Stability of spherically symmetric stationary solutions for the compressible Navier-Stokes equation

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BASED ON JOINT RESEARCHES WITH

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$$\begin{cases} \hat{\rho}_t + \mathrm{div}(\hat{\rho} U) = 0, \\ (\hat{\rho} U)_t + \mathrm{div}(\hat{\rho} U \otimes U) + \nabla p = \nu \Delta U + (\nu + \lambda) \nabla (\mathrm{div}\, U) \end{cases} \quad t > 0,\ x \in \Omega. \qquad (1)$$

- $\hat{\rho}(t,x)$: density.
- $U(t,x) = (u_1, u_2, \cdots, u_n)(t,x)$: velocity.
- $p(\hat{\rho})$: pressure, C^1 and $p'(\hat{\rho}) > 0$.
- ν, λ : viscosity coefficients satisfying $\nu > 0$, $2\nu + n\lambda > 0$.
- spherically symmetric solution $\Rightarrow$ $\hat{\rho}$, U are function of $r = |x|$,

$$\hat{\rho}(t,x) = \rho(t,r), \quad U(t,x) = \frac{x}{r} u(t,r). \qquad (2)$$

Substituting (2) in (1) $\Rightarrow$

$$\begin{cases} (r^{n-1}\rho)_t + \left(r^{n-1}\rho u\right)_r = 0, \\ (\rho u)_t + (\rho u^2 + p(\rho))_r + (n-1)\dfrac{\rho u^2}{r} = \mu \left(\dfrac{\left(r^{n-1}u\right)_r}{r^{n-1}} \right)_r, \end{cases} \quad \mu = 2\nu + \lambda > 0. \qquad (3)$$

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- Initial and boundary conditions.

$$\begin{aligned} &(\rho(0,r),u(0,r)) = (\rho_0(r),u_0(r)) \to (\rho_+,u_+) \text{ as } r \to \infty, \\ &u(t,1) = u_b < 0 \end{aligned} \tag{4}$$

$\rho_+ > 0$, $u_+ < 0$, $u_b < 0$ are constants. As $u_b < 0$, outflow problem.

Note. $u_b > 0$: inflow problem, $u_b = 0$: impermeable problem.

One boundary condition is nesessary and sufficient for well-posedness as characteristic speed u of 1st equation is negative near boundary. Compatibility condition is assumed to hold, i.e.,

$$u_0(1) = u_b,$$

$$\left\{ -\rho_0 u_0 (u_0)_r + \mu\Big(\frac{(r^{n-1}u_0)_r}{r^{n-1}}\Big)_r - P(\rho_0)_r \right\}\Big|_{r=1} = 0.$$

As stationary solution $(\tilde{\rho}, \tilde{u})(r)$ is independent of time variable t,

$$\begin{aligned} &\left(r^{n-1}\tilde{\rho}\tilde{u}\right)_r = 0, \\ &\tilde{\rho}\tilde{u}\tilde{u}_r + p(\tilde{\rho})_r = \mu\left(\frac{\left(r^{n-1}\tilde{u}\right)_r}{r^{n-1}}\right)_r, \quad r > 1. \end{aligned} \tag{5}$$

Boundary and far field condition. (same as (4))

$$(\tilde{\rho}, \tilde{u})(r) \to (\rho_+, u_+) \text{ as } r \to \infty, \quad \tilde{u}(1) = u_b.$$

We show the solution (ρ, u) exists globally in time and converges to the stationary solution $(\tilde{\rho}, \tilde{u})$

Related results

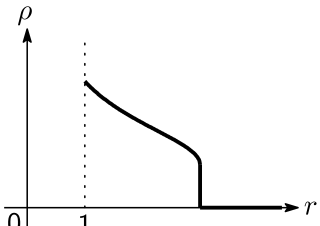
Impermeable problem on exterier domain ($n \geq 2$)

- S. Jiang (1996) $(u_+, u_b = 0)$
  - $p = R\rho\theta$.
  - Large initial data, No external force.

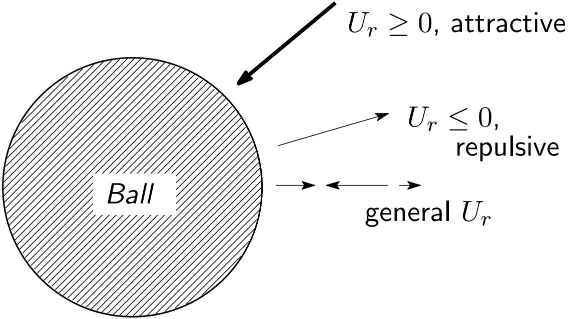
$\Rightarrow$ · Global solution.

· (Partial) asymptotics, i.e., $\lim_{t\to\infty} \|u(t)\|_{L^{2j}} = 0 \ (j = 2, 3, \dots)$ for $\mathbf{R}^3$.

- T. Makino, M. Okada (1993, 1995) + S. Matsusu-Necasova (1997) ·
  - $p = K\rho^\gamma$.
  - Initial finite Mass $\Rightarrow$ Vacuum.
  - Gravitational force. $\Rightarrow$ $^\exists$ Stationary solution with compact support.
  - $\gamma > 4/3, \ 0 < K \ll 1 \ \Rightarrow$ Stationary solution is stable.



- Nakamura-Nishibata-Yanagi (2004) ($u_b = 0$:Impermeable)
  Existence and asymptotic stability of stationary solution to the system with external potential forces for large initial data.
- Nakamura-Nishibata (2004) ($u_b = 0$)
  Same results as above for heat conductive model.
  If $U_r(r) \geq 0$, it can be arbitrary large.



II. Existence and property of stationary solution

Existence of the stationary solution is proved by Hashimoto-Matsumura.

Integrating 1st equation in (5) over $[1, r]$,

$$r^{-n+1}\tilde{\rho}(r)\tilde{u}(r) = \tilde{\rho}(1)u_b, \quad \tilde{u}(r) = \frac{\tilde{\rho}(1)u_b}{\tilde{\rho}(r)}r^{-n+1}.$$

It is necessary to assume $u_+ = 0$ as $u(r) \to u_+ = 0$ $(r \to \infty)$.

Substituting $\eta(r) := \dfrac{\tilde{u}(r)}{\rho(1)u_b}r^{n-1} - \dfrac{1}{\rho_+}$ in 2nd equation of (5),

$$\tilde{\rho}(1)u_b\mu\left(\frac{\eta_r(r)}{r^{n-1}}\right)_r = p(v_+ + \eta(r))_r + \frac{\tilde{\rho}^2(1)u_b^2 v_+}{2}\left(\frac{1}{r^{2(n-1)}}\right)_r + \frac{\tilde{\rho}^2(1)u_b^2}{r^{n-1}}\left(\frac{\eta(r)}{r^{n-1}}\right)_r,$$

Integrating,

$$\tilde{\rho}(1)u_b\mu\frac{\eta_r(r)}{r^{n-1}} - p(v_+ + \eta(r)) - \frac{\tilde{\rho}(1)^2u_b^2v_+}{2r^{2(n-1)}} - \frac{\tilde{\rho}(1)^2u_b^2\eta(r)}{r^{2(n-1)}} + \\ + \tilde{\rho}(1)^2u_b^2(n-1)\int_r^\infty \frac{\eta(s)}{s^{2n-1}}ds = C_0 \ : \text{constant}.$$

$$\lim_{r\to\infty}\frac{\eta_r(r)}{r^{n-1}} = \frac{1}{\tilde{\rho}(1)u_b\mu}(C_0 + p(v_+)) = 0 \quad \Longrightarrow \quad C_0 = -p(v_+)$$

