Global well-posedness to the isentropic compressible Navier-Stokes equations on 3D thin domains

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2D compressible Navier-Stokes equations

$3D$ compressible isentropic Navier-Stokes equations on thin domains $\Omega_\delta = \mathbb{T} \times (\delta\mathbb{T})^2 \quad (0 < \delta < 1)$:

$$\begin{cases} \partial_t \rho_\delta + \mathrm{div}(\rho_\delta \mathrm{u}_\delta) = 0, \\ \partial_t(\rho_\delta \mathrm{u}_\delta) + \mathrm{div}(\rho_\delta \mathrm{u}_\delta \otimes \mathrm{u}_\delta) - \mu\Delta \mathrm{u}_\delta - \xi\nabla \mathrm{div}\mathrm{u}_\delta + \nabla p(\rho_\delta) = 0, \\ \rho_\delta(0,x) = \rho_{0,\delta}(x),\ \mathrm{u}_\delta(0,x) = \mathrm{u}_{0,\delta}(x). \end{cases} \tag{1}$$

- 2π periodic interval: $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$.
- $\mu > 0,\ \mu + \xi > 0$.
- Pressure $p(\rho_\delta) = a\rho_\delta^\gamma,\ a > 0,\ \gamma \in [1, \infty)$.

Main results:

- Global-in-time well-posedness when δ is small.
- The 3D CNS can be well approximated by the 1D CNS.

Some known results on global existence

For certain small initial data:

- Matsumura-Nishida 1980, $\|\rho_0 - 1, u_0\|_{H^3} \ll 1$.
- Small energy with vaccum: Huang-Li-Xin 2012, Huang-Li 2018, Li-Xin 2019.
- Smallness in Besov Space: Danchin, Charve-Danchin, Chen-Miao-Zhang, Haspot, $\cdots$

For certain large initial velocity:

- Ding-Wen-Yao-Zhu 2012: Spherically symmetric solutions.
- Danchin-Mucha 2018, Chen-Zhai 2019.
- Fang-Zhang-Zi 2018, by dispersive property.
- He-Huang-Wang 2019, velocity with large incompressible part.
- L.-Zhang 2021, 2D with a slow variable.
- Yang 2023, 3D with two slow variables.

Some known results on global existence

Global smooth solutions for large initial data:

- $\mu = const$, $\mu' = \rho^\beta$, in 2D: Vaigant-Kazhikhov 1995 ($\beta > 3$), Huang-Li 2015 ($\beta > 4/3$)...

Blow-up smooth solutions:

- Density with compact support: Xin 1998, Xin-Yan 2013, Li-Wang-Xin 2019.

Blow-up criterion :

- Upper bound of the density: Sun-Zhang 2010, Sun-Wang-Zhang 2010, Huang-Li-Xin 2010, Wen-Zhu 2013, Huang-Li-Wang 2013...

Weak solutions:

P. L. Lions 1998, Jiang-Zhang 2001, Feireisl-Novotný-Petzeltová 2001.

Limit equations

Let $f \in C(\delta\mathbb{T})$. Then

$$f \to \bar{f} := \frac{1}{|\delta\mathbb{T}|}\int_{\delta\mathbb{T}} f.$$

Given a solution $(\mathrm{u}_\delta, \rho_\delta)$ in $\Omega_\delta = \mathbb{T} \times (\delta\mathbb{T})^2$, one would expect:

$$(\mathrm{u}_\delta, \rho_\delta)(x_1, x_h) \to (\bar{\mathrm{u}}_\delta, \bar{\rho}_\delta)(x_1) := \frac{1}{|(\delta\mathbb{T})^2|}\int_{(\delta\mathbb{T})^2} (\mathrm{u}_\delta, \rho_\delta)(x_1, x_h)\mathrm{d}x_h,$$

with $x_h = (x_2, x_3) \in (\delta\mathbb{T})^2$.

Limit equations

Taking the average of the equations in $x_h \in (\delta\mathbb{T})^2$ gives:

$$
\begin{aligned}
&\partial_t \overline{\rho_\delta} + \partial_{x_1}(\overline{\rho_\delta \mathrm{u}_\delta}) = 0,\\
&\partial_t \overline{\rho_\delta \mathrm{u}_\delta^1} + \partial_{x_1}\left(\overline{\rho_\delta \mathrm{u}_\delta^1 \cdot \mathrm{u}_\delta^1}\right) - (\mu+\xi)\partial_{x_1x_1}\overline{\mathrm{u}_\delta^1} + \partial_{x_1}\overline{p(\rho_\delta)} = 0,\\
&\partial_t \overline{\rho_\delta \mathrm{u}_\delta^2} + \partial_{x_1}\left(\overline{\rho_\delta \mathrm{u}_\delta^1 \cdot \mathrm{u}_\delta^2}\right) - \mu\partial_{x_1x_1}\overline{\mathrm{u}_\delta^2} = 0,\\
&\varphi_t \overline{\rho_\delta \mathrm{u}_\delta^3} + \partial_{x_1}\left(\overline{\rho_\delta \mathrm{u}_\delta^1 \cdot \mathrm{u}_\delta^3}\right) - \mu\partial_{x_1x_1}\overline{\mathrm{u}_\delta^3} = 0.
\end{aligned}
$$

