Sharp non-uniqueness for 3D hyperdissipative Navier-Stokes equations: above the Lions exponent

Deng Zhang

Shanghai Jiao Tong University

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I. Motivations

- Let us consider the 3D Navier-Stokes equations (NSE) on the torus $\mathbb{T}^3 := [-\pi, \pi]^3$

$$\partial_t u + \nu(-\Delta)u + (u \cdot \nabla)u + \nabla P = 0,$$

under the incompressible condition

$$\mathrm{div} u = 0.$$

where $u = (u_1, u_2, u_3)^\top(t, x) \in \mathbb{R}^3$ and $P = P(t, x) \in \mathbb{R}$ represent the velocity field and pressure of the fluid, respectively, $\nu > 0$ is the viscosity coefficient.

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- Consider also the 3D hyper-dissipative Navier-Stokes equations (hyper-NSE) on $\mathbb{T}^3$

$$\partial_t u + \nu(-\Delta)^\alpha u + (u \cdot \nabla)u + \nabla P = 0,$$

where $\alpha \in [1, 2)$, the hyper-viscosity $(-\Delta)^\alpha$ is defined by

$$\mathcal{F}((-\Delta)^\alpha u)(\xi) = |\xi|^{2\alpha} \mathcal{F}(u)(\xi), \quad \xi \in \mathbb{Z}^3.$$

I. Motivations

In 1934, Leray [Leray Acta Math.'34] constructed global weak solutions (Leray-Hopf solutions) to NSE, satisfying

- regularity

$$u \in C_{weak}\big([0,+\infty); L^2(\Omega)\big) \cap L^2\big([0,+\infty); \dot{H}^1(\Omega)\big),$$

  where $\Omega = \mathbb{R}^3$ or $\mathbb{T}^3$;

- energy inequality

$$\|u(t)\|_{L^2}^2 + 2\nu \int_{t_0}^{t} \|\nabla u(s)\|_{L^2}^2 \mathrm{d}s \leq \|u(t_0)\|_{L^2}^2$$

  for any $t > 0$ and a.e. $t_0 \geq 0$;

- Partial regularity: $\exists$ a closed set $S \subseteq \mathbb{R}^+$ of measure zero, s.t.
  - ◇ Leray solution is smooth on $\mathbb{R}^3 \times (\mathbb{R}^+ \setminus S)$;
  - ◇ The 1/2 Hausdorff measure $\mathcal{H}^{1/2}(S) = 0$.

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- Ladyžhenskaya-Prodi-Serrin condition: $2/\gamma + 3/p \leq 1$, $p > 3$;

  Weak solutions to NSE in the (sub)critical spaces $L^\gamma_t L^p_x$ are automatically unique regular Leray-Hopf solutions [Fabes-Jones-Riviere, 1972].

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  Weak solutions to NSE in the (sub)critical spaces $L^\gamma_t L^p_x$ are automatically unique regular Leray-Hopf solutions [Fabes-Jones-Riviere, 1972].
- NSE in critical spaces:

$$L^3 \hookrightarrow \dot{B}^{\frac{3}{p}-1}_{p|2\leq p<\infty,\infty} \hookrightarrow BMO^{-1}(= \dot{F}^{-1}_{\infty,2})$$

  [Fujita-Kato ARMA'64], [Escauriaza-Seregin-Šverák Uspekhi Mat.Nauk'03], [Koch-Tataru Adv.Math'01].

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- Partial regularity:
  - ◇ $\exists$ Leray-Hopf solutions to NSE with singular space-time sets of zero Hausdorff $\mathcal{H}^1$ measure [Caffarelli-Kohn-Nirenberg CPAM'82];
  - ◇ Extended to hyper-NSE with zero Hausdorff $\mathcal{H}^{5-4\alpha}$ measure [Colombo-De Lelli-Massaccesi CPAM'20].