Hence we have

$$\eta_r = \frac{r^{n-1}}{\varepsilon\mu}\left(P\left(v_+ + \eta\right) - P\left(v_+\right)\right) + \frac{\varepsilon v_+}{2\mu}\frac{1}{r^{n-1}} + \frac{\varepsilon\eta}{\mu r^{n-1}} - \frac{\varepsilon(n-1)r^{n-1}}{\mu}\int_r^\infty \frac{\eta(s)}{s^{2n-1}}ds, \quad r > 1, \qquad (6)$$

$$\lim_{r\to\infty}\eta(r) = 0,$$

where $\varepsilon := \tilde{\rho}(1)u_b$, $v_+ := \dfrac{1}{\rho_+}$, $P(v) := p\left(\dfrac{1}{v}\right)$

Some coefficients in (6) diverge as $r \to \infty$

$\Longrightarrow$ (6) is a singular differential integral equation.

By iteration method, it is proved the unieque existence of the stationary solution η in the space

$$Y = \{\eta \in C([1,\infty)); \|\eta\|_Y < \infty\}, \quad \|\eta\|_Y = \sup_{r\geq 1}|r^{2(n-1)}\eta(r)|.$$

Theorem 1.(Hashimoto-Matsumura)
(Unique existence of stationary solution)

Let $|u_b| \ll 1$, the stationary solution to (6) exists uniquely in

$$S_M = \left\{ \eta \in C([1,\infty)) \ ; \ \sup_{r \geqq 1} \left| r^{2(n-1)} \eta(r) \right| \leq M \right\}.$$

It satisfies

$$|\eta(r)| \leq C r^{-2n+2} |u_b|^2,$$

where M is a positive constant depending on $|u_b|$ and
C is a positive constant independent of r, $|u_b|$.

- $M \sim C|u_b|^2$
- Then we derive
  the properties and the decay rates of η_r and η_{rr}.

- Existence of $\eta(r)$ $\Rightarrow$ Existence of stationary solution $(\tilde{\rho}, \tilde{u})(r)$ to (3), (4): outflow problem $\cdots$ $u_b < 0$ as

$$(\tilde{\rho}, \tilde{u})(r) = \left(\frac{1}{\eta(r) + v_+}, \frac{u_b \left(v_+ + \eta(r) \right)}{\left(v_+ + \eta(1) \right) r^{n-1}} \right)$$

- $(\tilde{\rho}, \tilde{u})(r) \to (\rho_+, 0)$ as $r \to \infty$
- Convergent rates of 1st and 2nd order derivatives of $(\tilde{\rho}, \tilde{u})$ as $r \to \infty$ $\Rightarrow$ Asymptotic stability of stationary solution $(\tilde{\rho}, \tilde{u})$.

Theorem 2 (I.Hashimoto, S.Sugizaki, S.N.)

Let $|u_b| \ll 1$. $\eta(r)$ satisfies

$$\begin{array}{ll} \eta_r(r) < 0, \quad |\eta_r| \le Cr^{-2n+1}|u_b|^2 \\ \eta_{rr}(r) > 0, \quad |\eta_{rr}| \le Cr^{-2n}|u_b|^2, \end{array} \qquad r \ge 1$$

C is a positive constant independent of r and $|u_b|$.

- Convergence rate plays essential role in asymptotic ananysis.
- $|\eta(r)| \le Cr^{-2n+2}|u_b|^2$.
  These decay rates seem reasonable since standard O.D.E. satisfies
  $\partial_x u = f(u)$, $|u| \le C|x|^{-2n+2} \Longrightarrow |\partial_x^i u| \le C|x|^{-2n+2-i}$ for $i = 1, 2, \cdots$.

- Stationary solution $(\tilde{\rho}, \tilde{u})(r)$ is given by
  $(\tilde{\rho}, \tilde{u})(r) = \left(\dfrac{1}{\eta(r) + v_+}, \dfrac{u_b\,(v_+ + \eta(r))}{(v_+ + \eta(1))\, r^{n-1}} \right)$
  $\to$ [Theorem 2] yields the property of $(\tilde{\rho}, \tilde{u})$.

Theorem 3 ($(\tilde{\rho}, \tilde{u})(r)$ $(r \to \infty)$ decay rate, sign)
$\cdots$(I.Hashimoto, S.Sugizaki, S.N.)

Let $|u_b| \ll 1$. For $r \geq 1$,

$$\tilde{u}_r(r) > 0, \quad |\tilde{u}_r| \leq Cr^{-n}|u_b|,$$
$$\tilde{u}_{rr}(r) < 0, \quad |\tilde{u}_{rr}| \leq Cr^{-n-1}|u_b|.$$
$$|\tilde{\rho}_r(r)| \leq Cr^{-2n+1}|u_b|^2, \quad |\tilde{\rho}_{rr}(r)| \leq Cr^{-2n}|u_b|^2$$

where C is a positive constant independent of r, $|u_b|$.

- Decay rates of stationary solution in Theorem 3
  $\Rightarrow$ Asymptotic stability of the stationary solution.

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III. Asymptotic stability with small initial data

Theorem 4, Asymptotic stability (I.Hashimoto, S.Sugizaki, S.N.)

Let $\rho_+ > 0$ and $u_b < 0$. Suppose $|u_b| \ll 1$ and initial data (ρ_0, u_0) satisfy

$\rho_0 \in \mathcal{B}^{1+\sigma}[1,\infty)$, $u_0 \in \mathcal{B}^{2+\sigma}[1,\infty)$ for $0 < {}^{\exists}\sigma < 1$,

$\|r^{\frac{n-1}{2}}(\rho_0 - \tilde{\rho}, u_0 - \tilde{u})\|_1 \ll 1$.

$\Rightarrow$ (3), (4) has a time global solution (ρ, u). Namely,

for ${}^{\forall}T > 0$ $\quad (\rho, u) \in \mathcal{B}^{1+\sigma/2,1+\sigma} \times \mathcal{B}^{1+\sigma/2,2+\sigma}([0,T] \times [1,\infty))$,

$(\rho - \tilde{\rho}, u - \tilde{u}) \in C([0,\infty); H^1([1,\infty)))$, converging to $(\tilde{\rho}, \tilde{u})$, i.e.,

$$\lim_{t\to\infty} \sup_{r>1} |(\rho, u)(t,r) - (\tilde{\rho}, \tilde{u})(r)| = 0.$$

$\|\cdot\|_1$: H^1 Sobolev norm

Höder space

$f \in \mathcal{B}^{k+\alpha}(U) \Longleftrightarrow |f|_{\mathcal{B}^{k+\alpha}(U)} < \infty.$

$$|f|_{\mathcal{B}^{k+\alpha}(U)} = \sum_{i=0,\cdots k} \sup_{x\in U} |\partial_x^i f(x)| + \sup_{x,y\in U, x\neq y} \frac{|\partial_x^k f(x) - \partial_x^k u(y)|}{|x-y|^\alpha}$$

Proposition 1 : A priori estimate

Under the same assumptions as in Theorem 4, we have.

$$\|r^{\frac{n-1}{2}}(\phi,\psi)(t)\|_1^2 + \int_0^t \|r^{\frac{n-1}{2}}\phi_r(\tau)\|^2 + \|r^{\frac{n-1}{2}}\psi_r(\tau)\|_1^2 + \\ + |\phi(\tau,1)|^2 d\tau \le C\|r^{\frac{n-1}{2}}(\phi_0,\psi_0)\|_1^2$$

where C is a positive constant independent of initial data.

