Singularity formation for fluid models

Mimi Dai
Univ. of Illinois Chicago

BIRS-IASM 2023

August 8, 2023

Euler

Incompressible Euler:

$$\begin{aligned} u_t + u \cdot \nabla u + \nabla \Pi &= 0, \\ \nabla \cdot u &= 0. \end{aligned} \tag{1}$$

Vorticity form: $\omega = \nabla \times u$

$$\begin{aligned} \omega_t + u \cdot \nabla \omega - \omega \cdot \nabla u &= 0, \\ u &= \nabla \times (-\Delta)^{-1} \omega. \end{aligned} \tag{2}$$

MHD

Incompressible ideal magnetohydrodynamics (MHD):

$$\begin{aligned} u_t + u \cdot \nabla u - B \cdot \nabla B + \nabla \Pi &= 0, \\ B_t + u \cdot \nabla B - B \cdot \nabla u &= 0, \\ \nabla \cdot u = 0, \ \nabla \cdot B &= 0. \end{aligned} \tag{3}$$

Elsässer variables: $p = u + B, \quad m = u - B$

$$\begin{aligned} p_t + (m \cdot \nabla) p + \nabla \Pi &= 0, \\ m_t + (p \cdot \nabla) m + \nabla \Pi &= 0, \\ \nabla \cdot p = 0, \quad \nabla \cdot m &= 0. \end{aligned}$$

MHD

Vorticity current form: $\Omega = \nabla \times p, \quad \omega = \nabla \times m$

$$\begin{aligned} \Omega_t + (m\cdot\nabla)\Omega - (\Omega\cdot\nabla)m + \nabla\times(m\nabla p) &= 0, \\ \omega_t + (p\cdot\nabla)\omega - (\omega\cdot\nabla)p + \nabla\times(p\nabla m) &= 0, \\ p = \nabla\times(-\Delta)^{-1}\Omega, m &= \nabla\times(-\Delta)^{-1}\omega \end{aligned} \tag{4}$$

where $(m\nabla p)_j = m_i \partial_j p_i$ and $(p\nabla m)_j = p_i\partial_j m_i$ for $1 \le j \le 3$.
Note $\nabla\times(m\nabla p) = -\nabla\times(p\nabla m)$.

Hall MHD

Incompressible magnetohydrodynamics (MHD) with Hall effect:

$$\begin{aligned} u_t + u\cdot\nabla u - B\cdot\nabla B + \nabla p &= \nu\Delta u, \\ B_t + u\cdot\nabla B - B\cdot\nabla u + d_i\nabla\times((\nabla\times B)\times B) &= \mu\Delta B, \\ \nabla\cdot u &= 0. \end{aligned} \tag{5}$$

Subsystems and scalings

- $d_i > 0$: Hall MHD, no natural scaling
- Two nonlinear structures:

$$\nabla \times ((\nabla \times B) \times B) = \nabla \times \nabla \cdot (B \otimes B)$$

$$(u \cdot \nabla) \cdot u = \nabla \cdot (u \otimes u)$$

  different scalings; different "degrees of singular effect"; different geometry properties
- MHD and Hall MHD obey the same energy law:

$$\frac{1}{2}\frac{d}{dt}\left(\|u\|_{L^2}^2 + \|B\|_{L^2}^2\right) + \nu\|\nabla u\|_{L^2}^2 + \mu\|\nabla B\|_{L^2}^2 = 0$$

Electron MHD

$u \equiv 0$ and $d_i = 1$ in (5):

$$\begin{aligned} B_t + \nabla \times ((\nabla \times B) \times B) = \mu \Delta B, \\ \nabla \cdot B = 0. \end{aligned} \tag{6}$$

Current: $J = \nabla \times B$

$$B_t + B \cdot \nabla J - J \cdot \nabla B = \mu \Delta B. \tag{7}$$

Compare the Euler vorticity form with (12) ($\mu = 0$):

$$\omega_t + u \cdot \nabla \omega - \omega \cdot \nabla u = 0$$

with $B \sim u$ and $J \sim \omega$

Unanswered Questions (perspective of mathematics)

- (i) Global regularity / finite time singularity
  (ii) Uniqueness / non-uniqueness of Leray-Hopf solution
  (iii) Stability / instability
  (iv) Turbulence related questions: anomalous dissipation...
- Pure fluid VS MHD: similarity + complexity

$\downarrow$

interactions of u and $B+$ Hall nonlinearity

- Toy models to gain insights toward understanding the questions above: dimension reduction, symmetry reduction, dyadic models, ...

1D models for Euler

Constantin-Lax-Majda model for Euler equation (2):

$$\omega_t - \omega H\omega = 0, \quad u_x = H\omega$$

with Hilbert transform H defined

$$Hf(x) = \frac{1}{2\pi} P.V. \int_{-\pi}^{\pi} f(y) \cot\left(\frac{x-y}{2}\right) dy.$$

De Gregorio model:

$$\omega_t + u\omega_x - \omega H\omega = 0, \quad u_x = H\omega$$

and generalized De Gregorio model by Okamoto-Sakajo-Wunsch:

$$\omega_t + au\omega_x - \omega H\omega = 0, \quad u_x = H\omega$$

with $a \in \mathbb{R}$.

Well-understood, due to contributions of

- Morlet, Hou-Li-Shi-Wang-Yu, Jia-Stewart-Šverák, Lei-Liu-Ren
- Córdoba-Córdoba-Fontelos
- Elgindi-Jeong, Elgindi-Ghoul-Masmoudi
- Chen, Chen-Hou, Hou-Lei
- Lushnikov-Silantyev-Siegel
- ......

