Brill–Noether reconstruction for Fano 3folds and applications

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- The positive integer i such that $-K_X = iH$ is called the **index** of X.

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- index $i = 2$: $1 \leq d \leq 5$. Example: quartic double solids ($d = 2$) and cubic 3folds ($d = 3$).
- index $i = 1$: $2 \leq d \leq 22$ is even and $d \neq 20$. Example: Gushel–Mukai 3folds ($d = 10$) and codimension five linear sections of $\mathrm{Gr}(2, 6)$ ($d = 14$).

Kuznetsov components

- Let Y be an index 2 prime Fano 3fold, then we have a semi-orthogonal decomposition:

$$\mathrm{D}^b(Y) = \langle \mathcal{K}u(Y), \mathcal{O}_Y, \mathcal{O}_Y(H) \rangle.$$

In other words, $E \in \mathcal{K}u(Y)$ if and only if $\mathrm{RHom}(\mathcal{O}_Y(kH), E) = 0$ for $0 \leq k \leq 1$.

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- Let X be an index 1 prime Fano 3fold of degree $d \geq 10$, then we have a semi-orthogonal decomposition:

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  where $\mathcal{E}_X$ is a stable vector bundle on X. In other words, $E \in \mathcal{K}u(X)$ if and only if $\mathrm{RHom}(\mathcal{O}_X, E) = \mathrm{RHom}(\mathcal{E}, E) = 0$.

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The category $\mathcal{K}u$ is called the **Kuznetsov component**. It is a **non-commutative smooth projective variety** in the sense of Orlov.

Kuznetsov components

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Theorem (Bayer–Lahoz–Macri–Stellari)

Let X be a prime Fano 3fold of index 1 or 2. Then $\mathcal{K}u(X)$ admits a family of Bridgeland stability conditions.

Fano 3folds as Brill–Noether loci in moduli of bundles

Theorem (Mukai)

Let X be an index 1 prime Fano 3fold of degree $d = 12$ or 16. Then X is isomorphic to a (generalized) Brill–Noether locus of a moduli space of rank two stable bundles over C_X.

For $d = 12$, the Brill–Noether condition is given by $\mathcal{O}_{C_X}$ ($h^0(-) \geq ?$); for $d = 16$, the Brill–Noether condition is given by a rank two bundle $\mathcal{F}_X$ ($h^0(- \otimes \mathcal{F}_X) \geq ?$).

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Question

How about other prime Fano 3folds?

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Key observation: For an index 1 prime Fano 3fold of degree $d = 12$ or 16, the equivalence $\mathrm{D}^b(C_X) \simeq \mathcal{K}u(X)$ maps $\mathcal{O}_{C_X}$ (or $\mathcal{F}_X$) to $\mathcal{G}_X$.

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moduli of stable bundles over $C_X \Rightarrow$ moduli of stable objects in $\mathcal{K}u(X)$

generalised Brill–Noether condition $\Rightarrow$ Brill–Noether condition given by $\mathcal{G}_X$

Fano 3folds as Brill–Noether loci in moduli of objects

Theorem (Jacovski–Liu–Zhang)

Let X be an index 1 prime Fano 3fold of degree $d \geq 10$. Then

$$X \cong \{F : \hom(F, \mathcal{G}_X[k]) \geq n_d \text{ for some } k \in \mathbb{Z}\} \subset M_\sigma(\mathcal{K}u(X), [i^*\mathbb{C}_x]).$$

Here n_d is an integer only depends on d, σ is any Serre-invariant stability condition on $\mathcal{K}u(X)$, and $M_\sigma(\mathcal{K}u(X), [i^*\mathbb{C}_x])$ is the moduli space of σ-stable objects with the class $[i^*\mathbb{C}_x]$.

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Theorem (Feyzbakhsh–Liu–Zhang)

Let Y be an index 2 prime Fano 3fold of degree $d \geq 2$. Then

$$Y \cong \{F : \hom(F, \mathcal{G}_Y[k]) \geq d+1 \text{ for some } k \in \mathbb{Z}\} \subset M_\sigma(\mathcal{K}u(Y), [i^*\mathbb{C}_y]).$$

Here $\mathbb{C}_y$ is the skyscraper sheaf supported on $y \in Y$, σ is any Serre-invariant stability condition on $\mathcal{K}u(Y)$, and $M_\sigma(\mathcal{K}u(Y), [i^*\mathbb{C}_y])$ is the moduli space of σ-stable objects with the class $[i^*\mathbb{C}_y]$.

Application 1: Categorical Torelli theorems

Corollary (Refined categorical Torelli theorem)

Let X and X' be index 1 prime Fano 3folds of degree $d \geq 10$. If there is an equivalence $\mathcal{K}u(X) \simeq \mathcal{K}u(X')$ that maps $\mathcal{G}_X$ to $\mathcal{G}_{X'}$, then $X \cong X'$.

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Theorem (Feyzbakhsh–Liu–Zhang)

Let Y and Y' be index 2 prime Fano 3folds of degree $2 \leq d \leq 4$. Then up to some explicit auto-equivalences, any equivalence $\mathcal{K}u(Y) \simeq \mathcal{K}u(Y')$ maps $\mathcal{G}_Y$ to $\mathcal{G}_{Y'}$.

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When $d = 3$, this is proved by Bernardara-Macri-Mehrotra-Stellari, Pertusi-Yang and Bayer-Beentjes-Feyzbakhsh-Hein-Martinelli-Rezaee-Schmidt. When $d = 2$, Altavilla-Petković-Rota proved this corollary under certain generic assumption.

Application 2: Auto-equivalences of Kuznetsov components

Corollary

Let Y be an index 2 prime Fano 3fold of degree $2 \leq d \leq 3$. Then

$$\mathrm{Aut}_{\mathrm{FM}}(\mathcal{K}u(Y)) = \langle \mathrm{Aut}(Y), \mathbf{O}, [1] \rangle.$$

Here $\mathbf{O}$ is the **rotation functor**. The $d = 3$ case is also proved by Ziqi Liu independently, using a different method.

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Corollary

Let X be an index 1 prime Fano 3fold of degree $d = 10$ or 14. When $d = 10$, we furthermore assume that X is general. Then $\mathrm{Aut}_{\mathrm{FM}}(\mathcal{K}u(X)) = \langle \mathrm{Aut}(X), S_{\mathcal{K}u(X)}, [1] \rangle$.

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Corollary

Let X be an index 1 prime Fano 3fold of degree $d = 14$, and Y be the Phaffian cubic threefold associated with X. Then we have $\mathrm{Aut}(X) \cong \mathrm{Aut}(Y)$.

Application 3: Kuznetsov's Fano 3fold conjecture

We denote the moduli stack of index i degree d prime Fano 3folds by M^i_d.

Conjecture (Kuznetsov)

For any $1 \leq d \leq 5$, there is a correspondence $Z \subset \mathrm{M}^1_{4d+2} \times \mathrm{M}^2_d$ dominants each factor, such that for any point $(X_{4d+2}, Y_d) \in Z$, there is an equivalence $\mathcal{K}u(X_{4d+2}) \simeq \mathcal{K}u(Y_d)$.

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The case $3 \leq d \leq 5$ is proved by Kuznetsov. For $d = 1$, the conjecture does not hold for $\mathrm{HH}(-)$ reason. The remaining case $d = 2$ is disproved by Bayer–Perry and Zhang using different methods.

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Corollary

Let X be a general Gushel–Mukai 3fold and Y a quartic double solid. Then $\mathcal{K}u(X)$ is not equivalent to $\mathcal{K}u(Y)$.

Thanks!