$\|\cdot\|_1$: H^1 Sobolev norm
$\|\cdot\|$: L^2 norm.

- Outline of the proof of Theorem 4.
  - Time local solution in Hölder space in Eularian coordinate.

(Ref:Tani.1977)

Note. To construct solution in H^1 Sobolev space is difficult.

Rewriting the equations in Eulerian coordinate to Lagrangian coordinate.

Derive the estimate in Hölder norm from a priori H^1 estimate with the aid of Schauder theory for parabolic equations.

$\Rightarrow$ Time global solution in Lagrangian coordinate

with moving boundary

$\Rightarrow$ Time global solution in Eularian coordinate. Namely for $^{\forall}T > 0$,

$$r^{\frac{n-1}{2}}(\rho-\tilde{\rho}), r^{\frac{n-1}{2}}(u-\tilde{u}, r^{\frac{n-1}{2}}(\rho-\tilde{\rho})_r, r^{\frac{n-1}{2}}(u-\tilde{u})_r \in C([0,T]; L^2(1,\infty$$

$$\rho \in \mathcal{B}^{1+\frac{\sigma}{2},1+\sigma}([0,T]\times[1,\infty)), u \in \mathcal{B}^{1+\frac{\sigma}{2},2+\sigma}([0,T]\times[1,\infty))$$

(Ref. Nakamura-Nishibata-Yanagi.2004, Kawashima-Nishibata-P.Zhu.2003)

H^1 a priori estimate yields the asymptotic stability

$$\lim_{t\to\infty}\sup_{r>1}|(\rho,u)(t,r)-(\tilde{\rho},\tilde{u})(r)| = 0.$$

H^1 a priori estimate

- From (3), (5), Perturbation $(\phi, \psi)(t, r) := (\rho(t, r) - \tilde{\rho}(r), u(t, r) - \tilde{u}(r))$ satisfies

$$\begin{cases} \phi_t + u\phi_r + \rho\psi_r = \mathcal{F}, \\ \rho(\psi_t + u\psi_r) + P'(\rho)\phi_r - \mu\psi_{rr} = \mathcal{G} \end{cases} \tag{7}$$

$$\mathcal{F} := -\tilde{\rho}_r\psi - \tilde{u}_r\phi - \frac{n-1}{r}(\phi u + \tilde{\rho}\psi)$$

$$\mathcal{G} := -(\phi\psi + \tilde{u}\phi + \tilde{\rho}\psi)\tilde{u}_r - (P'(\rho) - P'(\tilde{\rho}))\tilde{\rho}_r + \mu(n-1)(\frac{\psi}{r})_r$$

- Initial and boundary data

$$\phi(0, r) = \phi_0(r) := \rho_0(r) - \tilde{\rho}(r),$$
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- Define energy form $\mathcal{E}$ by

$$\mathcal{E} = \frac{1}{2}(u - \tilde{u})^2 + \int_{\tilde{\rho}}^{\rho} \frac{p(y) - p(\tilde{\rho})}{y^2} dy \sim |(\phi, \psi)|^2$$

if $|(\phi, \psi)|$ is small.
$\mathcal{E}$ satisfies

$$\begin{aligned}
&(\rho\mathcal{E})_t + \left\{ \rho u \mathcal{E} + (P(\rho) - P(\tilde{\rho}))\psi - \mu\psi\psi_r - \mu(n-1)\frac{\psi^2}{2r} \right\}_r \\
&\quad + \mu\psi_r^2 + \frac{n-1}{2}\left(\frac{\rho u}{r}\psi^2 + \mu\frac{\psi^2}{r^2}\right) \\
&\quad + (n-1)\left\{ \frac{\rho u}{r} \int_{\tilde{\rho}}^{\rho} \frac{P(\eta) - P(\tilde{\rho})}{\eta^2} d\eta + \frac{P(\rho) - P(\tilde{\rho})}{r\rho}(\phi u + \tilde{\rho}\psi) \right\} \\
&\quad + (n-1)\frac{\tilde{\rho}\tilde{u}}{r}\left(\frac{P(\rho) - P(\tilde{\rho})}{\rho} - P'(\tilde{\rho})\frac{\rho - \tilde{\rho}}{\tilde{\rho}} \right) \\
&= -\left(\rho\psi^2 + P(\rho) - P(\tilde{\rho}) - P'(\tilde{\rho})(\rho - \tilde{\rho})\right)\tilde{u}_r - \frac{\mu}{\tilde{\rho}}\phi\psi\left(\frac{(r^{n-1}\tilde{u})_r}{r^{n-1}}\right)_r.
\end{aligned} \tag{8}$$

We cannot have estimate by integrating above. $\Rightarrow$ Weight function.

Multiplying (8) by r^{n-1}, we have

$$\begin{aligned}
&(r^{n-1}\rho\mathcal{E})_t + \left[r^{n-1}\left\{\rho u\mathcal{E} + (P(\rho)-P(\tilde{\rho}))\psi - \mu\psi\psi_r - \mu(n-1)\frac{\psi^2}{2r}\right\}\right]_r \\
&\quad + \frac{1}{2}\mu(n-1)(r^{n-2}\psi^2)_r + \mu(n-1)r^{n-3}\psi^2 + \mu r^{n-1}\psi_r^2 + r^{n-1}\rho\tilde{u}_r\psi^2 \qquad (9) \\
&= -(r^{n-1}\tilde{u})_r\left(P(\rho)-P(\tilde{\rho})-P'(\tilde{\rho})\phi\right) - \mu\tilde{v}\phi\psi r^{n-1}\left(\frac{(r^{n-1}\tilde{u})_r}{r^{n-1}}\right)_r .
\end{aligned}$$

Note. $\psi(1,t)=0$. $\rho u\mathcal{E}(1,t) \sim \rho(1,t)u_b\phi^2(1,t) < 0$

$\Rightarrow$ Integration of the 2nd term yields good term.

$\tilde{u}_r > 0 \Rightarrow$ the last term in L.H.S. is good term.

Remark. If f is symmetric, $\int\cdots\int_{|r|>1} f \sim \int_1^\infty r^{n-1} f$

- Integrating (9) over $[1,\infty)\times[0,t]$.

$$
\begin{aligned}
&\int_1^\infty r^{n-1}\phi^2(t,r)+r^{n-1}\psi^2(t,r)dr+|u_b|\int_0^t \phi(\tau,1)^2 d\tau \\
&\quad+\int_0^t\int_1^\infty r^{n-3}\psi^2+r^{n-1}\psi_r^2+|u_b|r^{-1}\psi^2 dr d\tau \\
&\le C\int_1^\infty r^{n-1}\phi_0^2(r)+r^{n-1}\psi_0^2(r)dr \\
&+\int_0^t\int_1^\infty (r^{n-1}\tilde{u})_r\left(P(\rho)-P(\tilde{\rho})-P'(\tilde{\rho})(\rho-\tilde{\rho})\right)drd\tau \\
&-\int_0^t\int_1^\infty \frac{\mu}{\tilde{\rho}}\phi\psi r^{n-1}\left(\frac{(r^{n-1}\tilde{u})_r}{r^{n-1}}\right)_r drd\tau
\end{aligned}
$$