Formally passing $\delta \to 0$ and assuming $(\overline{\rho_\delta}, \overline{\mathrm{u}_\delta}) \to (r, \mathrm{v}) = (r, \mathrm{v}_1, \mathrm{v}_2, \mathrm{v}_3)$,

$$
\begin{aligned}
&\partial_t r + \partial_{x_1}(r\mathrm{v}^1) = 0,\\
&\partial_t(r\mathrm{v}_\delta^1) + \partial_{x_1}(r\mathrm{v}^1\mathrm{v}^1) - (\mu+\xi)\partial_{x_1x_1}\mathrm{v}^1 + \partial_{x_1}p(r) = 0,\\
&\partial_t(r\mathrm{v}^2) + \partial_{x_1}(r\mathrm{v}^1\mathrm{v}^2) - \mu\partial_{x_1x_1}\mathrm{v}^2 = 0,\\
&\partial_t(r\mathrm{v}^3) + \partial_{x_1}(r\mathrm{v}^1\mathrm{v}^3) - \mu\partial_{x_1x_1}\mathrm{v}^3 = 0.
\end{aligned}
\tag{2}
$$

1D CNS on $\mathbb{T}$: globally well-posedness.

Well-posedness for 1D CNS

Theorem (Kazhikhov 1979, Straškraba-Zlotnik 2002)

Let $(r_0, \mathrm{v}_0) \in H^5(\mathbb{T})$ satisfy

$$\frac{1}{2\pi}\int_{\mathbb{T}} r_0(x_1)\mathrm{d}x_1 = 1, \quad \int_{\mathbb{T}} (r_0\mathrm{v}_0)(x_1)\mathrm{d}x_1 = 0, \quad 0 < a_0 \le r_0 \le b_0 < \infty.$$

Then (2) admits a unique global-in-time strong solution $(r, \mathrm{v}) \in C([0,\infty); H^5(\mathbb{T}))$ with initial datum (r_0, v_0) so that

$$\begin{aligned} &0 < a_1 \le r \le b_1 < \infty, \\ &\|(r-1)(t,\cdot)\|_{H^5(\mathbb{T})} + \|\mathrm{v}(t,\cdot)\|_{H^5(\mathbb{T})} \le Ce^{-\alpha t}, \ \alpha > 0. \end{aligned} \tag{3}$$

Main results

Assumptions on initial data:

Let $\kappa \in (0, \frac{1}{2})$ and assume

$$(\rho_{0,\delta},\, u_{0,\delta}) = (r_0, v_0) + (\tau_{0,\delta}, U_{0,\delta}),$$

where $(\tau_{0,\delta}, U_{0,\delta}) \in H^2(\Omega_\delta)$ with

$$\begin{aligned} &\int_{(\delta\mathbb{T})^2} \tau_{0,\delta}(x_1, x_h)\mathrm{d}x_h = 0, \quad \int_{(\delta\mathbb{T})^2} U_{0,\delta}(x_1, x_h)\mathrm{d}x_h = 0, \ \forall\, x_1 \in \mathbb{T}, \\ &\int_{\Omega_\delta} \tau_{0,\delta} U_{0,\delta}(x)\mathrm{d}x = 0, \quad \left\|(\nabla^2 \tau_{0,\delta}, \nabla^2 U_{0,\delta})\right\|_{L^2(\Omega_\delta)} \leq \delta^{-\kappa}. \end{aligned} \tag{4}$$

There holds

$$\frac{1}{|\Omega_\delta|}\int_{\Omega_\delta} \rho_{0,\delta}\mathrm{d}x = 1, \quad \int_{\Omega_\delta} \rho_{0,\delta} u_{0,\delta}\mathrm{d}x = 0.$$

Main results

Pertubations

$$\tau_\delta := \rho_\delta - r, \quad \mathrm{U}_\delta := u_\delta - \mathrm{v}.$$

Energy functional: $T > 0$, $\omega_\delta := \operatorname{curl} \mathrm{U}_\delta$, $D_t\mathrm{U}_\delta := \partial_t \mathrm{U}_\delta + \mathrm{u}_\delta \cdot \nabla \mathrm{U}_\delta$,

$$\begin{aligned} E_\delta(T) := & \sup_{0<t<T} \int_{\Omega_\delta} \left(|\mathrm{U}_\delta|^2 + |\tau_\delta|^2 + |\nabla \mathrm{U}_\delta|^2 + |D_t \mathrm{U}_\delta|^2 + |\nabla \omega_\delta|^2 \right) \mathrm{d}x \\ & + \int_0^T \int_{\Omega_\delta} \left(|\nabla \mathrm{U}_\delta|^2 + |D_t \mathrm{U}_\delta|^2 + |\nabla \omega_\delta|^2 + |\nabla D_t \mathrm{U}_\delta|^2 \right) \mathrm{d}x\, \mathrm{d}t, \\ I_\delta(T) := & \sup_{0<t<T} \|\tau_\delta(t)\|_{L^\infty(\Omega_\delta)}. \end{aligned}$$

Theorem 2 (Li-L.-Sun 2023). Let $\delta > 0$ be small. The problem (1) admits a unique global-in-time strong solution $(\rho_\delta, \mathrm{u}_\delta) \in C([0,\infty); H^2(\Omega_\delta))$ with

$$E_\delta(\infty) \le C\delta^{-2\kappa}, \quad I_\delta^2(\infty) \le C\delta^{\beta}, \quad \beta \in (0, 1-2\kappa).$$