Remark

- Hyper-NSE is invariant under the scaling

$$u(t,x) \mapsto \lambda^{2\alpha-1} u(\lambda^{2\alpha} t, \lambda x), \quad P(t,x) \mapsto \lambda^{4\alpha-2} u(\lambda^{2\alpha} t, \lambda x).$$

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- $C_t L^2$ is the critical space when $\alpha = \frac{5}{4}$ (Lions exponent).

- Critical space $L^\gamma_t \dot{W}^{s,p}_x$ with

$$\frac{2\alpha}{\gamma} + \frac{3}{p} = 2\alpha - 1 + s.$$

In particular, when $s = 0$, it is the LPS condition.

Negative results (Ill-posedness)

- 3D NSE: $\exists$ non-unique weak solutions in $C_tL^2_x$ [Buckmaster-Vicol Ann.Math'19], via intermittent convex integration.
    - ◇ Convex integration was introduced to 3D Euler equations [De Lellis-Székelyhidi, Ann.Math.'09].

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- 3D NSE at one LPS endpoint: non-uniqueness in $L_t^\gamma L_x^\infty$ for any $\gamma \in [1, 2)$, which is sharp as $L_t^2 L_x^\infty$ is the LPS endpoint space [Cheskidov-Luo Invent.Math.'22].

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- 3D forced NSE has non-unique Leray-Hopf solutions [Albritton-Brué-Colombo Ann. of Math.'22].

Questions

Enlightened by these progresses, we consider three non-uniqueness questions:

1. In view of well-posedness in the high dissipative case $\alpha \geq 5/4$ [Lions 1969],

   is it possible to find non-unique and non-Leray-Hopf weak solutions, even in the high dissipative regime beyond the Lions exponent ?

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2. In view of the generalized LPS, as in [Cheskidov-Luo, Invent.Math.'22] for NSE,

   do there exist non-unique weak solutions to hyper-NSE in the supercritical spaces $L^\gamma_t W^{s,p}_x$, where $2\alpha/\gamma + 3/p > 2\alpha - 1 + s$?

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2. In view of the generalized LPS, as in [Cheskidov-Luo, Invent.Math.'22] for NSE,

   do there exist non-unique weak solutions to hyper-NSE in the supercritical spaces $L_t^\gamma W_x^{s,p}$, where $2\alpha/\gamma + 3/p > 2\alpha - 1 + s$?

3. In view of extensive well-posedness results in critical spaces $C_t\mathbb{X}$ [Fujita-Kato ARMA'64], [Koch-Tataru Adv.Math'01],

   do there exist non-unique weak solutions to hyper-NSE in the supercritical spaces $C_t\mathbb{X}$, where $\mathbb{X}$ might be Lebesgue, Besov, Triebel-Lizorkin spaces ?

II. Main results

Definition (weak solutions)

Given any weakly divergence-free datum $u_0 \in L^2(\mathbb{T}^3)$, we say that $u \in L^2([0,T] \times \mathbb{T}^3)$ is a weak solution for the hyperdissipative Navier-Stokes equations if u is divergence-free for a.e. $t \in [0,T]$, and

$$\int_{\mathbb{T}^3} u_0 \varphi(0,x) dx = -\int_0^T \int_{\mathbb{T}^3} u(\partial_t \varphi - \nu(-\Delta)^\alpha \varphi + (u \cdot \nabla)\varphi) dx dt$$

for any divergence-free test function $\varphi \in C_0^\infty([0,T) \times \mathbb{T}^3)$.