- The 3rd term in (R.H.S.) is estimated by using [Theorem 2]

$$|(r^{n-1}\tilde{u})_r| = |\frac{(r^{n-1}\tilde{u})}{\tilde{v}^2\tilde{\rho}}\eta_r| \le C|u_b|^3 r^{-2n+1}$$

Integration by parts of 3rd term yields

$$\begin{aligned}
&\int_0^t \int_1^\infty |(\tilde{v}\phi\psi r^{n-1})_r \frac{(r^{n-1}\tilde{u})_r}{r^{n-1}}| dr d\tau \\
&\le C|u_b|^3 \left\{ \int_0^t \|\phi_r\|^2 + \|\psi_r\|^2 d\tau + \int_0^t \int_1^\infty r^{-2n}(\phi^2 + \psi^2) dr d\tau \right\} \\
&\le C|u_b|^3 \int_0^t (\phi^2(1,\tau) + \|\phi_r^2\| + \|\psi_r^2\|) d\tau \qquad \text{as } \psi^2(1,\tau) = 0,
\end{aligned}$$

where we have used the poincaré type inequality with $2n \ge 4$ $(n \ge 2)$.

- Since $f(r) = f(1) + \int_1^r f_r(r)dr$, $|f(r)| \le |f(1)| + \sqrt{r-1}\|f_r\|_{L^2}$ $(k \ge 3) \Rightarrow$

$$\int_1^\infty r^{-k} f^2 dr \le C \int_1^\infty r^{-k}(|f(1)|^2 + (r-1)\|f_r\|_{L^2}^2) dr \le C(|f(1)|^2 + \|f_r\|_{L^2}).$$

- The other terms in (R.H.S.) are handled similarly.

- Basic estimate of (ϕ, ψ)

$$\begin{aligned}
&\|(r^{\frac{n-1}{2}}\phi, r^{\frac{n-1}{2}}\psi)(t)\|^2 + |u_b| \int_0^t \phi(\tau,1)^2 d\tau \\
&\quad + \int_0^t \int_1^\infty r^{n-3}\psi^2 + r^{n-1}\psi_r^2 + |u_b| r^{-1}\psi^2 dr d\tau \\
&\le C\|(r^{\frac{n-1}{2}}\phi_0, r^{\frac{n-1}{2}}\psi_0)\|^2 + C|u_b|^3 \int_0^t \|\phi_r(\tau)\|^2 d\tau.
\end{aligned} \tag{10}$$

The last term is handled by the estimate of higher derivatives by using $|u_b| \ll 1$.

- Higher order estimate of ϕ

$$\begin{cases} \phi_t + u\phi_r + \rho\psi_r = \mathcal{F}, \\ \rho(\psi_t + u\psi_r) + P'(\rho)\phi_r - \mu\psi_{rr} = \mathcal{G} \end{cases} \tag{7}$$

• We have to use difference quatient in place of ∂_r

$$\varphi_h = \varphi_h(r) := \frac{\varphi(r+h) - \varphi(r)}{h},$$

for $h > 0$. After driving the nesessary estimates, we let $h \to 0$.
Here we avoid these (tedious) discussions.

• 1st step. Differentiate 1st equation in (7) by r and multiply $\mu\phi_r$.
2nd step. Multiply 2nd equation in (7) by $\rho\phi_r$ and add to above.

• Multipl r^{n-1} and integrate over $[0,\infty) \times [0,t]$.

$$
\begin{aligned}
&\int_1^\infty r^{n-1}\phi_r^2 dr + |u_b| \int_0^t \phi_r(\tau,1)^2 d\tau \\
&\quad + \int_0^t \int_1^\infty \left\{ |u_b| \frac{\phi_r^2}{r} + r^{n-1}\phi_r^2 + r^{n-1}|\tilde{u}_r|\phi_r^2 \right\} dr d\tau \\
&\leq C \int_1^\infty r^{n-1}\phi_0^2(r) + r^{n-1}\psi_0^2(r) dr \\
&\quad + \int_1^\infty r^{n-1}\phi_{0\,r}^2(r) dr + C|u_b|^3 \int_0^t \|\phi_r(\tau)\|^2 d\tau \\
&\quad + N(t) \int_0^t |u_b|^2 \phi(\tau,1)^2 d\tau + CN(t) \int_0^t \|\psi_r(\tau)\|_1^2 d\tau \\
&\quad + C(|u_b| + N(t)) \int_0^t \int_1^\infty r^{n-1}\phi_r^2 + r^{n-3}\psi^2 + r^{n-1}\psi_r^2 dr d\tau,
\end{aligned}
\tag{11}
$$

where $N(t) := \sup_{0\le\tau\le t} \|(\phi,\psi)\|_1^2(\tau)$.

- <u>Higher order estimate of ψ</u>
  - Multiply 2nd equation in (7) by $-\psi_{rr}/\rho$.
    Integrate the result over $[0,\infty) \times [0,t]$

$$
\begin{aligned}
&\int_1^\infty r^{n-1}\psi_r^2 dr + \int_0^t \int_1^\infty r^{n-1}\psi_{rr}^2 dr d\tau \\
&\le C\int_1^\infty r^{n-1}\phi_0^2(r) + r^{n-1}\psi_0^2(r) dr \\
&\quad + C\int_1^\infty r^{n-1}\phi_{0\,r}^2(r) + r^{n-1}\psi_{0\,r}^2(r) dr \\
&\quad + C|u_b|^3 \int_0^t \|\phi_r(\tau)\|^2 d\tau \\
&\quad + C(|u_b| + N(t)) \int_0^t \int_1^\infty r^{n-1}\phi_r^2 + r^{n-3}\psi^2 + r^{n-1}\psi_r^2 dr d\tau \\
&\quad + CN(t)\int_0^t r^{n-1}\psi_{rr}^2 d\tau,
\end{aligned}
\tag{12}
$$

Multiplying suitable constants on
(10) (Basic estimate),
(11) (Higher order estimate of ϕ),
(12) (Higher order estimate of ψ) respectively, summing up,
and then using the smallness, i.e., $|u_b| \ll 1$, $N(t) \ll 1$,
we have H^1 a priori estimate.

$$\|r^{\frac{n-1}{2}}(\phi,\psi)(t)\|_1^2 + \int_0^t \|r^{\frac{n-1}{2}}\phi_r(\tau)\|^2 + \|r^{\frac{n-1}{2}}\psi_r(\tau)\|_1^2 + \\ + |\phi(\tau,1)|^2 d\tau \le C\|r^{\frac{n-1}{2}}(\phi_0,\psi_0)\|_1^2.$$

Thus under the assumption that initail data $\|(\phi_0,\psi_0)\|$ is small,
the asymptotic stability of the stationary solution holds.
In the proof, the positivity of ρ holds as

$$\sup_{r\ge 1}|(\phi,\psi)(t)| \le N(t) \ll 1 \quad \Rightarrow \rho = \tilde{\rho} + \phi \fallingdotseq \tilde{\rho} > 0.$$

Difficulty to handle large initial data.
$\cdots$ Positivity of ρ for large initial data?
Point wise estime of ρ by representation formula in Lagrangian coodinate.

IV. Inflow problem

Initial and boundary condition.

$$(\rho(0,r),u(0,r)) = (\rho_0(r),u_0(r)) \to (\rho_+,0) \text{ as } r\to\infty,$$
$$(\rho(t,1),u(t,1)) = (\rho_b,u_b).$$

ρ_+, ρ_b, u_b are positive constants. Compatibility condition is assumed. $u_b > 0$ $\Longrightarrow$ inflow problem.

