Recent results on steady vortex rings of small cross section

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The incompressible Euler equation and its vorticity formulation

The 3D incompressible Euler equation:

$$\begin{cases} \partial_t \mathbf{v} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla P, \quad x\in\mathbb{R}^3,\ t>0, \\ \nabla\cdot\mathbf{v} = 0. \end{cases} \tag{1}$$

By introducing the vorticity function:

$$\boldsymbol{\omega} = \nabla\times\mathbf{v} = (\partial_2 v_3 - \partial_3 v_2, \partial_3 v_1 - \partial_1 v_3, \partial_1 v_2 - \partial_2 v_1),$$

and taking curl operator $\nabla\times$ at both side of (1), we can obtain

$$\partial_t\boldsymbol{\omega} = \nabla\times(\mathbf{v}\times\boldsymbol{\omega}) \quad \text{or} \quad \partial_t\boldsymbol{\omega} + (\mathbf{v}\cdot\nabla)\boldsymbol{\omega} = (\boldsymbol{\omega}\cdot\nabla)\mathbf{v}, \tag{2}$$

which is known as the vorticity formulation of (1) or Helmholtz equation.

The Biot-Savart law

In (2), the velocity $\mathbf{v}$ be recovered by $\boldsymbol{\omega}$ using Biot–Savart law:

$$\begin{aligned}\mathbf{v}(\boldsymbol{x}) &= \nabla \times (-\Delta)^{-1}\boldsymbol{\omega} \\ &= \frac{1}{4\pi}\int_{\mathbb{R}^3} \frac{\boldsymbol{x}-\boldsymbol{y}}{|\boldsymbol{x}-\boldsymbol{y}|^3} \times \boldsymbol{\omega}(\boldsymbol{y})d\boldsymbol{y}.\end{aligned} \tag{3}$$

Then by (1) we can obtain the pressure scalar P.

Note that in Helmholtz equation (2), $\boldsymbol{\omega}$ will not only be influenced by $(\mathbf{v}\cdot\nabla)\boldsymbol{\omega}$, but also stretchs or twists due to the nonlinear term$(\boldsymbol{\omega}\cdot\nabla)\mathbf{v}$, which leads to possible finite time blow up of L^∞ norm.

The case for axi-symmetric flow

Let $\{\mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_z\}$ be the standard 3D Cylindrical coordinate. Then velocity $\mathbf{v}$ can be written as:

$$\mathbf{v} = v^r(r,z,t)\mathbf{e}_r + v^\theta(r,z,t)\mathbf{e}_\theta + v^z(r,z,t)\mathbf{e}_z.$$

Since the flow is axi-symmetric, the velocity $\mathbf{v}$ is independent on θ. We further asuume $v^\theta \equiv 0$. Helmholtz equation (2) is then transformed to

$$\partial_t\left(\frac{\omega^\theta}{r}\right) + (\mathbf{v}\cdot\nabla)\left(\frac{\omega^\theta}{r}\right) = 0, \tag{4}$$

which is a transform equation on ω^θ/r.

Vortex rings in reality

Vortex rings

Vortex rings can be characterized as an axi-symmetric flow with a 'thin' or 'fat' toroidal vortex tube. By letting $\zeta := \omega^\theta/r$ be , ζ satisfies the transport equation

$$\partial_t\zeta + \mathbf{v}\cdot\nabla\zeta = 0, \tag{5}$$

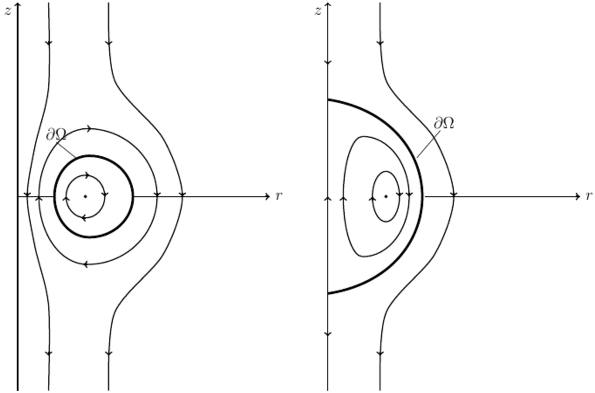
Especially, a steady vortex ring ζ has following form:

$$\zeta(\boldsymbol{x}, t) = \zeta(\boldsymbol{x} + t\mathbf{v}_\infty),$$

where $\mathbf{v}_\infty = -\mathbf{W}\mathbf{e}_z$ is a constant translational speed. Combining this with (5), we have a steady equation

$$(\mathbf{v}_\infty + \mathbf{v})\cdot\nabla\zeta = 0, \quad \mathbf{v} = \nabla\times(-\Delta)^{-1}(r\zeta\mathbf{e}_\theta).$$

2 kinds of vortex rings in axi-symmetric flows



(a) Streamline pattern for vortex ring of small cross-section.