II. Main results

We mainly focus on two supercritical regimes $L^\gamma_t W^{s,p}_x$:

- When $\alpha \in [5/4, 2)$,

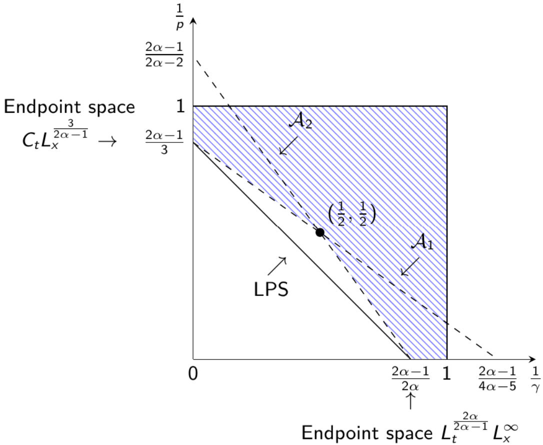
$$\mathcal{A}_1 := \left\{ (s,\gamma,p) \in [0,3) \times [1,\infty] \times [1,\infty] : 0 \leq s < \frac{4\alpha-5}{\gamma} + \frac{3}{p} + 1 - 2\alpha \right\}.$$

Note: the borderline of $\mathcal{A}_1$ contains the endpoint $(s,\gamma,p) = (0, \infty, \frac{3}{2\alpha-1})$.

- When $\alpha \in [1, 2)$,

$$\mathcal{A}_2 := \left\{ (s,\gamma,p) \in [0,3) \times [1,\infty] \times [1,\infty] : 0 \leq s < \frac{2\alpha}{\gamma} + \frac{2\alpha-2}{p} + 1 - 2\alpha \right\}.$$

Note: the borderline of $\mathcal{A}_2$ contains the endpoint $(s,\gamma,p) = (0, \frac{2\alpha}{2\alpha-1}, \infty)$.


$\frac{1}{p}$
$\frac{2\alpha-1}{2\alpha-2}$
Endpoint space
$C_tL_x^{\frac{3}{2\alpha-1}}$ →
1
$\frac{2\alpha-1}{3}$
$\mathcal{A}_2$
$(\frac{1}{2}, \frac{1}{2})$
$\mathcal{A}_1$
LPS
0
$\frac{2\alpha-1}{2\alpha}$
1
$\frac{2\alpha-1}{4\alpha-5}$
$\frac{1}{\gamma}$
Endpoint space $L_t^{\frac{2\alpha}{2\alpha-1}}L_x^\infty$

Theorem 1

Let $\tilde{u}$ be any smooth, divergence-free and mean-free vector field on $[0, T] \times \mathbb{T}^3$. Then, $\exists \beta' \in (0,1)$ s.t. $\forall \varepsilon_*, \eta_* > 0$, $\forall (s, p, \gamma) \in \mathcal{A}_1$ (resp.$\mathcal{A}_2$) if $\alpha \in [5/4, 2)$ (resp.$\alpha \in [1, 2)$), there exist a velocity field u and a "good" set

$$\mathcal{G} = \bigcup_{i=1}^{\infty} (a_i, b_i) \in [0, T],$$

s.t.

- Weak solution: u is a weak solution to hyper-NSE with $u(0) = \widetilde{u}(0)$;

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- Regularity: $u \in H^{\beta'}_{t,x} \cap L^\gamma_t W^{s,p}_x$, and

$$u|_{\mathcal{G} \times \mathbb{T}^3} \in C^\infty(\mathcal{G} \times \mathbb{T}^3);$$

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$$u|_{\mathcal{G}\times\mathbb{T}^3} \in C^\infty(\mathcal{G} \times \mathbb{T}^3);$$

- Small Hausdorff dimension of the singular set $\mathcal{B} = [0, T]/\mathcal{G}$:

$$d_{\mathcal{H}}(\mathcal{B}) < \eta_*;$$

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- Small deviations of temporal support:

$$\mathrm{supp}_{\,t} u \subseteq N_{\varepsilon_*}(\mathrm{supp}_{\,t} \tilde{u});$$

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$$u|_{\mathcal{G} \times \mathbb{T}^3} \in C^\infty(\mathcal{G} \times \mathbb{T}^3);$$

- Small Hausdorff dimension of the singular set $\mathcal{B} = [0, T]/\mathcal{G}$:

$$d_{\mathcal{H}}(\mathcal{B}) < \eta_*;$$

- Small deviations of temporal support:

$$\mathrm{supp}_{\,t} u \subseteq N_{\varepsilon_*}(\mathrm{supp}_{\,t} \tilde{u});$$