•Two boundary conditions is nesessary and sufficient for well-posedness as characteristic speed u of 1st equation is positive near boundary.

• Existence of stationary solution $(\tilde\rho,\tilde u)$ $\cdots$ A.Matsumura, I.Hashimoto.

• Convergence rate $\cdots$ S.Sugizaki and S.N.

Theorem 5, Asymptotic stability (Y. Huang, S.N.)

$|(\rho_b-\rho_+,u_b)| \ll 1$ and $\rho_0\in\mathcal{B}^{1+\sigma}[\mathbf{R}_+)$, $u_0\in\mathcal{B}^{2+\sigma}[\mathbf{R}_+)$ for $0<{}^\exists\sigma<1$,
$\|r^{\frac{n-1}{2}}(\rho_0-\tilde\rho,u_0-\tilde u)\|_1 \ll 1$. $\Rightarrow$ a global solution, ${}^\forall T>0$
$(\rho,u)\in\mathcal{B}^{1+\sigma/2,1+\sigma}\times\mathcal{B}^{1+\sigma/2,2+\sigma}([0,T]\times[1,\infty))$,
$(\rho-\tilde\rho,u-\tilde u)\in C([0,\infty);H^1(\mathbf{R}_+))$, converging to $(\tilde\rho,\tilde u)$, i.e.,
$$\lim_{t\to\infty}\sup_{r>1}|(\rho,u)(t,r)-(\tilde\rho,\tilde u)(r)| = 0.$$

(Proof.) By using Lagrangian coodinate, not Euilarian.

V. Asymptotic stability for large initial data

Theorem 5 Outfow problem (Y.Huang, S.N.)

Let $p(\rho) = k\rho^\gamma$ ($1 \le \gamma \le 2$, adiabatic constant), $0 < {}^\exists c < \rho_0$,

$$r^{\frac{n-1}{2}}(\rho_0 - \tilde{\rho}, u_0 - \tilde{u}) \in H^1(1,\infty), \quad \rho_0 \in \mathcal{B}^{1+\sigma}_{\text{loc}}[1,\infty),\ u_0 \in \mathcal{B}^{2+\sigma}_{\text{loc}}[1,\infty)$$

for $0 < {}^\exists\sigma < 1$. $|u_b| \ll 1$

$\Longrightarrow {}^{\exists 1}$ time global solution $(\rho, u)(r,t)$;

$$\lim_{t\to\infty} \sup_{r\in[1,\infty)} |(\rho(r,t) - \tilde{\rho}(r), u(r,t) - \tilde{u}(r))| = 0.$$

† Smallness of initial data is NOT necessary.

$$\mathcal{B}^{m+\sigma}_{\text{loc}}[1,\infty) := \{f \in \mathcal{B}^{m+\sigma}[a,b]\ ;\ {}^\forall[a,b] \subset [1,\infty)\},$$

$$\mathcal{B}^{m+\sigma}[a,b] := \left\{ f \in C^m[a,b]\ ;\ \frac{|f^{(m)}(x) - f^{(m)}(y)|}{|x-y|^\sigma} < \infty \right\}.$$

- Difficulty: To show positivity of density $0 < \underline{\rho} < \rho < \overline{\rho} < \infty$

  $\Rightarrow$ Representation formula of density ρ in Lagrangian coordinate.

Outline of proof for large initial data.

1. $(\rho(r,t), u(r,t))$ in Eulerian coordinate
   $\Downarrow$ transformation
   $(v(x,t), u(x,t))$ in Lagrangian coordinate.
2. Time local solution over $(0,\infty)$ by using cut off function in space.
   (c.f., [Kazhikhov 1981], [S. Jiang 1996], [T.Nakamura, S.N.])
3. Energy estimate.
   Pointwise estimate $0 < \underline{v} \le v(x,t) \le \overline{v}$. $\Rightarrow$ A priori estimate in H^1
4. Apply Schauder theory for parabolic equations.
   $\Rightarrow$ Hölder estimate.
   $\Rightarrow$ Time global solution, Asymptotic state in Eulerian coordinate,
   i.e., $(\rho, u) \to (\tilde{\rho}, \tilde{u}) \quad (t \to \infty)$.

Lagrangian mass coordinate

Transformation $(r,t) \to (x,t)$ given by

$$x = B(t) + \int_1^r \xi^{n-1}\rho(\xi,t)\,d\xi, \quad B(t) := -u_b \int_0^t \rho(1,s)\,ds \tag{T}$$

$$r_t = u, \quad r_x = \frac{v}{r^{n-1}}. \quad (v := \frac{1}{\rho} : \text{ specific volume})$$

(E) $\Longrightarrow$

$$v_t - (r^{n-1}u)_x = 0, \quad x > 0,\ t > 0, \tag{L.a}$$

$$u_t + r^{n-1}p_x = \mu r^{n-1}\left(\frac{(r^{n-1}u)_x}{v}\right)_x. \tag{L.b}$$

· Initial data

$$(v,u)(x,0) = (v_0,u_0)(x). \quad (v_0(x) := 1/\rho_0(r(x,0)))$$

· Boundary data

$$u(B(t),t) = u_b.$$

† $\{r > 1\} \ \mapsto\ \{x > B(t)\}$. $B'(t) > 0$ as $u_b < 0$ and $\rho > 0$.

† Stationary solution $(\tilde\rho,\tilde u)(r)$ is not stationary in Lagarangian coordinate (x,t) as $r = r(x,t)$, defined by inverse of (T).

- Difficulty in Lagrangian coordinate

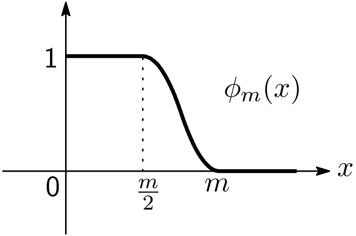
$$r(x,t) \to \infty \ \text{ as } \ x \to \infty.$$

$$\Downarrow$$

Coefficients in (L) are unbounded in $x > 0$.

$$\Downarrow$$

Need to use "cut off function" $B(t) + \phi_m(x)$.



$$\begin{cases} \phi_m(x) = 1 & \text{for } x \in [0, m/2], \\ 0 \le \phi_m(x) \le 1 & \text{for } x \in (m/2, m), \\ \phi_m(x) = 0 & \text{for } x \in [m, \infty). \end{cases}$$

$$\phi_m \in C^3[0,\infty), \quad |\partial_x^i \phi_m| \le C/m^i \text{ for } i = 1, 2, 3.$$

(c.f., [S. Jiang 1996], [T.Nakamura, S.N. Yanagi 2004])

After deriving estimates in $H^1(0,m)$, we let $m \to \infty$.

Energy form $\mathcal{E}_L$ defined by

$$\mathcal{E}_L := \frac{1}{2}\psi^2 + G(v,\tilde{v}),$$

$$G(v,\tilde{v}) := \int_{\tilde{v}^{-1}}^{v^{-1}} \frac{p(z^{-1}) - p(\tilde{v})}{z^2}\,\mathrm{d}z = \tilde{v}p(\tilde{v})g\Big(\frac{v}{\tilde{v}}\Big), \quad v = \frac{1}{v},\ \tilde{v} := 1\tilde{\rho}$$

$$g(s) := s - 1 - \int_1^s \eta^{-\gamma}\,d\eta \geq s - 1 - \log s\ (s > 0).$$

Let $(\phi,\psi)(x,t) := (v(x,t) - \tilde{v}(r(x,t)), u(x,t) - \tilde{u}(r(x,t))$.