Remarks

- Second order derivatives of $(\tau_\delta, \mathrm{U}_\delta)$ are large as $\delta \to 0$. While, by Poincaré inequalities in thin domains of size δ gives

$$\|\nabla \tau_{0,\delta}\|_{L^2(\Omega_\delta)} + \|\nabla \mathrm{U}_{0,\delta}\|_{L^2(\Omega_\delta)} \leq C\delta^{(1-\kappa)},$$
$$\|\tau_{0,\delta}\|_{L^2(\Omega_\delta)} + \|\mathrm{U}_{0,\delta}\|_{L^2(\Omega_\delta)} \leq C\delta^{(2-\kappa)}.$$

- No size restriction on (r_0, v_0).
- The initial data can be generalized as

$$(\rho_{0,\delta},\ \mathrm{u}_{0,\delta}) = (r_{0,\delta}, \mathrm{v}_{0,\delta}) + (\tau_{0,\delta}, \mathrm{U}_{0,\delta}),$$

with

$$(r_{0,\delta}, \mathrm{v}_{0,\delta}) \to (r_0, \mathrm{v}_0) \text{ in } H^5(\mathbb{T}).$$

Perturbed equations

Let $(\rho_\delta, \mathrm{u}_\delta)$ be the solution to (1) with maximal existence time T_δ^*. Recall

$$\tau_\delta := \rho_\delta - r, \quad \mathrm{U}_\delta := u_\delta - \mathrm{v}.$$

Then

$$\begin{aligned}
&\partial_t \tau_\delta + \mathrm{div}\,(\rho_\delta \mathrm{U}_\delta) + \mathrm{div}\,(\tau_\delta \mathrm{v}) = 0,\\
&\rho_\delta D_t \mathrm{U}_\delta - \mu\Delta \mathrm{U}_\delta - \xi\nabla \mathrm{div}\,\mathrm{U}_\delta + \nabla(p(\rho_\delta) - p(r))\\
&\quad + \rho_\delta \mathrm{U}_\delta \cdot \nabla \mathrm{v} + \tau_\delta(\partial_t \mathrm{v} + \mathrm{v}\cdot\nabla \mathrm{v}) = 0,\\
&\tau_\delta(0,x) = \tau_{0,\delta}(x),\ \mathrm{U}_\delta(0,x) = \mathrm{U}_{0,\delta}(x).
\end{aligned} \tag{5}$$

Define

$$T_\delta^{**} = \sup\{T < T_\delta^*,\ I_\delta(T) = \sup_{0<t<T} \|\tau_\delta(t)\|_{L^\infty(\Omega_\delta)} \le \frac{a_1}{2}\},$$

Then for all $t < T_\delta^{**}$,

$$0 < a_1 - \frac{a_1}{2} \le \rho_\delta \le b_1 + \frac{a_1}{2} < \infty.$$

Basic energy estimates

Energy functional

$$\mathcal{E}(t) = \int_{\Omega_\delta} \left(\frac{1}{2}\rho_\delta |\mathsf{U}_\delta|^2 + P(\rho_\delta) - P(r) - P'(r)\tau_\delta \right)(t)\mathrm{d}x,$$

where

$$P(\rho) = \frac{a}{\gamma - 1}\rho^\gamma, \text{ if } \gamma > 1; \quad P(\rho) = a\rho \log \rho, \text{ if } \gamma = 1.$$

Relative entropy inequality: for all $t < T_\delta^{**}$,

$$\mathcal{E}(t) + \int_0^t \int_{\Omega_\delta} (\mu |\nabla \mathsf{U}_\delta|^2 + \xi |\mathrm{div}\, \mathsf{U}_\delta|^2)\, \mathrm{d}x\, \mathrm{d}t = \mathcal{E}(0) + \int_0^t \mathcal{R}(t')\mathrm{d}t',$$

$$\mathcal{R} = -\int_{\Omega_\delta} \rho_\delta D_t \mathsf{v} \cdot \mathsf{U}_\delta dx - \int_{\Omega_\delta} (\mu \nabla \mathsf{v} : \nabla \mathsf{U}_\delta + \xi \mathrm{div}\, \mathsf{v} \mathrm{div}\, \mathsf{U}_\delta)\, \mathrm{d}x$$
$$- \int_{\Omega_\delta} (\tau_\delta \partial_t P'(r) - (r\mathsf{v} - \rho_\delta \mathsf{u}_\delta)\nabla P'(r))\, \mathrm{d}x - \int_{\Omega_\delta} \mathrm{div}\, \mathsf{v} \cdot (p(\rho_\delta) - p(r))\mathrm{d}x.$$

Basic energy estimates

By Gronwall's inequality

$$\mathcal{E}(t) + \int_0^t \int_{\Omega_\delta} \mu |\nabla \mathsf{U}_\delta|^2 \, \mathrm{d}x \, \mathrm{d}t \leq C\mathcal{E}(0).$$