- Small deviations on average:

$$\|u - \tilde{u}\|_{L^1_t L^2_x} + \|u - \tilde{u}\|_{L^\gamma_t W^{s,p}_x} \leq \varepsilon_*.$$

Corollary 1 (Strong non-uniqueness in $L^\gamma_t W^{s,p}_x$)

Let $\alpha \in [5/4, 2)$. Then, for any weak solution $\widetilde{u}$ to hyper-NSE, there exists a different weak solution u in $L^\gamma_t W^{s,p}_x$ with the same initial data, where $(s, \gamma, p) \in \mathcal{A}_1 \cup \mathcal{A}_2$.

Remark:

Non-uniqueness holds in the strong sense in $L^\gamma_t W^{s,p}$, as in [Cheskidov-Luo, Invent.Math.'22], i.e., any weak solution to hyper-NSE in $L^\gamma_t W^{s,p}_x$ is not unique.

Corollary 2 (Non-uniqueness in $C_t\mathbb{X}$)

Let $\alpha \in [5/4, 2)$. Then, there exist non-unique weak solutions to hyper-NSE in $C_t\mathbb{X}$, where $\mathbb{X}$ can be any of the following three supercritical spaces:

- Lebesgue space L^p: $p \in [1, \frac{3}{2\alpha-1})$;
- Besov space $B^s_{p,q}$: $s \in (-\infty, \frac{3}{p} + 1 - 2\alpha)$, $p \in (1, \infty)$, $q \in [1, \infty]$;
- Triebel-Lizorkin space $F^s_{p,q}$: $s \in (-\infty, \frac{3}{p} + 1 - 2\alpha)$, $p \in (1, \infty)$, $q \in [1, \infty]$.

Theorem 2 (Strong vanishing viscosity limit)

Let $\alpha \in (1,2)$ and $u \in H^{\widetilde{\beta}}([-2T, 2T] \times \mathbb{T}^3)$, $\widetilde{\beta} > 0$, be any mean-free weak solution to the Euler equation

$$\partial_t u + (u \cdot \nabla) u + \nabla P = 0$$

under the incompressible condition.

Then, there exist $\beta' \in (0, \widetilde{\beta})$ and a sequence of weak solutions $u^{(\nu_n)} \in H^{\beta'}_{t,x}$ to the hyper-NSE, where ν_n is the viscosity coefficient, s.t.

$$u^{(\nu_n)} \to u \quad \text{strongly in } H^{\beta'}_{t,x}, \quad as\ \nu_n \to 0.$$

Some comments:

- Non-uniqueness beyond the Lions exponent.
  - ◇ By [Luo-Titi, Calc.Var.PDE'20], [Buckmaster-Colombo-Vicol JEMS'21], $\alpha = 5/4$ is the critical threshold for the well-posedness of NSE in $C_t L^2_x$.
  - ◇ By Theorem 1, even for $\alpha > 5/4$, uniqueness would still fail in $L^\gamma_t W^{s,p}_x$, where $(s, \gamma, p) \in \mathcal{A}_1 \cup \mathcal{A}_2$.
  - ◇ When $\alpha = 5/4$, uniqueness would fail in $C_t L^p_x$ for any $p < 2$ (sharp).

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  - ◇ When $\alpha = 5/4$, uniqueness would fail in $C_t L_x^p$ for any $p < 2$ (sharp).
- Partial regularity

  In the high dissipative regime $\alpha \in [5/4, 1)$, $\forall \eta_* > 0$, $\exists$ weak solutions s.t.
  - ◇ Coincide with the Leray-Hopf solutions near $t = 0$;
  - ◇ Close to Leray-Hopf solutions in $L_t^\gamma W_x^{s,p}$, $(s, \gamma, p) \in \mathcal{A}_1 \cup \mathcal{A}_2$;
  - ◇ Singular temproal sets have zero Hausdorff $\mathcal{H}^{\eta_*}$ measure.