Note. $(\hat{\phi},\hat{\psi})(r,t) := (\rho(r,t) - \tilde{\rho}(r), u(r,t) - \tilde{u}(r))$.

Multiplying (L.b) by ψ and using (L.a), we have the equation for $\mathcal{E}_L$,

$$\mathcal{E}_{Lt} + \mu\frac{(r^{n-1}\psi)_x^2}{v} + (\gamma-1)\tilde{\rho}(1)|u_b|\frac{\partial_r\tilde{\rho}}{r^{n-1}\tilde{\rho}^2}G(v,\tilde{v}) + \partial_r\tilde{u}|\psi|^2$$
$$= \Big\{\Big(\mu\frac{(r^{n-1}\psi)_x}{v} + p(\tilde{v}) - p(v)\Big)r^{n-1}\psi\Big\}_x + \tilde{L}\psi\phi, \quad \text{(E)}$$

where $\tilde{L} := \mu\partial_r(r^{1-n}\partial_r(r^{n-1}\tilde{u}))$.

For simplicity, assume perturbation (ϕ, ψ) in Lagrangian coodinate satisfy

$$\phi, \psi, r^{n-1}\phi_x, r^{n-1}\psi_x \in C([0,T] : L^2(0,\infty)),$$

$\Longleftrightarrow$ perturbation $(\hat{\phi}, \hat{\psi})$ in Eularian coodinate satisfy

$$r^{\frac{r-1}{2}}(\hat{\phi}, \hat{\psi}) \in C([0,T] : H^1(1,\infty)).$$

Integrate (E) over $\mathcal{L}(T) := \{(x,t) \in \mathbf{R} \times [0,T] \mid B(t) \le x\}$.

- As $\tilde{\rho}_r > 0$, $\tilde{u}_r > 0$, (L.H.S.) yields good terms.
- 1st term in (R.H.S.) disappears after integration as $\psi(B(t),t) = 0$.

$$\int_{B(t)}^{\infty} \mathcal{E}_L(x,t)\,\mathrm{d}x + \mu \iint_{\mathcal{L}(T)} \left\{ \frac{n-1}{2} \frac{v\psi^2}{r^2} + \frac{r^{2n-2}}{v}\psi_x^2 \right\} \mathrm{d}x\mathrm{d}t + |u_b| \int_0^T \frac{G(v,\tilde{v})}{v}(B(t),t)\,\mathrm{d}t$$
$$+ c \iint_{\mathcal{L}(T)} \left\{ |u_b| \frac{\psi^2}{r^n} + |u_b|^3 \frac{G(v,\tilde{v})}{r^{3n-2}} \right\} \mathrm{d}x\mathrm{d}t \le \int_0^{\infty} \mathcal{E}_L(x,0)\,\mathrm{d}x + \iint_{\mathcal{L}(T)} \tilde{L}\psi\phi dx$$

- The last integration is handled as follows.

$$\left| \iint_{\mathcal{L}(T)} \tilde{L}\psi\phi dx \right| \le C|u_b|^3 \iint_{\mathcal{L}(T)} \frac{|\phi| \cdot |\psi|}{r^{3n-1}} dx \le \iint_{v \le \tilde{v}} \bullet\, dx + \iint_{\tilde{v} < v} \bullet\, dx,$$

as $|\tilde{L}| = |\mu \partial_r (r^{1-n} \partial_r (r^{n-1}\tilde{u}))| \le C|u_b|^3 r^{-3n+1}$.

Lemma

If $1 \le \gamma \le 2$, then for v, $\tilde{v} \ge 0$,

$$\frac{\gamma K}{2}|\phi|^2 = \frac{\gamma K}{2}|v-\tilde{v}|^2 \le \begin{cases} \tilde{v}^{1+\gamma} G(v,\tilde{v}) & \text{if } v \le \tilde{v}, \\ \tilde{v}^{\gamma} v G(v,\tilde{v}) & \text{if } \tilde{v} < v. \end{cases}$$

For $v \le \tilde{v}$, it holds

$$|u_b|^3 \int_{v \le \tilde{v}} \frac{|\phi| \cdot |\psi|}{r^{3n-1}} \, \mathrm{d}x \le C|u_b|^3 \int_{v \le \tilde{v}} \frac{|\psi|}{r^{3n-1}} \sqrt{\tilde{v}^{1+\gamma} G(v,\tilde{v})} \, \mathrm{d}x$$
$$\le C|u_b|^3 \int_{B(t)}^{\infty} \frac{G(v,\tilde{v})}{r^{3n-2}} \, \mathrm{d}x + C|u_b| \int_{B(t)}^{\infty} \frac{\psi^2}{r^n} \, \mathrm{d}x,$$

For $\tilde{v} \le v$, it holds

$$|u_b|^3 \int_{v > \tilde{v}} \frac{|\phi| \cdot |\psi|}{r^{3n-1}} \, \mathrm{d}x \le C|u_b|^3 \int_{S(t) \cap \{v > \tilde{v}\}} \frac{|\psi|}{r^{3n-1}} \sqrt{\tilde{v}^{\gamma} v G(v,\tilde{v})} \, \mathrm{d}x$$
$$\le \frac{(n-1)\mu}{2} \int_{S(t)} \frac{v\psi^2}{r^2} \, \mathrm{d}x + C|u_b|^3 \int_{B(t)}^{\infty} \frac{G(v,\tilde{v})}{r^{6n-4}} \, \mathrm{d}x,$$

These terms are absored in (L.H.S.) by taking $|u_b| \ll 1$,

$$\int_{B(t)}^{\infty} \mathcal{E}_L(x,t)\,\mathrm{d}x + \mu \iint_{\mathcal{L}(T)} \Big\{ \frac{n-1}{2}\frac{v\psi^2}{r^2} + \frac{r^{2n-2}}{v}\psi_x^2 \Big\} \mathrm{d}x\mathrm{d}t +$$

$$+ |u_b| \int_0^T \frac{G(v,\tilde{v})}{v}(B(t),t)\,\mathrm{d}t + c \iint_{\mathcal{L}(T)} \Big\{ |u_b| \frac{\psi^2}{r^n} + |u_b|^3 \frac{G(v,\tilde{v})}{r^{3n-2}} \Big\} \mathrm{d}x\mathrm{d}t$$

$$\le \int_0^{\infty} \mathcal{E}_L(x,0)\,\mathrm{d}x, \quad \text{(E)}$$

(L.H.S.) contains $v(x,t)$, which may be zero eventhough $0 < v_0(x)$.
We derive $c_0 \le v(x,t) \le C_0$, c_0 C_0 constant depending on (v_0, u_0).
Once it is shown, we have L^2 estimate as $\mathcal{E}_L \sim \phi^2 + \psi^2$

$$\int_{B(t)}^{\infty} (\phi^2, \psi^2)(x,t)\,\mathrm{d}x + \iint_{\mathcal{L}(T)} \frac{\psi^2}{r^2} + r^{2n-2}\psi_x^2 \mathrm{d}x\mathrm{d}t + \int_0^T \phi^2(B(t),t)\,\mathrm{d}t +$$