Observe:

$$\mathcal{E}(t) \sim \int_{\Omega_\delta} \left(|\mathsf{U}_\delta|^2 + \tau_\delta^2\right)(t,x)\mathrm{d}x,$$

$$\mathcal{E}(0) \sim \int_{\Omega_\delta} \left(|\mathsf{U}_{0,\delta}|^2 + \tau_{0,\delta}^2\right)(x)\mathrm{d}x \leq C\delta^{(4-2\kappa)}.$$

Then for $t < T_\delta^{**}$,

$$\int_{\Omega_\delta} \left(|\mathsf{U}_\delta|^2 + \tau_\delta^2\right)(t,x)\mathrm{d}x + \int_0^t \int_{\Omega_\delta} |\nabla \mathsf{U}_\delta|^2 \mathrm{d}x\mathrm{d}t' \leq C\delta^{(4-2\kappa)}.$$

Decay estimates of L^2 norms

Proposition 1

For each $t < T_\delta^{**}$, there exist positive constants C and α independent of T_δ^{**} such that

$$\int_{\Omega_\delta} \left(|\mathrm{U}_\delta|^2 + |\tau_\delta|^2\right)(t,x) \le C\delta^2 e^{-\alpha t}.$$

Proposition 2

For each $t < T_\delta^{**}$, there exist positive constants C and α independent of T_δ^{**} such that

$$\int_{\Omega_\delta} \left(|\mathrm{u}_\delta|^2 + |\rho_\delta - 1|^2\right)(t,x) \le C\delta^2 e^{-\alpha t}.$$

Proposition 2 and the decay estimates of (r, v) implies Proposition 1.

Decay estimates of L^2 norms

Basic idea:

$$f'(t) + \alpha f(t) \leq 0 \Longrightarrow f(t) \leq f(0)e^{-\alpha t}.$$

Basic energy equality

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_1(t) + \int_{\Omega_\delta} (\mu|\nabla \mathrm{u}_\delta|^2 + \xi|\mathrm{div}\, \mathrm{u}_\delta|^2)\mathrm{d}x = 0, \tag{7}$$

where

$$\mathcal{E}_1(t) = \int_{\Omega_\delta} \left(\frac{1}{2}\rho_\delta|\mathrm{u}_\delta|^2 + P(\rho_\delta) - P(1) - P'(1)(\rho_\delta - 1)\right)(t)\mathrm{d}x.$$

Observe:

$$P(\rho_\delta) - P(1) - P'(1)(\rho_\delta - 1) \sim (\rho_\delta - 1)^2.$$

Decay estimates of L^2 norms

Conservation of mass and linear momentum:

$$\frac{1}{|\Omega_\delta|}\int_{\Omega_\delta}\rho_\delta(t,x)\mathrm{d}x = 1, \quad \int_{\Omega_\delta}\rho_\delta \mathrm{u}_\delta(t,x)\mathrm{d}x = 0.$$

Poincaré type Lemma (Zhang-Zi 2020): if $\rho_\delta \le \bar{\rho} < \infty$, then

$$\int_{\Omega_\delta}\rho_\delta|\mathrm{u}_\delta|^2\,\mathrm{d}x \le \bar{\rho}^2\int_{\Omega_\delta}|\nabla \mathrm{u}_\delta|^2\,\mathrm{d}x.$$

Indeed, using $\frac{1}{|\Omega_\delta|}\int_{\Omega_\delta}\rho_\delta = 1$, $\int_{\Omega_\delta}\rho \mathrm{u} = 0$,

$$\begin{aligned}
\int_\Omega \rho \mathrm{u}^2\,\mathrm{d}x &= \int_{\Omega_\delta}\rho|\mathrm{u}|^2\,\mathrm{d}x - |\langle\rho \mathrm{u}\rangle|^2 = \frac{1}{2}\frac{1}{|\Omega_\delta|}\int_{\Omega_\delta}\int_{\Omega_\delta}\rho(x)\rho(x')|\mathrm{u}(x)-\mathrm{u}(x')|^2 \\
&\le \frac{1}{2}\frac{1}{|\Omega_\delta|}\bar{\rho}^2\int_{\Omega_\delta}\int_{\Omega_\delta}|\mathrm{u}(x)-\mathrm{u}(x')|^2\,\mathrm{d}x\,\mathrm{d}x' \\
&= \bar{\rho}^2\int_{\Omega_\delta}|\mathrm{u}-\langle \mathrm{u}\rangle|^2\,\mathrm{d}x \le \bar{\rho}^2\int_{\Omega_\delta}|\nabla \mathrm{u}|^2\,\mathrm{d}x.
\end{aligned}$$

Decay estimates of L^2 norms

Bogovskii operator: let $p \in (1,\infty)$, $f \in \mathcal{D}_0'(\Omega_\delta)$, $g \in (\mathcal{D}'(\Omega_\delta))^3$, then

$$\|B[f]\|_{W^{1,p}(\Omega_\delta)} \le C\|f\|_{L^p(\Omega_\delta)},$$
$$\|B[\operatorname{div} g]\|_{L^p(\Omega_\delta)} \le C\|g\|_{L^p(\Omega_\delta)}.$$