(b) Streamline pattern for Hill's vortex.

The history for investigation of existence

1858: Helmholtz detected that vortex rings travel with a large constant velocity along the axis of the ring.

1860s: Lord Kelvin (Thomson) and Hicks obtained the well-known Kelvin–Hicks formula.

1894: Hill discovered the Hill's spherical vortex.

1972: Fraenkel constructed a series of vortex rings of small cross-section.

1972: Norbury constructed a class of vortex rings near Hill's spherical vortex.

1974: Fraenkel and Berger gave a general theory for construction of steady vortex rings.

1980–: Fredman et al., Ambrosetti et al., de Valeriola et al., Cao et al.

Notations for vortex rings

A scalar function $\vartheta : \mathbb{R}^3 \to \mathbb{R}$ is axi-symmetric if $\vartheta(\boldsymbol{x}) = \vartheta(r, z)$, and a subset $\Omega \subset \mathbb{R}^3$ is axi-symmetric if $\mathbf{1}_\Omega$ is axi-symmetric. The cross-section parameter σ of an axi-symmetric set $\Omega \subset \mathbb{R}^3$ can be defined as

$$\sigma(\Omega) := \frac{1}{2} \cdot \sup \{\boldsymbol{\delta}_z(\boldsymbol{x}, \boldsymbol{y}) \mid \boldsymbol{x}, \boldsymbol{y} \in \Omega\}$$

with the axi-symmetric distance

$$\boldsymbol{\delta}_z(\boldsymbol{x}, \boldsymbol{y}) := \inf \{|\boldsymbol{x} - Q(\boldsymbol{y})| \ \mid \ Q \text{ is a rotation around } \mathbf{e}_z\}.$$

Let $\mathcal{C}_r = \{\boldsymbol{x} \in \mathbb{R}^3 \,|x_1^2 + x_2^2 = r^2, x_3 = 0\}$ be a circle of radius r on the plane perpendicular to $\mathbf{e}_z$. For an axi-symmetric set $\Omega \subset \mathbb{R}^3$, we define the axi-symmetric distance between Ω and $\mathcal{C}_r$ as

$$\mathrm{dist}_{\mathcal{C}_r}(\Omega) = \sup_{\boldsymbol{x} \in \Omega} \inf_{\boldsymbol{x}' \in \mathcal{C}_r} |\boldsymbol{x} - \boldsymbol{x}'|.$$

The Kelvin–Hicks formula

The circulation of a steady vortex ring ζ is given by

$$\frac{1}{2\pi}\int_{\mathbb{R}^3}\zeta(\boldsymbol{x})d\boldsymbol{x}.$$

A steady vortex ring ζ is said to be *centralized* if ζ is symmetric non-increasing in z, namely,

$$\zeta(r,z)=\zeta(r,-z), \quad \text{and}$$
$$\zeta(r,z) \text{ is a non-increasing function of } z \text{ for } z>0, \quad \text{for each fixed } r>0.$$

Kelvin–Hicks formula: a vortex ring with uniform density throughout the core would approximately move at the velocity

$$\frac{\kappa}{4\pi r^*}\left(\ln\frac{8r^*}{\sigma}-\frac{1}{4}\right)$$

Our existence result

Let κ and W be two positive numbers. Then there exists a small number $\varepsilon_0 > 0$ such that, for every $\varepsilon \in (0, \varepsilon_0]$ there is a centralized steady vortex ring ζ_ε with fixed circulation κ and translational velocity $W \ln \varepsilon \, \mathbf{e}_z$. Moreover,

(i) $\zeta_\varepsilon = \varepsilon^{-2} \mathbf{1}_{\Omega_\varepsilon}$ for some axi-symmetric topological torus $\Omega_\varepsilon \subset \mathbb{R}^3$.