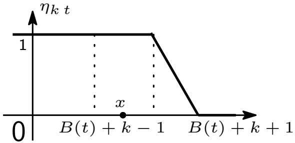
$$+ \iint_{\mathcal{L}(T)} \Big\{ \frac{\psi^2}{r^n} + \frac{\phi^2}{r^{3n-2}} \Big\} \mathrm{d}x\mathrm{d}t \le C \int_0^{\infty} (\phi_0, \psi_0)(x)\,\mathrm{d}x.$$

Representation formula of density

For $k = 1, 2, \dots,$ define "cut off function" by

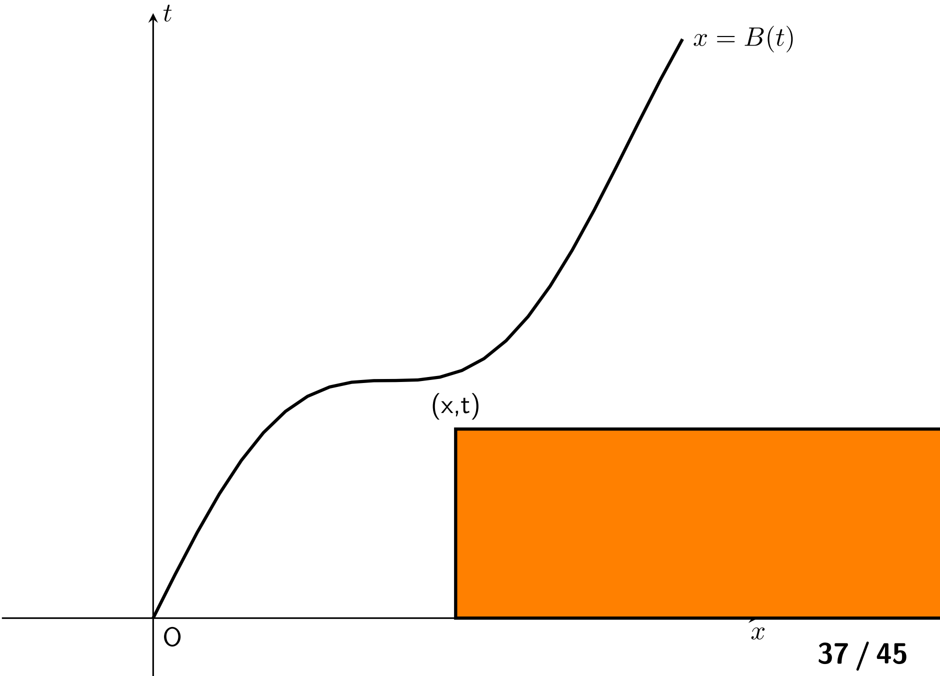
$$\eta_{k\,t}(x) = \begin{cases} 1, & B(t)+k-1 \le x \le B(t)+k, \\ 1-x+B(t)+k, & B(t)+k \le x \le B(t)+k+1, \\ 0, & B(t)+k+1 \le x \end{cases}$$

Note. $B(t)$ is monotonically increasing function.



For ${}^\forall(x,t) \in \mathcal{L}(T)$, ${}^\exists k$ s.t. $x \in [B(t)+k, B(t)+k+1]$.
Multiply (L.b) by $r^{n-1}\eta_{k\,t}$, integrate over $[x,\infty) \times [0,t]$ and take $\exp$.

As $B(t)$ is increasing,



$$v(x,t)^\gamma = \frac{v_0(x)^\gamma + \frac{K\gamma}{\mu}\int_0^t A(x,s)D(x,s)\mathrm{d}s}{A(x,t)D(x,t)}, \quad (x,t) \in \mathcal{L}(T)$$

$$A_(x,t) = \exp\Big(\frac{K\gamma}{\mu}\int_0^t\int_{B(t)+k}^{B(t)+k+1} v^{-\gamma}\,\mathrm{d}s - \frac{(n-1)\gamma}{\mu}\int_0^t\int_x^\infty \eta_{k\,t}\frac{|\tilde{u}|^2}{r^n}\,\mathrm{d}y\mathrm{d}s\Big),$$

$$D(x,t) = \exp\Big(\frac{\gamma}{\mu}\int_x^\infty \eta_{k\,t}\{\frac{\psi}{r^{n-1}} - f(r)\}(y,s)\mathrm{d}y\Big|_{s=0}^{s=t} + \frac{(n-1)\gamma}{\mu}\int_0^t\int_x^\infty \eta_{k\,t}\frac{\psi^2}{r^n}\mathrm{d}y\mathrm{d}s - \int_{B(t)+k}^{B(t)+k+1}\log\frac{v}{v_0}dx\Big)$$

$$f(r) := \tilde{\rho}(1)|u_b|\int_1^r \frac{\tilde{v}'(s)}{s^{2(n-1)}}\mathrm{d}s.$$

Pointwise value $v(x,t)$ is given by the integrals.
Showing the arguments of $\exp$ in (R.H.S.) bounded, term by term, from (E) gives

$$0 < c_0 \le v(x,t) \le C_0 < \infty \quad \Rightarrow \quad 0 < \underline{\rho} \le \rho(x,t) \le \overline{\rho} < \infty.$$

Pointwise boundedness of $v(x,t)$

Proposition 3

$$0 < c_0 \le v(x,t) \le C_0 \quad \text{for} \quad (x,t) \in \mathcal{L}(T).$$

c_0 & C_0 depend only on initial data.

† Prop.3 is proved by several lemmas by using

$$\int_{B(t)}^{\infty} \mathcal{E}_L(x,t)\,\mathrm{d}x + \mu \iint_{\mathcal{L}(T)} \Big\{ \frac{n-1}{2} \frac{v\psi^2}{r^2} + \frac{r^{2n-2}}{v}\psi_x^2 \Big\} \mathrm{d}x\mathrm{d}t +$$
$$+ |u_b| \int_0^T \frac{G(v,\tilde{v})}{v}(B(t),t)\,\mathrm{d}t + c \iint_{\mathcal{L}(T)} \Big\{ |u_b| \frac{\psi^2}{r^n} + |u_b|^3 \frac{G(v,\tilde{v})}{r^{3n-2}} \Big\}\,\mathrm{d}x\mathrm{d}t$$
$$\le \int_0^{\infty} \mathcal{E}_L(x,0)\,\mathrm{d}x, \quad \text{(E)}$$

Lemma 1

$$\int_0^T \sup_{x\geq B(t)} |r^{n-2}\psi^2(x,t)\,\mathrm{d}t \leq C_0,\ C_0 \text{ is const. depending on } \|(u_0,v_0)\|.$$

$$(r^{n-2}\psi^2)(x,t) = \int_{B(t)}^x (r^{n-2}\psi^2)_x \mathrm{d}x = \int_{B(t)}^x (n-2)r^{n-3}r_x\psi^2 + 2r^{n-2}\psi\psi_x \mathrm{d}x,$$

Substituting $r_x = v/r^{n-1}$, taking absolute value, integrating in t and using Schwartz ineqaluty, we have from (E),

$$\begin{aligned}\int_0^T \sup_{x\geq B(t)} |r^{n-2}\psi^2(x,t)|\,\mathrm{d}t &\leq \iint_{\mathcal{L}(T)} \Big\{(n-1)\frac{v}{r^2}\psi^2 + \frac{r^{2n-2}}{v}\psi_x^2\Big\}(x,t)\,\mathrm{d}x \\ &\leq 2\int_0^\infty \mathcal{E}_L(x,0)\,\mathrm{d}x. \qquad (14)\end{aligned}$$

