joint future efforts between different communities.

- Participants identified some open challenging problems in the fields, and share with each other the ideas/thoughts from his/her own background. This lays a basis for future collaboration and exploration.
- Participants from geometric background and stochastic background discussed the recent advances in mean curvature flows and stochastic Allen-Cahn equations, and had a brief intention to explore together collaboratively.
- Participants discussed possible application of methods from PDEs with singular randomness to some challenging inverse problem questions.

Overall, this workshop gives an opportunity and environment to initiate collaborations between different communities as well as potential applications of theoretical methods to concrete practical problems.

5 Outcome of the Meeting

This meeting successfully created a lively and welcoming atmosphere, bringing together a diverse group of participants, including senior experts, early career researchers, and many PhD students. This mix of backgrounds (both in career stage and field) added diversity to the discussions and encouraged fresh ideas, making it a truly enriching experience for everyone involved.

The size of the meeting is relatively small, and this allows participants to build meaningful relationships that could lead to future collaborations.

Participants appreciated the hybrid format, which allowed smooth communication between those attending in person and those joining online. The virtual aspect also broadened participation, allowing those who might not have been able to attend in person to contribute meaningfully to discussions (we have two online lectures by Jean-Christophe Mourrant and Yu Deng, and more online participants).

Feedback from attendees was overwhelmingly positive, with many stating it was one of the most rewarding events they had participated in recently. They highlighted the variety of topics covered, from scientific presentations, to relaxed environment for free discussions, and also excellent staff support. All these stimulate inspirations. Many discussions led to ideas for potential research projects and collaborations, with participants expressing enthusiasm about the new directions they could explore together.

Additionally, attendees appreciated the well-organised structure of the meeting, which balanced informative sessions with opportunities for social interaction. Many commented on how the social activities helped to break down barriers and facilitated a more relaxed atmosphere, making it easier to engage with fellow participants.

In summary, the meeting successfully advanced important topics in PDEs with singular randomness and inverse problems, while fostering a strong sense of community among participants. The positive experiences shared by attendees indicate a strong interest in maintaining this spirit, paving the way for future interactions and collaborations in the field. The connections made during the meeting are likely to lead to fruitful partnerships and continued engagement, contributing to the growth and development of the community.

Acknowledgement

We (the organisers and the participants) have benefited from the ideal academic environment of BIRS-IASM, and all of us agree that the profession and hospitality of the IASM staff and the support from BIRS-IASM are essential for the success of the fruitful meeting. We thank all the staff for their professional work and for creating such a warming and welcoming environment.

List of participants

There are totally 53 participants of the meeting, with 42 on-site (which reaches the maximum number) and 11 participating remotely.

On-site:

Xu, Weijun (Peking University)
Fan, Chenjie (Chinese Academy of Sciences)
Zhou, Ting (Zhejiang University)
Ban, Yingzhe (Institute of Applied Physics and Computational Mathematics)
Chen, Yilin (Peking University)
Chong, Jacky Jia Wei (Peking University)
Deng, Yangkendi (Chinese Academy of Sciences)
Di, Boning (Chinese Academy of Sciences)
Du, Wenkui (University of Toronto)
Fehrman, Benjamin (Louisiana State University)
Funaki, Tadahisa (Beijing Institute of Mathematical Sciences and Applications)
Gao, Lingxi (Zhejiang University)
Gu, Chenlin (Tsinghua University)
Kwon, Soonsik (Korea Advanced Institute of Science and Technology)
Li, Li (University of California Irvine)
Liang, Guangqiu (Emory University)
Liu, Baoping (Peking University)
Liu, Shao (University of Bonn)
Qiu, Lingyun (Tsinghua University)
Qiu, Yanqi (University of Chinese Academy of Sciences)
Ruizhe, Xu (Zhejiang University)
Shan, Minjie (Minzu University of China)
Shen, Hao (University of Wisconsin - Madison)
Tong, Jiajun (Peking University)
Uhlmann, Gunther (University of Washington and HKUST)
Wang, Xu (Chinese Academy of Sciences)
Wang, Yiran (Emory University)
Wang, Haiyi (Peking University)
Wu, Wei (NYU Shanghai)
Xi, Yakun (Zhejiang University)
Xiang, Shengquan (Peking University)
Yang, Yang (Johns Hopkins)
Ye, Hongcan (Zhejiang University)
Yue, Haitian (ShanghaiTech University)
Zhai, Jian (Fudan University)
Zhang, Ting (Zhejiang Univeristy)
Zhang, Zhifei (Peking University)
Zhang, Xicheng (Beijing Institute of Technology)
Zhao, Zehua (Beijing Institute of Technology)
Zhao, Wenhao (EPFL)
Zheng, Siqin (Tsinghua University)
Zhou, Shuhan (Peking University)

Online:

Bao, Haotian (Zhejiang University)
Chen, Huiping (Academy of Mathematics and Systems Science - Chinese Academy of Sciences)
Deng, Yu (University of Southern California)
He, Lingbing (Tsinghua University)
Kong, Fanhao (Peking University)
Li, Guopeng (Maxwell Institute for Mathematical Sciences/Beijing Institute of Technology)

Mourrat, Jean-Christophe (ENS Lyon and CNRS)
Peng, Nanxiong (Zhejiang University)
Phung, Cheng Fei (National University of Singapore)
Xiao, Baoming (Zhejiang University)

References

[1] S. Armstrong and T. Kuusi, Elliptic homogenization from qualitative to quantitative, *arXiv preprint*, 2022.