- Sharp non-uniqueness at two LPS endpoints.

  ◇ By [Cheskidov-Luo, Invent.Math.'22], sharp non-uniqueness for 3D NSE holds at the LPS endpoint $(s, \gamma, p) = (0, 2, \infty)$, i.e., for any $\gamma < 2$, there exist non-unique solutions in $L^\gamma_t L^\infty_x$ to NSE $(\alpha = 1)$.

  ◇ By [Cheskidov-Luo, Ann.PDE'23], sharp non-uniqueness for 2D NSE holds at another LPS endpoint $(s, \gamma, p) = (0, \infty, 2)$.

  ◇ In view of the LPS condition, Theorem 1 provides the sharp non-uniqueness for the hyper-NSE at two LPS endpoints:

    ◇ $(2\alpha/\gamma + 1 - 2\alpha, \gamma, \infty)$ for $\alpha \in (1, 2)$;

    ◇ $(3/p + 1 - 2\alpha, \infty, p)$ for $\alpha \in [5/4, 2)$.

- Non-uniqueness in supercritical Lebesgue and Besov spaces.

  ◇ Corollary 2 appears to provide the first non-unique weak solutions for hyper-NSE in $C_t\mathbb{X}$, where $\mathbb{X}$ can be the supercritical Lebesgue, Besov and Triebel-Lizorkin spaces.

  ◇ Non-uniqueness is sharp in the supercritical Lebesgue spaces and Besov spaces, due to the well-posedness result in $L_x^{3/(2\alpha-1)}$, and in $\dot{B}_{2,q}^{\frac{5}{2}-2\alpha}$ for $1 < q \leq \infty$, $\alpha > 1/2$ [Wu, 2006].

  ◇ By Corollary 2 and embeddings of Besov spaces, non-uniqueness of weak solutions holds in $B^s_{\infty,q}$, $\forall s < 1 - 2\alpha$, $1 \leq q \leq \infty$.

  It complements to the ill-posedness in [Cheskidov-Dai Indiana Univ.Math.J.'14], where norm-inflation instability was proved for hyper-NSE in $B^s_{\infty,q}$, $\forall s \leq -\alpha$, $2 < q \leq \infty$.

  For norm-inflation of quintic forth-order nonlinear Schrödinger equation, see [Xia-Z., arXiv:2202.03020].

III. Further works

1. Non-uniqueness of MHD [Li-Zeng-Zhang, JMPA'22]

$$\begin{cases} \partial_t u - \nu_1 \Delta u + (u\cdot\nabla)u - (B\cdot\nabla)B + \nabla P = 0, \\ \partial_t B - \nu_2 \Delta B + (u\cdot\nabla)B - (B\cdot\nabla)u = 0, \\ \mathrm{div} u = 0, \quad \mathrm{div} B = 0, \end{cases}$$

- ◇ Previous intermittent jets for NSE are not applicable to MHD.

  ⇒ require to construct new intermittent flows to respect the geometric of MHD.
- ◇ 1D intermittent shear flows for ideal MHD [Beekie-Buckmaster-Vicol., Ann.PDE'20].
- ◇ New intermittent flows: 2D spatial intermittency + 1D temporal intermittency.

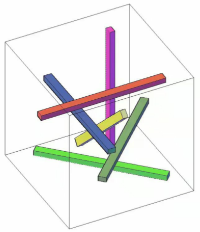

FIGURE 2. Intermittent flows

2. Non-uniqueness of hyper-MHD $\mathbb{T}^3$: [Li-Zeng-Z., ArXiv:2208.00624]

$$\begin{cases} \partial_t u + \nu_1(-\Delta)^\alpha u + (u\cdot\nabla)u - (B\cdot\nabla)B + \nabla P = 0, \\ \partial_t B + \nu_2(-\Delta)^\alpha B + (u\cdot\nabla)B - (B\cdot\nabla)u = 0, \\ \mathrm{div}\, u = 0, \quad \mathrm{div}\, B = 0, \end{cases}$$

where $\nu_1, \nu_2 \geq 0$ and $\alpha \in [1, 3/2)$.