Testing the momentum equations by $B[\rho_\delta - 1]$ implies

$$\int_{\Omega_\delta} p(\rho_\delta)(\rho_\delta - 1)\mathrm{d}x - \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_\delta} \rho_\delta \mathrm{u}_\delta \cdot B[\rho_\delta - 1]\mathrm{d}x$$
$$= -\int_{\Omega_\delta} \rho_\delta \mathrm{u}_\delta B[\partial_t \rho_\delta]\mathrm{d}x - \int_{\Omega_\delta} \rho_\delta \mathrm{u}_\delta \otimes \mathrm{u}_\delta : \nabla B[\rho_\delta - 1]\mathrm{d}x$$
$$+ \int_{\Omega_\delta} (\mu \nabla \mathrm{u}_\delta : \nabla B[\rho_\delta - 1] + \xi \operatorname{div} \mathrm{u}_\delta(\rho_\delta - 1))\mathrm{d}x.$$

Observe:

$$\int_{\Omega_\delta} p(\rho_\delta)(\rho_\delta - 1)\,\mathrm{d}x = \int_{\Omega_\delta} (p(\rho_\delta) - 1)(\rho_\delta - 1)\mathrm{d}x \sim \int_{\Omega_\delta} (\rho_\delta - 1)^2\,\mathrm{d}x.$$

Decay estimates of L^2 norms

One has

$$-\int_{\Omega_\delta} \rho_\delta \mathrm{u}_\delta B[\partial_t \rho_\delta]\mathrm{d}x = \int_{\Omega_\delta} \rho_\delta \mathrm{u}_\delta B[\mathrm{div}\,(\rho_\delta \mathrm{u}_\delta)]\mathrm{d}x$$

$$\leq C \int_{\Omega_\delta} \rho_\delta |\mathrm{u}_\delta|^2 \mathrm{d}x \leq C \int_{\Omega_\delta} |\nabla \mathrm{u}_\delta|^2 \mathrm{d}x.$$

Also,

$$-\int_{\Omega_\delta} \rho_\delta \mathrm{u}_\delta \otimes \mathrm{u}_\delta : \nabla B[\rho_\delta - 1]\mathrm{d}x \leq C\|\mathrm{u}_\delta\|^2_{L^4(\Omega_\delta)} \|\nabla B[\rho_\delta - 1]\|_{L^2(\Omega_\delta)}$$

$$\leq C\|\mathrm{u}_\delta\|^2_{L^4(\Omega_\delta)} \|\rho_\delta - 1\|_{L^2(\Omega_\delta)} \leq C\delta \|\mathrm{u}_\delta\|^2_{L^4(\Omega_\delta)} \|\rho_\delta - 1\|_{L^\infty(\Omega_\delta)}$$

$$\leq C\delta \|\mathrm{u}_\delta\|^2_{L^4(\Omega_\delta)}.$$

Decay estimates of L^2 norms

Gagliardo-Nirenberg type inequalities:

$$
\begin{aligned}
\|\mathrm{u}_\delta\|_{L^4(\Omega_\delta)} &\leq C\delta^{(\frac{3}{4}-\frac{1}{2})}\|\nabla \mathrm{u}_\delta\|_{L^2(\Omega_\delta)} \\
&\quad + C\delta^{-2(\frac{1}{2}-\frac{1}{4})}(\|\mathrm{u}_\delta\|_{L^2(\Omega_\delta)} + \|\mathrm{u}_\delta\|_{L^2(\Omega_\delta)}^{\frac{1}{2}+\frac{1}{4}}\|\nabla \mathrm{u}_\delta\|_{L^2(\Omega_\delta)}^{\frac{1}{2}-\frac{1}{4}}) \\
&\leq C\delta^{-\frac{1}{2}}(\|\sqrt{\rho_\delta}\mathrm{u}_\delta\|_{L^2(\Omega_\delta)} + \|\nabla \mathrm{u}_\delta\|_{L^2(\Omega_\delta)}) \\
&\leq C\delta^{-\frac{1}{2}}\|\nabla \mathrm{u}_\delta\|_{L^2(\Omega_\delta)}.
\end{aligned} \tag{8}
$$

Thus,

$$
-\int_{\Omega_\delta} \rho_\delta \mathrm{u}_\delta \otimes \mathrm{u}_\delta : \nabla B[\rho_\delta - 1]\mathrm{d}x \leq C\delta\|\mathrm{u}_\delta\|_{L^4(\Omega_\delta)}^2 \leq C\|\nabla \mathrm{u}_\delta\|_{L^2(\Omega_\delta)}^2.
$$

Decay estimates of L^2 norms

Hence

$$\int_{\Omega_\delta} (p(\rho_\delta) - 1)(\rho_\delta - 1)\mathrm{d}x - \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega_\delta} \rho_\delta \mathrm{u}_\delta \cdot B[\rho_\delta - 1]\mathrm{d}x$$
$$\leq C\varepsilon^{-1}\|\nabla \mathrm{u}_\delta\|^2_{L^2(\Omega_\delta)} + \varepsilon\|(\rho_\delta - 1)\|^2_{L^2(\Omega_\delta)}.$$

Together with basic energy equality

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_1(t) + \int_{\Omega_\delta} (\mu|\nabla \mathrm{u}_\delta|^2 + \xi|\mathrm{div}\, \mathrm{u}_\delta|^2)\mathrm{d}x = 0,$$

where

$$\mathcal{E}_1(t) = \int_{\Omega_\delta} \left(\frac{1}{2}\rho_\delta|\mathrm{u}_\delta|^2 + P(\rho_\delta) - P(1) - P'(1)(\rho_\delta - 1)\right)(t)\mathrm{d}x,$$

one deduces the exponential decay estimates.