(ii) It holds $C_1 \varepsilon \le \sigma(\Omega_\varepsilon) < C_2 \varepsilon$ for some constants $0 < C_1 < C_2$.

(iii) As $\varepsilon \to 0$, $\mathrm{dist}_{\mathcal{C}_{r^*}}(\Omega_\varepsilon) \to 0$, where $r^* = \kappa / 4\pi W$.

A related semi-linear elliptic equation

Using $\nabla \cdot \mathbf{v} = 0$, the velocity field can be written as

$$\mathbf{v} = \frac{1}{r}\left(-\frac{\partial \psi}{\partial z}\mathbf{e}_r + \frac{\partial \psi}{\partial r}\mathbf{e}_z\right)$$

wth ψ the Stokes stream function. Then (5) is transformed to an equation on the meridional half plane $\Pi = \{(r,z) \mid r > 0\}$

$$\begin{cases} \mathcal{L}\psi_\varepsilon = \frac{1}{\varepsilon^2}\mathbf{1}_{A_\varepsilon}, \quad \text{in } \Pi, \\ \psi_\varepsilon(0,z) = 0, \\ \psi_\varepsilon, \quad |\nabla \psi_\varepsilon|/r \to 0 \ \text{ as } \ r^2 + z^2 \to \infty, \end{cases} \tag{6}$$

with

$$\mathcal{L} := -\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}\right) - \frac{1}{r^2}\frac{\partial^2}{\partial z^2}$$

an elliptic operator, and

$$A_\varepsilon := \left\{ \boldsymbol{x} = (r,z) \in \Pi \mid \psi_\varepsilon - \frac{W}{2}|\ln \varepsilon| r^2 - \mu_\varepsilon > 0 \right\}$$

the cross-section of vortex ring.

The Green's function and its asymptotic property

The Green's function $G_*(\boldsymbol{x}, \boldsymbol{x}')$ for operator $\mathcal{L}$ (with boundary condition in (6)) has following asymptotic behavior:

$$G_*(\boldsymbol{x}, \boldsymbol{x}') = \frac{r^{1/2} r'^{3/2}}{4\pi} \left(\ln\left(\frac{1}{\rho}\right) + 2\ln 8 - 4 + O\left(\rho \ln \frac{1}{\rho}\right)\right), \quad \text{as } \rho \to 0,$$

with

$$\rho = \frac{(r-r')^2 + (z-z')^2}{rr'}.$$

Thus we can split G_* as

$$G_*(\boldsymbol{x}, \boldsymbol{x}') = q_1^2 G(\boldsymbol{x}, \boldsymbol{x}') + H(\boldsymbol{x}, \boldsymbol{x}'),$$

where

$$G(\boldsymbol{x}, \boldsymbol{x}') = \frac{1}{4\pi} \ln \frac{(r+r')^2 + (z-z')^2}{(r-r')^2 + (z-z')^2}.$$

is the Green's function for $-\Delta$ on the meridional half plane Π.

Approximate solutions

By introducing the scaled stream function of Rankine vortex:

$$V_{\boldsymbol{q},\varepsilon}(\boldsymbol{x}) = \begin{cases} \frac{a}{2\pi}\ln\frac{1}{\varepsilon} + \frac{z_1^2}{4\varepsilon^2}(s^2 - |\boldsymbol{x}-\boldsymbol{q}|^2), & |\boldsymbol{x}-\boldsymbol{q}| \le s, \\ \frac{a}{2\pi}\ln\frac{1}{\varepsilon}\cdot\frac{\ln|\boldsymbol{x}-\boldsymbol{q}|}{\ln s}, & |\boldsymbol{x}-\boldsymbol{q}| \ge s, \end{cases} \tag{7}$$

and regular part $\mathcal{H}_{\boldsymbol{q},\varepsilon}(\boldsymbol{x})$, we can write the Stokes stream function ψ_ε for vortex ring of small cross-section as

$$\psi_\varepsilon(\boldsymbol{x}) = V_{\boldsymbol{q},\varepsilon}(\boldsymbol{x}) - V_{\bar{\boldsymbol{q}},\varepsilon}(\boldsymbol{x}) + \mathcal{H}_{\boldsymbol{q},\varepsilon}(\boldsymbol{x}) + \phi_\varepsilon(\boldsymbol{x}),$$

where ϕ_ε is the **error term**, and the barycenter $\boldsymbol{q} = (q_1, 0)$ of $V_{\boldsymbol{q},\varepsilon}(\boldsymbol{x})$ determined later.