- ◇ Non-uniqueness in $L^\gamma_t W^{s,p}_x$ above the Lions exponent;
- ◇ Sharp non-uniqueness at one LPS endpoint space $L^2_t L^\infty_x$;
- ◇ Partial regularity.

3. Non-uniqueness of compressible hypo-NSE on $\mathbb{T}^d$ [Li-Qu-Zeng-Z., arXiv:2212.05844]

$$\begin{cases} \partial_t \rho + \mathrm{div}\, m = 0, \\ \partial_t m + \mu(-\Delta)^\alpha(\rho^{-1}m) - (\mu+\nu)\nabla \mathrm{div}(\rho^{-1}m) + \mathrm{div}(\rho^{-1}m \otimes m) + \nabla P(\rho) = 0, \end{cases}$$

where $d \geq 2$, $\alpha \in (0,1)$.

- ◇ Non-unique weak solutions $(\rho, m) \in C_t C_x^1 \times L_t^p C_x^s$, $\alpha + s - \frac{2\alpha}{p} < 0$;
- ◇ The pressure term ∇P can not be removed simply by Leray projection;
- ◇ New intermittent flows for density and momentum: spatial homogeneity + temporal intermittency.

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4. stochastic MHD (in preparation)

- ◇ Non-uniqueness of probabilistically strong, analytically weak solutions;
- ◇ Global existence of probabilistically strong solutions: related to open problem in the SPDE field to construct global solutions to stochastic NSE [Hofmanova-Zhu-Zhu, AOP'23].

IV. Sketch of proof

- **Sketch of proof:** Two main stages

  ◇ Gluing stage:

  based on the gluing technique developed in the resolution of the Onsager conjecture.

  ◇ Intermittent convex integration stage:

  based on approximating equations with Reynolds stress.

In the intermittent convex integration stage:

- The hyperdissipative Navier-Stokes-Reynolds system, for every integer $q \geq 0$,

$$\begin{cases} \partial_t u_q + \nu(-\Delta)^\alpha u_q + \mathrm{div}(u_q \otimes u_q) + \nabla P_q = \mathrm{div} \mathring{R}_q, \\ \mathrm{div}\, u_q = 0, \end{cases}$$

where the Reynolds stress $\mathring{R}_q$ is a symmetric traceless 3×3 matrix.

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  where the Reynolds stress $\mathring{R}_q$ is a symmetric traceless 3×3 matrix.

- Aim: to prove that $\mathrm{div} \mathring{R}_q \to 0$ in an appropriate space as q tends to infinity.

  This procedure is quantified in the following iterative estimates:

$$\|u_q\|_{L_t^\infty H_x^3} \lesssim \lambda_q^5, \quad \|\partial_t u_q\|_{L_t^\infty H_x^2} \lesssim \lambda_q^8,$$
$$\|\mathring{R}_q\|_{L_t^\infty H_x^3} \lesssim \lambda_q^9, \quad \|\mathring{R}_q\|_{L_t^\infty H_x^4} \lesssim \lambda_q^{10}, \quad \|\mathring{R}_q\|_{L^1_{t,x}} \leq \lambda_q^{-\varepsilon_R} \delta_{q+1}.$$

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$$\|\mathring{R}_q\|_{L^\infty_t H^3_x} \lesssim \lambda_q^9, \quad \|\mathring{R}_q\|_{L^\infty_t H^4_x} \lesssim \lambda_q^{10}, \quad \|\mathring{R}_q\|_{L^1_{t,x}} \leq \lambda_q^{-\varepsilon_R}\delta_{q+1}.$$

- Construction of velocity increment $w_{q+1} := u_{q+1} - u_q$.

Thank you very much