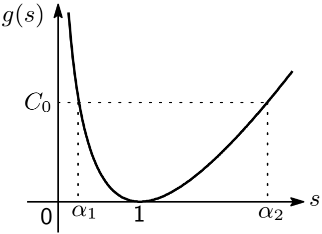
Lemma 2

$$0 < {}^{\exists}c_1 < {}^{\exists}c_2 < \infty \text{ s.t. } \quad c_1 \le \int_a^{a+1} v(x,t)\,dx \le c_2 \text{ for } {}^{\forall}a \ge B(t).$$

(Proof) From (E), we have

$$\int_a^{a+1} g\left(\frac{v}{\tilde{v}}\right) dx \le \int_{B(t)}^{\infty} g\left(\frac{v}{\tilde{v}}\right) dx \le \int_{B(t)}^{\infty} \mathcal{E}_L(x,t)\,\mathrm{d}x \le \int_0^{\infty} \mathcal{E}_L(x,0)\,\mathrm{d}x =: C_0.$$

Note $g''(s) > 0$ $(s > 0)$ and $g(s) \to \infty$ as $s \searrow 0$ or $s \to \infty$. $(g(s) = s - 1 - \int_1^s \eta^{-\gamma}\,d\eta)$



By Jensen's inequality,

$$g\left(\int_a^{a+1} \frac{v}{\tilde{v}} dx\right) \le \int_a^{a+1} g\left(\frac{v}{\tilde{v}}\right) dx \le C_0, \quad \Longrightarrow \quad \alpha_1 \le \int_a^{a+1} \left(\frac{v}{\tilde{v}}\right)(x,t)dx \le \alpha_2.$$

$\tilde{v} = 1/\tilde{\rho}$ is bounded, i.e., $0 < c \le \tilde{v} \le C$.

For example, 2nd term in $D(x,t)$ is handled by Lemma 1.

$$\left| \int_0^t \int_x^\infty \eta_{k\,t} \frac{|\psi|^2}{r^n} \, \mathrm{d}y \mathrm{d}s \right| \le C \int_0^t \int_{B(t)}^\infty \sup \psi^2 \le C_0$$

By Jensen's inequality, we have the upper bound for 3rd term in $D(x,t)$,

$$\int_{B(t)+k}^{B(t)+k+1} \log \frac{v(y,\tau)}{v_0(y)} \, \mathrm{d}y \le \log \Big(\int_{B(t)+k}^{B(t)+k+1} \frac{v(y,\tau)}{v_0(y)} \, \mathrm{d}y \Big) \le C_0.$$

The lower bound follows from Lemma 2,

$$\int_{B(t)+k}^{B(t)+k+1} \log \frac{v(y,\tau)}{v_0(y)} \, \mathrm{d}y = \int_{B(t)+k}^{B(t)+k+1} \Big\{ \log \frac{v(y,\tau)}{\tilde{v}(y,\tau)} + \log \frac{\tilde{v}(y,\tau)}{v_0(y)} \Big\} \, \mathrm{d}y$$

$$\ge - \int_{B(t)+k}^{B(t)+k+1} \Big\{ g\big(\frac{v}{\tilde{v}}\big) - \frac{v}{\tilde{v}} + 1 - \log \frac{\tilde{v}}{v_0} \Big\}(y,\tau) \, \mathrm{d}y \ge -C_0.$$

Thus we have $0 < c_0 \le v(x,t) \le C_0 \Rightarrow \int_{B(t)}^\infty \mathcal{E}_L(t)\, dx \sim \int_{B(t)}^\infty |(\phi,\psi)(x,t)|^2 \, dx.$

Corollary $\|(\phi,\psi)(t)\|^2 + \int_0^t \|r^{-1}\psi\|^2 + \|r^{n-1}\psi_x\|^2 \, d\tau \le C\|(\phi_0,\psi_0)\|^2.$

Weighted H^1 estimate in Lagrangian coodinate

- $r^{\frac{n-1}{2}}(\rho-\tilde\rho, u-\tilde u) \in C(H^1) \Longleftrightarrow (v-\tilde v, u-\tilde u), r^{n-1}((\rho-\tilde\rho)_x, (u-\tilde u)_x) \in C(L^2)$.

Let $F := \mu\dfrac{\phi_x}{v} - \dfrac{\psi}{r^{n-1}}, \quad (\phi,\psi) = (v-\tilde v, u-\tilde u)$.

Substituting (L.a) in (L.b) and muptiply by $r^{-(n-1)}$ and using $r_t = u$,

$$F_t + \frac{\gamma K}{\mu}\frac{F}{v^\gamma} = (n-1)\frac{\psi^2}{r^n} + \gamma\frac{p(v)-p(\tilde v)}{v-\tilde v}\frac{\partial_r\tilde\rho}{r^{n-1}\tilde\rho^2}\phi + Q\frac{\psi}{r^{n-1}}$$

$$Q := \partial_r\tilde u + (n-1)\frac{\tilde u}{r} - \frac{\gamma K}{\mu}v^{-\gamma} + \mu r^{n-1}\partial_r\Big(\frac{\partial_r\tilde\rho}{r^{n-1}\tilde\rho^2}\Big)$$

Multiplying by $r^{2(n-2)}F$ and integrating over $\mathcal{L}(T)$

$$\int_{B(t)}^\infty r^{2n-2}F^2(x,t)\,\mathrm{d}x + |u_b|\int_0^T\phi_x^2(B(t),t)\mathrm{d}t + \int_0^T\int_{B(t)}^\infty r^{2n-4}F^2\,\mathrm{d}x\mathrm{d}t \le C_0.$$

As ψ is estimated in Corollary,

$$\int_{B(t)}^\infty r^{2n-2}\phi_x^2(x,t)\,\mathrm{d}x + |u_b|\int_0^T\phi_x^2(B(t),t)\,\mathrm{d}t + \int_0^T\int_{B(t)}^\infty r^{2n-4}\phi_x^2\,\mathrm{d}x\mathrm{d}t \le C_0.$$

Estimate of ψ_x in Lagrangian coodinate

Multiply (L.b) by $r^{n-1}\psi_{xx}$ and integrate to obtain

$$\int_{B(t)}^{\infty} r^{2n-4}\psi_x^2(x,t)\mathrm{d}x + \int_0^t \int_{B(\tau)}^{\infty} r^{4n-6}\psi_{xx}^2\,\mathrm{d}x\mathrm{d}\tau \leq C_0.$$

Let $u_b \ll 1$.

$$\int_{B(t)}^{\infty} (\phi^2, \psi^2, r^{2n-2}\phi_x, r^{2n-2}\psi_x)(x,t)\,\mathrm{d}x + \int_0^T (\phi^2, \phi_x^2)(B(t),t)\,\mathrm{d}t + \\ + \iint_{\mathcal{L}(T)} r^{2n-4}\phi_x^2 + \frac{\psi^2}{r^2} + r^{2n-2}\psi_x^2 \mathrm{d}x\mathrm{d}t \leq C_0.$$

Proposition [Asimptotic behavior]

$$\sup_{x\in(B(t),\infty)} |(u(x,t) - \tilde{u}(r(x,t))), v(x,t) - \tilde{v}(r(x,t)))| \to 0 \ \text{ as } \ t \to \infty.$$

Thank you for your attention.