The equation for error term

By chosing the parameter a in (7) appropriately, equation (6) is then transformed to an semilinear elliptic equation on ϕ_ε:

$$\mathbb{L}_\varepsilon \phi_\varepsilon = R_\varepsilon(\phi_\varepsilon), \tag{8}$$

where

$$\mathbb{L}_\varepsilon \phi_\varepsilon = -r\mathcal{L}\phi_\varepsilon - \frac{2}{sq_1}\phi_\varepsilon(s,\theta)\boldsymbol{\delta}_{|\boldsymbol{x}-\boldsymbol{q}|=s}.$$

is the linear operator,

$$R_\varepsilon(\phi_\varepsilon) = \frac{1}{\varepsilon^2}\left(r\mathbf{1}_{A_\varepsilon} - r\mathbf{1}_{\{V_{\boldsymbol{q},\varepsilon} > \frac{a}{2\pi}\ln\frac{1}{\varepsilon}\}} - \frac{2}{sq_1}\phi_\varepsilon(s,\theta)\boldsymbol{\delta}_{|\boldsymbol{x}-\boldsymbol{q}|=s}\right)$$

is the nonlinear term.

Lyapunov-Schmidt reduction

Notice that $\mathbb{L}_\varepsilon$ is not invertible, and is approximated by linear operator

$$\mathbb{L}_\varepsilon^* = -\frac{1}{q_1}\Delta\phi_\varepsilon - \frac{2}{sq_1}\phi_\varepsilon(s,\theta)\boldsymbol{\delta}_{|\boldsymbol{x}-\boldsymbol{q}|=s},$$

whose kernel is asymptotically spanned by $Z_{\boldsymbol{q},\varepsilon} = \chi_\varepsilon \cdot \frac{\partial V_{\boldsymbol{q},\varepsilon}}{\partial x_1}$.
(χ_ε is a smooth truncation to make $Z_{\boldsymbol{q},\varepsilon}$ satisfy the boundary condition).
We can first consider the **projective problem**

$$\begin{cases} \mathbb{L}_\varepsilon\phi = \mathbf{h}(\boldsymbol{x}) - \Lambda r\mathcal{L}Z_{\boldsymbol{q},\varepsilon}, & \text{in } \Pi, \\ \int_\Pi \frac{\nabla\phi}{r}\cdot\nabla Z_{\boldsymbol{q},\varepsilon}d\boldsymbol{x} = 0, & \\ \phi(\boldsymbol{x}) = 0, & \text{on } r=0, \\ \phi,\ |\nabla\phi|/r \to 0, & \text{as } |\boldsymbol{x}|\to\infty, \end{cases} \tag{9}$$

Letting $\mathbf{h}(\boldsymbol{x}) = R_\varepsilon(\phi_\varepsilon)$, (8) is then transformed to a fixed point problem

$$\phi_\varepsilon = \mathcal{T}_\varepsilon R_\varepsilon(\phi_\varepsilon).$$

The finite dimensional problem

The last step of proving existence is to fined $\boldsymbol{q}$ such that the coefficient Λ in (9) equals 0, namely,

$$\varepsilon^2 \int_{\Pi} \frac{1}{r} \nabla \psi_\varepsilon \cdot \nabla Z_{\boldsymbol{q},\varepsilon} d\boldsymbol{x} - \int_{A_\varepsilon} r \cdot Z_{\boldsymbol{q},\varepsilon} d\boldsymbol{x} = 0,$$

By further analysis, the above reduction condition is

$$q_1 - \frac{\kappa}{4\pi W} = O\left(\frac{1}{|\ln \varepsilon|}\right).$$

Hence we find the desired $\boldsymbol{q}$, and prove the existence for a series of vortex rings of small cross-section ζ_ε.

The history for investigation of uniqueness

Compared with existence result, there are fewer works on uniqueness.

Amick, Fraenkel (Arch. Ration. Mech. Anal: 92(2) 91–119,1986) obtained the uniqueness of Hill's spherical vortex by the method of moving plane.