Estimates of the energy functional

Recall:

$$E_\delta(T) := \sup_{0<t<T} \int_{\Omega_\delta} \left(|\mathrm{U}_\delta|^2 + |\tau_\delta|^2 + |\nabla \mathrm{U}_\delta|^2 + |D_t \mathrm{U}_\delta|^2 + |\nabla \omega_\delta|^2\right) \mathrm{d}x$$
$$+ \int_0^T \int_{\Omega_\delta} \left(|\nabla \mathrm{U}_\delta|^2 + |D_t \mathrm{U}_\delta|^2 + |\nabla \omega_\delta|^2 + \ |\nabla D_t \mathrm{U}_\delta|^2\right) \mathrm{d}x\, \mathrm{d}t,$$
$$I_\delta(T) := \sup_{0<t<T} \|\tau_\delta(t)\|_{L^\infty(\Omega_\delta)}.$$

After a long journey (idea of D. Hoff): If $T < T_\delta^{**}$:

$$E_\delta(T) \le C\delta^{-2\kappa} + C \int_0^T \int_{\Omega_\delta} |\nabla \mathrm{U}_\delta|^4 \,\mathrm{d}x\, \mathrm{d}t.$$

Estimates on velocity

Effective viscous flux:

$$F_\delta := \nu \operatorname{div} \mathsf{U}_\delta - \big(p(\rho_\delta) - p(r)\big), \quad \nu := \mu + \xi.$$

Then

$$\begin{aligned} \nu \Delta \mathsf{U}_\delta &= \nu \nabla \operatorname{div} \mathsf{U}_\delta + \nu \nabla^\perp \omega_\delta \\ &= \nabla F_\delta + \nabla \big(p(\rho_\delta) - p(r)\big) + \nu \nabla^\perp \omega_\delta. \end{aligned}$$

Then

$$\|\nabla \mathsf{U}_\delta\|_{L^4(\Omega_\delta)} \le C \big(\|\omega_\delta\|_{L^4(\Omega_\delta)} + \|F_\delta\|_{L^4(\Omega_\delta)} + \|p(\rho_\delta) - p(r)\|_{L^4(\Omega_\delta)} \big).$$

Estimates of effective viscous flux:

Firstly,

$$\|F_\delta\|_{L^2(\Omega_\delta)} \le C\|\nabla \mathrm{U}_\delta\|_{L^2(\Omega_\delta)} + C\|\rho_\delta - r\|_{L^2(\Omega_\delta)}$$
$$\le C\|\nabla \mathrm{U}_\delta\|_{L^2(\Omega_\delta)} + C\delta^{(2-\kappa)}e^{-\alpha t}.$$

From the momentum equation,

$$\Delta F_\delta = \mathrm{div}\,(\mu\Delta \mathrm{U}_\delta + \xi\nabla \mathrm{div}\,\mathrm{U}_\delta - \nabla(p(\rho_\delta) - p(r)))$$
$$= \mathrm{div}\,(\rho_\delta D_t \mathrm{U}_\delta + \rho_\delta \mathrm{U}_\delta \cdot \nabla \mathrm{v} + \tau_\delta(\partial_t \mathrm{v} + \mathrm{v}\cdot\nabla \mathrm{v}))\,.$$

Consequently,

$$\|\nabla F_\delta\|_{L^2(\Omega_\delta)} \le C\|D_t \mathrm{U}_\delta\|_{L^2(\Omega_\delta)} + Ce^{-\alpha t}(\|\mathrm{U}_\delta\|_{L^2(\Omega_\delta)} + \|\tau_\delta\|_{L^2(\Omega_\delta)})$$
$$\le C\|D_t \mathrm{U}_\delta\|_{L^2(\Omega_\delta)} + Ce^{-\alpha t}\delta^{(2-\kappa)}.$$

Estimates of effective viscous flux

Gagliardo-Nirenberg inequality: for $f \in H^1(\Omega_\delta)$ and $p \in [2,6]$,

$$\|f\|_{L^p(\Omega_\delta)} \le C\delta^{-2(\frac{1}{2}-\frac{1}{p})}\big(\|f\|_{L^2(\Omega_\delta)} + \|f\|_{L^2(\Omega_\delta)}^{\frac{3}{p}-\frac{1}{2}}\|\nabla f\|_{L^2(\Omega_\delta)}^{\frac{3}{2}-\frac{3}{p}}\big).$$