[2] S. Armstrong, T. Kuusi and J.-C. Mourrat, *Quantitative Stochastic Homogenization and Large-Scale Regularity*, vol.352, Grundlehren der mathematischen Wissenschaften, Springer Nature, 2019.

[3] G. Bal, Inverse Problems in random media: a kinetic approach, *J. Phys. Conf. Series*, **124** (2008): 012001.

[4] G. Bal, A. Jollivet, Generalized stability estimates in inverse transport theory, *Inverse Probl. Imaging*, **12**-1 (2018): 5990.

[5] N. Barashkov, M. Gubinelli, A variational method for Φ^4_3, *Duke Math. J.*, **169**-17 (2020): 3339-3415.

[6] J. Bourgain, Invariant measures for the 2D-defocusing nonlinear Schrödinger equation, *Comm. Math. Phys.*, **176** (1996): 421445.

[7] B. Bringmann, Y. Deng, A. Nahmod, H. Yue, Invariant Gibbs measures for the three dimensional cubic nonlinear wave equation, *Invent. Math.*, **236**-3 (2024): 11331411.

[8] Y. Bruned, F. Gabriel, M. Hairer, L. Zambotti, Geometric stochastic heat equations., *J. Amer. Math. Soc.*, **35**-1 (2022): 180.

[9] N. Burq, N. Tzvetkov, Random data Cauchy theory for supercritical wave equations I: local theory, *Invent. Math.*, **173**-3 (2008): 449–475.

[10] A. Chandra, I. Chevyrev, M. Hairer, H. Shen, Langevin dynamic for the 2D YangMills measure, *Publ. Math.-IHES*, **136** (2022): 1147.

[11] A. Chandra, I. Chevyrev, M. Hairer, H. Shen, Stochastic quantisation of YangMillsHiggs in 3D, *Invent. Math.*, to appear (2024).

[12] I. Chevyrev, H. Shen, Invariant measure and universality of the 2D Yang-Mills Langevin dynamic, arXiv preprint, 2023.

[13] Y. Deng, Z. Hani, On the derivation of the wave kinetic equation for NLS, *Forum Math. Pi*, **9**-6 (2021).

[14] Y. Deng, Z. Hani, Full derivation of the wave kinetic equation, *Invent. Math.*, **233**-2 (2023): 543724.

[15] Y. Deng, Z. Hani, X. Ma, Long time derivation of Boltzmann equation from hard sphere dynamics, *arXiv preprint*, 2024.

[16] Y. Deng, A. Nahmod, H. Yue, Random tensors, propagation of randomness, and nonlinear dispersive equations, Invent. Math., **228**-2 (2022): 539686.

[17] Y. Deng, A. Nahmod, H. Yue, Invariant Gibbs measures and global strong solutions for nonlinear Schrdinger equations in dimension two, Ann. Math., **200**-2 (2024): 399-486.

[18] P. Duch, Flow equation approach to singular stochastic PDEs, arXiv preprint, 2021.

[19] B. Fehrman, B. Gess, Well-posedness of the DeanKawasaki and the nonlinear DawsonWatanabe equation with correlated noise, *Arch. Ration. Mech. Anal.*, **248**-20 (2024).

[20] B. Fehrman and B. Gess, Non-equilibrium large deviations and parabolic-hyperbolic PDE with irregular drift, *Invent. Math.*, **234** (2023): 573636.

[21] T. Funaki, Interface motion from Glauber-Kawasaki dynamics of non-gradient type, *arXiv preprint*, 2024.

[22] T. Funaki, C. Gu and H. Wang, Quantitative homogenization and hydrodynamic limit of non-gradient exclusion process, arXiv preprint, 2024.

[23] T. Funaki, K. Uchiyama and H.-T. Yau, Hydrodynamic limit for lattice gas reversible under Bernoulli measures. In *Nonlinear stochastic PDEs (Minneapolis, MN, 1994)*, IMA Vol. Math. Appl., **77**, 1-40, Springer, New York, 1996.

[24] A. Giunti, C. Gu and J.-C. Mourrat, Quantitative homogenization of interacting particle systems, *Ann. Probab.*, 50-5 (2022): 1885-1946.

[25] C. Gu, J.-C. Mourrat and M. Nitzschner, Quantitative equilibrium fluctuations for interacting particle systems, *arXiv preprint*, 2024.

[26] M. Gubinelli, P. Imkeller, N. Perkowski, Paracontrolled distributions and singular PDEs, *Forum. Math. Pi*, **3**-6 (2015).

[27] M. Hairer, A theory of regularity structures, *Invent. Math.*, **198**-2 (2014): 269-504.

[28] A. Kupiainen, Renormalization Group and Stochastic PDEs, *Ann. Henri Poincaré Probab. Stat.*, **17**-3 (2016): 497-535.

[29] M. Lassas, Y. Kurylev and G. Uhlmann, Inverse problems for Lorentzian manifolds and non-linear hyperbolic equations, *Invent. Math.*, **212**-3 (2018): 781857.

[30] Z. Liu, D. Wei, S. Xiang, J.-C. Zhao, Z. Zhang, Exponential mixing for random nonlinear wave equations: weak dissipation and localized control, *arXiv preprint*, 2024.

[31] M. Röckner, Stochastic analysis on configuration spaces: basic ideas and recent results. In New directions in Dirichlet forms, AMS/IP Stud. Adv. Math., **8**, 157-231, American Mathematical Society, Providence, RI, 1998.