Amick, Fraenkel (Arch. Ration. Mech. Anal: 100(3) 207–241,1988) obtained the the local uniqueness of Norbury's vortex rings by asymptotic estimates.

The limitations: their techniques depend strongly on specific distribution of vorticity in cross-section.

Our uniqueness result

Let κ and W be two positive numbers. Let $\{\zeta_\varepsilon^{(1)}\}_{\varepsilon>0}$ and $\{\zeta_\varepsilon^{(2)}\}_{\varepsilon>0}$ be two families of centralized steady vortex rings with the same circulation κ and translational velocity $W \ln \varepsilon \, \mathbf{e}_z$. If, in addition,

(i) $\zeta_\varepsilon^{(1)} = \varepsilon^{-2}\mathbf{1}_{\Omega_\varepsilon^{(1)}}$ and $\zeta_\varepsilon^{(2)} = \varepsilon^{-2}\mathbf{1}_{\Omega_\varepsilon^{(2)}}$ for certain axi-symmetric topological tori $\Omega_\varepsilon^{(1)}$, $\Omega_\varepsilon^{(2)} \subset \mathbb{R}^3$.

(ii) As $\varepsilon \to 0$, $\sigma\left(\Omega_\varepsilon^{(1)}\right) + \sigma\left(\Omega_\varepsilon^{(2)}\right) \to 0$.

(iii) There exists a $\delta_0 > 0$ such that
$$\Omega_\varepsilon^{(1)} \cup \Omega_\varepsilon^{(2)} \subset \left\{ \boldsymbol{x} \in \mathbb{R}^3 \mid \sqrt{x_1^2 + x_2^2} \geq \delta_0 \right\} \text{ for all } \varepsilon > 0.$$

Then there exists a small $\varepsilon_0 > 0$ such that $\zeta_\varepsilon^{(1)} \equiv \zeta_\varepsilon^{(2)}$ for all $\varepsilon \in (0, \varepsilon_0]$.

Two facts leading to uniqueness

a. The non-degeneracy of Hessian for regular part in stream function at vortex location q_ε:
Let $\bar{\psi}_\varepsilon(r,z) = -V_{\bar{\boldsymbol{q}},\varepsilon} + \mathcal{H}_{\boldsymbol{q},\varepsilon} - \frac{W}{2}|\ln\varepsilon|r^2$, then it holds $\partial_r^2\bar{\psi}_\varepsilon(r,0) \neq 0$.

(Fot the planer k-point vortex case, the regular part in stream function is the Kirchhoff-Routh function

$$\mathcal{K}_k(\boldsymbol{x}_1,\boldsymbol{x}_2,..,\boldsymbol{x}_k) := \frac{1}{2}\sum_{i\neq j}^{k}\kappa_i\kappa_j G(\boldsymbol{x}_i,\boldsymbol{x}_j) + \frac{1}{2}\sum_{i=1}^{k}\kappa_i^2 h(\boldsymbol{x}_i,\boldsymbol{x}_i)$$

and in particular, $(\boldsymbol{x}_1,\boldsymbol{x}_2,..,\boldsymbol{x}_k)$ must be a critical point of $\mathcal{K}_k$ when the system constitutes an equilibrium. If it holds the non-degeneracy condition $\det\nabla^2_{\boldsymbol{x}_i}\mathcal{K}_k(\boldsymbol{x}_1,\boldsymbol{x}_2,..,\boldsymbol{x}_k)\neq 0$, then a uniqueness property can be showed for corresponding regularization series.
For example: Cao et al.: JMPA 2019 for localized vortex patches, JFA 2022 for localized smooth vortices)

Two facts leading to uniqueness

b. The non-degeneracy of limiting linear problem:
Let

$$w(\boldsymbol{y}) := \begin{cases} \frac{1}{4}(1 - |\boldsymbol{y}|^2), & \text{if } |y| \le 1, \\ \frac{1}{2}|\ln \frac{1}{|\boldsymbol{y}|}, & \text{if } |y| > 1. \end{cases}$$

Suppose that $v \in L^\infty(\mathbb{R}^2) \cap C(\mathbb{R}^2)$ solves

$$-\nabla v - 2v(1, \theta)\boldsymbol{\delta}_{|\boldsymbol{y}|=1} = 0$$

in $\mathbb{R}^2$. Then

$$v \in \mathrm{span}\left\{\frac{\partial w}{\partial y_1}, \frac{\partial w}{\partial y_2}\right\}.$$

(Cao et al. adv. Math. 2015: Planar vortex patch problem in incompressible steady flow)

The proof of uniqueness

We first derive a precise estimate for vortex ring parameters. Then we use the Pohozaev identity to derive a contradiction for multi-solution situation.