Thus,

$$\int_0^t \|F_\delta\|_{L^4(\Omega_\delta)}^4 \mathrm{d}t \le \int_0^t C\delta^{-2}(\|F_\delta\|_{L^2(\Omega_\delta)}^4 + \|F_\delta\|_{L^2(\Omega_\delta)}\|\nabla F_\delta\|_{L^2(\Omega_\delta)}^3)\mathrm{d}t$$

$$\le \int_0^t C\delta^{-2}\big(\|\nabla \mathrm{U}_\delta\|_{L^2(\Omega_\delta)} + C\delta^{(2-\kappa)}e^{-\alpha t}\big)\big(\|D_t\mathrm{U}_\delta\|_{L^2(\Omega_\delta)}^3 + Ce^{-\alpha t}\delta^{(6-3\kappa)}\big)\mathrm{d}t$$

$$\le \int_0^t C\delta^{-2}\|\nabla \mathrm{U}_\delta\|_{L^2(\Omega_\delta)}\|D_t\mathrm{U}_\delta\|_{L^2(\Omega_\delta)}^3 \mathrm{d}t + \int_0^t C\delta^{-\kappa}\|D_t\mathrm{U}_\delta\|_{L^2(\Omega_\delta)}^3 e^{-\alpha t}\mathrm{d}t$$

$$\le C\delta^{-\kappa}E_\delta^{\frac{3}{2}}(t),$$

where we used

$$\int_0^t \int_{\Omega_\delta} |\nabla \mathrm{U}_\delta|^2 \mathrm{d}x\mathrm{d}t' \le C\delta^{(4-2\kappa)}.$$

Estimates of effective viscous flux

Refined GN inequality: for $f \in H^1(\Omega_\delta)$ and $p \in [2,6]$,

$$\begin{aligned}
\|f\|_{L^p(\Omega_\delta)} \le\ & C\delta^{-(\frac{1}{2}-\frac{1}{p})}\|f\|_{L^2(\Omega_\delta)}^{\frac{2}{p}}\|\nabla f\|_{L^2(\Omega_\delta)}^{1-\frac{2}{p}} \\
&+ C\delta^{-2(\frac{1}{2}-\frac{1}{p})}\Big(\|f\|_{L^2(\Omega_\delta)} + \|f\|_{L^2(\Omega_\delta)}^{\frac{1}{2}+\frac{1}{p}}\|\nabla f\|_{L^2(\Omega_\delta)}^{\frac{1}{2}-\frac{1}{p}}\Big) \\
&+ C\|f\|_{L^2(\Omega_\delta)}^{\frac{3}{p}-\frac{1}{2}}\|\nabla f\|_{L^2(\Omega_\delta)}^{\frac{3}{2}-\frac{3}{p}},
\end{aligned}$$

$$\begin{aligned}
\|f\|_{L^p(\Omega_\delta)} \le\ & C\delta^{(\frac{3}{p}-\frac{1}{2})}\|\nabla f\|_{L^2(\Omega_\delta)} \\
&+ C\delta^{-2(\frac{1}{2}-\frac{1}{p})}(\|f\|_{L^2(\Omega_\delta)} + \|f\|_{L^2(\Omega_\delta)}^{\frac{1}{2}+\frac{1}{p}}\|\nabla f\|_{L^2(\Omega_\delta)}^{\frac{1}{2}-\frac{1}{p}}).
\end{aligned}$$

Consequently,

$$\int_0^t \|F_\delta\|_{L^4(\Omega_\delta)}^4 \mathrm{d}t \le C\delta^{(2-2\kappa)}E_\delta(t) + C\delta^{(2-\kappa)}E_\delta^{\frac{3}{2}}(t).$$

No singular coefficients.

Estimates of density

Initially,

$$\|\rho_{0,\delta} - r_0\|_{L^\infty(\Omega_\delta)} \leq C\delta^{(\frac{1}{2}-\kappa)}.$$

Proposition. Let $\beta \in (0, \frac{1}{2} - \kappa)$. If $T < T_\delta^{**}$,

$$I_\delta(T) = \sup_{0<t<T} \|(\rho_\delta - r)(t,\cdot)\|_{L^\infty(\Omega_\delta)} \leq C\delta^\beta.$$

Idea: renormalized continuity equation

$$D_t(\log \rho_\delta - \log r) + \mathrm{U}_\delta \cdot \nabla \log r + \mathrm{div}\, \mathrm{U}_\delta = 0,$$

together withe the definition of F_δ implies

$$D_t(\log \rho_\delta - \log r) + \mathrm{U}_\delta \cdot \nabla \log r + \nu^{-1}(p(\rho_\delta) - p(r)) = -\nu^{-1} F_\delta.$$

Refined Gagliardo-Nirenberg inequalities

Recall

$$\|f\|_{L^p(\Omega_\delta)} \le C\delta^{-(\frac{1}{2}-\frac{1}{p})}\|f\|_{L^2(\Omega_\delta)}^{\frac{2}{p}}\|\nabla f\|_{L^2(\Omega_\delta)}^{1-\frac{2}{p}}$$
$$+ C\delta^{-2(\frac{1}{2}-\frac{1}{p})}\Big(\|f\|_{L^2(\Omega_\delta)} + \|f\|_{L^2(\Omega_\delta)}^{\frac{1}{2}+\frac{1}{p}}\|\nabla f\|_{L^2(\Omega_\delta)}^{\frac{1}{2}-\frac{1}{p}}\Big)$$
$$+ C\|f\|_{L^2(\Omega_\delta)}^{\frac{3}{p}-\frac{1}{2}}\|\nabla f\|_{L^2(\Omega_\delta)}^{\frac{3}{2}-\frac{3}{p}},$$

$$\|f\|_{L^p(\Omega_\delta)} \le C\delta^{(\frac{3}{p}-\frac{1}{2})}\|\nabla f\|_{L^2(\Omega_\delta)}$$
$$+ C\delta^{-2(\frac{1}{2}-\frac{1}{p})}(\|f\|_{L^2(\Omega_\delta)} + \|f\|_{L^2(\Omega_\delta)}^{\frac{1}{2}+\frac{1}{p}}\|\nabla f\|_{L^2(\Omega_\delta)}^{\frac{1}{2}-\frac{1}{p}}).$$

Trade-off between the regularity and thinness of domain.