For detail proof, see
Cao et al. (ArXiv: 2201.08232): Existence, uniqueness and stability of steady vortex rings of small cross-section

Cao et al. (preprint): Uniqueness and stability of traveling vortex patch-pairs for the incompressible Euler equation

The kinetic energy and impulse

We denote $BC([0,\infty);X)$ as the space of all bounded continuous functions from $[0,\infty)$ into a Banach space X, and define the weighted space $L^1_{\mathrm{w}}(\mathbb{R}^3)$ by $L^1_{\mathrm{w}}(\mathbb{R}^3)=\{\vartheta:\mathbb{R}^3\to\mathbb{R}\text{ measurable}\mid r^2\vartheta\in L^1(\mathbb{R}^3)\}$. We also introduce the kinetic energy

$$E[\zeta]:=\frac{1}{2}\int_{\mathbb{R}^3}|\mathbf{v}(\boldsymbol{x})|^2d\boldsymbol{x},\quad \mathbf{v}=\nabla\times(-\Delta)^{-1}(r\zeta),$$

and its impulse

$$\mathcal{P}[\zeta]=\frac{1}{2}\int_{\mathbb{R}^3}r^2\zeta(\boldsymbol{x})d\boldsymbol{x}=\pi\int_{\Pi}r^3\zeta drdz.$$

Well-posedness for the Cauchy problem

For any non-negative axi-symmetric function $\zeta_0 \in L^1 \cap L^\infty \cap L^1_{\mathrm{w}}(\mathbb{R}^3)$ satisfying $r\zeta_0 \in L^\infty(\mathbb{R}^3)$, there exists a unique weak solution $\zeta \in BC([0,\infty); L^1 \cap L^\infty \cap L^1_{\mathrm{w}}(\mathbb{R}^3))$ of (5) for the initial data ζ_0 such that $\zeta(\cdot,t) \geq 0$ and is axi-symmetric,

$$\begin{aligned}&\|\zeta(\cdot,t)\|_{L^p(\mathbb{R}^3)} = \|\zeta_0\|_{L^p(\mathbb{R}^3)}, \quad 1 \leq p \leq \infty,\\ &\mathcal{P}[\zeta(\cdot,t)] = \mathcal{P}[\zeta_0], \quad E[\zeta(\cdot,t)] = E[\zeta_0], \quad \text{for all } t > 0,\end{aligned}$$

and, for any $0 < \upsilon_1 < \upsilon_2 < \infty$ and for all $t > 0$,

$$\int_{\{\boldsymbol{x}\in\mathbb{R}^3 \ | \ \upsilon_1<\zeta(\boldsymbol{x},t)<\upsilon_2\}} \zeta(\boldsymbol{x},t)d\boldsymbol{x} = \int_{\{\boldsymbol{x}\in\mathbb{R}^3 \ | \ \upsilon_1<\zeta_0(\boldsymbol{x})<\upsilon_2\}} \zeta_0(\boldsymbol{x})d\boldsymbol{x}.$$

The variational description for vortex rings

By introducing the admissible class

$$\mathcal{A}_\varepsilon := \left\{\zeta \in L^\infty(\mathbb{R}^3) \mid \zeta : \text{axi-symmetric},\ 0 \le \zeta \le 1/\varepsilon^2,\ \|\zeta\|_{L^1(\mathbb{R}^3)} \le 2\pi\kappa\right\},$$

we consider the maximization problem:

$$\mathcal{E}_\varepsilon = \sup_{\zeta\in\mathcal{A}_\varepsilon} \left(E[\zeta] - W \ln\frac{1}{\varepsilon}\mathcal{P}[\zeta]\right).$$

Using the L^∞ restriction, maximizer $\hat{\zeta}_\varepsilon$ is attainable. For arbitrary $\zeta \in \mathcal{A}_\varepsilon$, we let

$$\zeta_\tau = \hat{\zeta}_\varepsilon + \tau(\zeta - \hat{\zeta}_\varepsilon), \quad \tau \in [0,1].$$