Fourier analysis on $\Omega_\delta = \mathbb{T} \times (\delta\mathbb{T})^2$

Notations:

$$\widetilde{\mathbb{Z}} = \mathbb{Z}/\delta, \quad k = (k_1, k_h) \in \mathbb{Z} \times \widetilde{\mathbb{Z}}^2, \quad k_h = (k_2, k_3) \in \widetilde{\mathbb{Z}}^2.$$

For $f \in L^2(\Omega_\delta)$,

$$f = \sum_{k \in \mathbb{Z} \times \widetilde{\mathbb{Z}}^2} \hat{f}_k \frac{e^{ik\cdot x}}{\sqrt{8\pi^3\delta^2}}, \quad \text{with } \hat{f}_k = \int_{\Omega_\delta} \frac{e^{-ik\cdot x}}{\sqrt{8\pi^3\delta^2}} f \,\mathrm{d}x.$$

Define

$$\mathcal{M}_1 f = \sum_{k_h=0} \hat{f}_k \frac{e^{ik\cdot x}}{\sqrt{8\pi^3\delta^2}}, \quad \mathcal{M}_2 f = \sum_{k_2\neq 0, k_3=0} \hat{f}_k \frac{e^{ik\cdot x}}{\sqrt{8\pi^3\delta^2}},$$

$$\mathcal{M}_3 f = \sum_{k_2=0, k_3\neq 0} \hat{f}_k \frac{e^{ik\cdot x}}{\sqrt{8\pi^3\delta^2}}, \quad \mathcal{M}_4 f = \sum_{k_2\neq 0, k_3\neq 0} \hat{f}_k \frac{e^{ik\cdot x}}{\sqrt{8\pi^3\delta^2}},$$

$$\mathcal{M}_1^\perp f = \mathcal{M}_2 f + \mathcal{M}_3 f + \mathcal{M}_4 f.$$

Littlewood-Paley decomposition on $\Omega_\delta = \mathbb{T} \times (\delta\mathbb{T})^2$

Let φ, χ be smooth and radial functions satisfying

$$\chi \in C_c^\infty(B(0, \frac{4}{3})),\ 0 \le \chi \le 1,\ \chi(x) = 1 \text{ for } |x| \le \frac{3}{4},$$
$$\varphi(x) = \chi(\frac{x}{2}) - \chi(x),$$
$$\chi(\xi) + \sum_{j \ge 0} \varphi(2^{-j}\xi) = 1 \text{ for any } \xi \in \mathbb{R},$$
$$\sum_{j \in \mathbb{Z}} \varphi(2^{-j}\xi) = 1 \text{ for any } \xi \in \mathbb{R}\backslash\{0\}.$$

For $f \in \mathcal{D}'(\Omega_\delta)$ and $j \in \mathbb{Z}$, define

$$\Delta_j f = \sum_{k \in \mathbb{Z} \times \widetilde{\mathbb{Z}}^2} \varphi(2^{-j}|k|)\hat{f}_k \frac{e^{ik\cdot x}}{\sqrt{8\pi^3\delta^2}},\ S_j f = \frac{\hat{f}_0}{\sqrt{8\pi^3\delta^2}} + \sum_{j' \le j-1} \Delta_{j'} f.$$

Bernstein type inequalities

Proposition. Let $(p, q) \in [1, \infty]^2$, $p \leq q$, $s \in [0, \infty)$.

$$\|\Delta_j \nabla^\alpha f\|_{L^p(\Omega_\delta)} \leq C 2^{j|\alpha|} \|\Delta_j f\|_{L^p(\Omega_\delta)},$$

$$C^{-1} \|\Delta_j \nabla f\|_{L^p(\Omega_\delta)} \leq 2^j \|\Delta_j f\|_{L^p(\Omega_\delta)} \leq C \|\Delta_j \nabla f\|_{L^p(\Omega_\delta)},$$

and

$$\|\Delta_j \mathcal{M}_1 f\|_{L^q(\Omega_\delta)} \leq C \delta^{-2(\frac{1}{p} - \frac{1}{q})} 2^{j(\frac{1}{p} - \frac{1}{q})} \|\Delta_j \mathcal{M}_1 f\|_{L^p(\Omega_\delta)}$$

$$\|\Delta_j \mathcal{M}_l f\|_{L^q(\Omega_\delta)} \leq C \delta^{-(\frac{1}{p} - \frac{1}{q})} 2^{2j(\frac{1}{p} - \frac{1}{q})} 2^{js} \delta^s \|\Delta_j \mathcal{M}_l f\|_{L^p(\Omega_\delta)}, \ \ l = 2, 3,$$

$$\|\Delta_j \mathcal{M}_4 f\|_{L^q(\Omega_\delta)} \leq C 2^{3j(\frac{1}{p} - \frac{1}{q})} 2^{js} \delta^s \|\Delta_j \mathcal{M}_4 f\|_{L^p(\Omega_\delta)}.$$

Thank you very much!