Since $\hat{\zeta}_\varepsilon$ is the maximizer, it must hold

$$\frac{d}{d\tau}\left(E[\zeta_\tau] - W \ln\frac{1}{\varepsilon}\mathcal{P}[\zeta_\tau]\right)\Bigg|_{\tau=0^+} \le 0,$$

The variational description for vortex rings

which is equivalent to

$$\int_{\Pi} \hat{\zeta}_\varepsilon \left(\hat{\psi}_\varepsilon - \frac{W}{2} r^2 \ln \frac{1}{\varepsilon} \right) rdrdz \geq \int_{\Pi} \zeta \left(\hat{\psi}_\varepsilon - \frac{W}{2} r^2 \ln \frac{1}{\varepsilon} \right) rdrdz.$$

Here $\hat{\psi}_\varepsilon$is the Stokes stream function corresponding to $\hat{\zeta}_\varepsilon$. According to bathtub lemma, it holds

$$\begin{cases} \hat{\psi}_\varepsilon - \frac{W}{2} r^2 \ln \frac{1}{\varepsilon} > \mu_\varepsilon, & \text{if } \hat{\zeta}_\varepsilon = 1/\varepsilon^2, \\ \hat{\psi}_\varepsilon - \frac{W}{2} r^2 \ln \frac{1}{\varepsilon} = \mu_\varepsilon, & \text{if } 0 < \hat{\zeta}_\varepsilon < 1/\varepsilon^2, \\ \hat{\psi}_\varepsilon - \frac{W}{2} r^2 \ln \frac{1}{\varepsilon} < \mu_\varepsilon, & \text{if } \hat{\zeta}_\varepsilon = 0. \end{cases}$$

By Steiner symmetrization, the middile part does not exist. Thus $\hat{\zeta}_\varepsilon$ gives a vortex ring. (The limiting behavior for $\hat{\zeta}_\varepsilon$ as $\varepsilon \to 0^+$ can be derived by Riesz rearrangement inequality).

Our nonlinear stability result

The steady vortex ring of small cross-section ζ_ε is stable up to translations in the following sense:
For any $\eta > 0$, there exists $\delta_1 > 0$ such that for any non-negative axi-symmetric function ζ_0 satisfying $\zeta_0, r\zeta_0 \in L^\infty(\mathbb{R}^3)$ and

$$\|\zeta_0 - \zeta_\varepsilon\|_{L^1\cap L^2(\mathbb{R}^3)} + \|r^2(\zeta_0 - \zeta_\varepsilon)\|_{L^1(\mathbb{R}^3)} \leq \delta_1,$$

the corresponding solution $\zeta(\boldsymbol{x}, t)$ of (5) for the initial data ζ_0 satisfies

$$\inf_{\tau\in\mathbb{R}} \left\{\|\zeta(\cdot - \tau\mathbf{e}_z, t) - \zeta_\varepsilon\|_{L^1\cap L^2(\mathbb{R}^3)} + \|r^2(\zeta(\cdot - \tau\mathbf{e}_z, t) - \zeta_\varepsilon\|_{L^1(\mathbb{R}^3)}\right\} \leq \eta,$$

for all $t > 0$. Here, $\|\cdot\|_{L^1\cap L^2(\mathbb{R}^3)}$ means $\|\cdot\|_{L^1(\mathbb{R}^3)} + \|\cdot\|_{L^2(\mathbb{R}^3)}$.

The proof for the nonlinear stability

Before our work, the only rigorous stability result for vortex rings is the orbital stability for Hill's vortex by Choi (CPAM 2022).

In view of the uniqueness result, we know $\hat{\zeta}_\varepsilon$ is unique under a translation in z direction. Using the conservation law and compactness of maximization sequences (modified version of Concentrate compactness theorem by Choi), we can show that $\hat{\zeta}_\varepsilon$ is nonlinear stable in $L^1 \cap L^\infty \cap L^1_{\mathrm{w}}$ norm.

For the general idea, see also:
G.R. Burton (J. Differential Equations: 270 547–572,2021): Compactness and stability for planar vortex-pairs with prescribed impulse

G. Wang (Mathematische Annalen 2023): Stability of two-dimensional steady Euler flows with concentrated vorticity.

Thank you for your attention!