1D models for MHD

1D model to mimic MHD system (4) (MD-Vyas-Zhang, 2021):

$$\begin{aligned}\Omega_t + m\Omega_x - \Omega m_x + \frac{1}{2}\omega p_x - \frac{1}{2}\Omega m_x &= 0,\\ \omega_t + p\omega_x - \omega p_x + \frac{1}{2}\Omega m_x - \frac{1}{2}\omega p_x &= 0,\\ p_x = H\Omega, \quad m_x &= H\omega.\end{aligned} \tag{8}$$

A generalized version of (8):

$$\begin{aligned}\Omega_t + am\Omega_x - \omega p_x &= 0,\\ \omega_t + ap\omega_x - \Omega m_x &= 0,\\ p_x = H\Omega, \quad m_x &= H\omega,\end{aligned} \tag{9}$$

with $a \in \mathbb{R}$.

Analytical results and numerical simulation

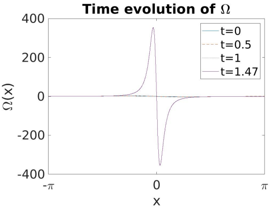
MD-Vyas-Zhang, 2021:

- local well-posedness of (9)
- Beale-Kato-Majda criterion
- numerical evidence for finite time singularity formation for cases with certain a and some initial data

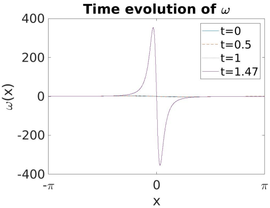
Initial data:

$$\Omega_0 = \omega_0 = -\frac{4}{3}\left(\sin x + \frac{1}{2}\sin(2x)\right). \tag{10}$$

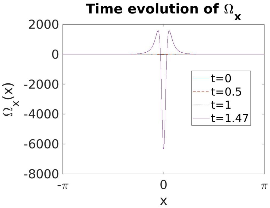
$a = 0.5$ in (9) with initial data (10)



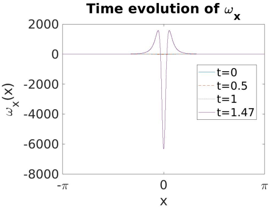
(a)



(b)

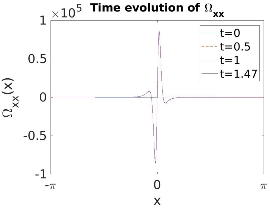


(c)

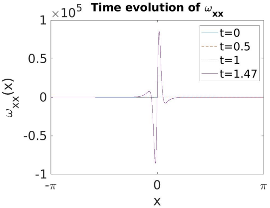


(d)

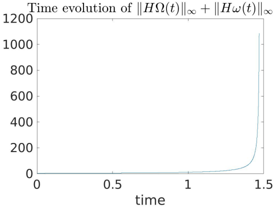
$a = 0.5$ in (9) with initial data (10)



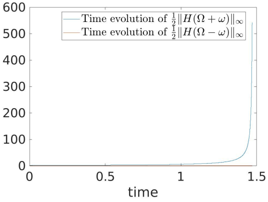
(e)



(f)



(g)



(h)

1D electron MHD model

MD, 2022: 1D simplified model for electron MHD (6)

$$\begin{aligned} B_t + aBJ_x + JB_x + \Lambda^\alpha B &= 0, \qquad a \in \mathbb{R} \\ B_x &= HJ \end{aligned} \tag{11}$$

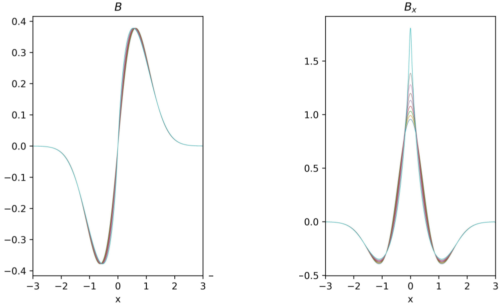
with $\Lambda = H\partial_x$ and $\alpha \geq 0$.

Theorem (MD, 2022)

(i) general a: local well-posed in Sobolev space for $\alpha > 2$
(ii) $a = 0$*: local well-posed in Sobolev space for* $\alpha > 1$
(iii) $a = 0$, $\alpha \geq 0$*: local analytic solution*
(iv) $a = 0$, $\alpha = 0$*: there exists initial data* B_0 *such that the solution develops singularity in finite time.*

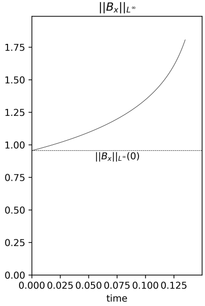
Numerical solution

$$B_0(x) = e^{-x^2} \tan^{-1} x.$$



(i) (j)

Norm growth



(k)

$2\frac{1}{2}$D electron MHD

Consider

$$B(x,y,t) = \nabla \times (a\vec{e_z}) + b\vec{e_z}, \quad \vec{e_z} = (0,0,1).$$

with scalar-valued functions

$$a = a(x,y,t), \quad b = b(x,y,t).$$

System satisfied by a and b:

$$\begin{aligned} a_t + (a_y b_x - a_x b_y) &= \mu\Delta a, \\ b_t - (a_y \Delta a_x - a_x \Delta a_y) &= \mu \Delta b. \end{aligned} \tag{12}$$

Theorem (MD-Wu, 2022)

Assume $(a(t), b(t))$ *is a regular solution of (12) on* $[0, T)$. *If either*

$$\int_0^T \|\nabla b_{\leq Q(b)}(t)\|_{B^1_{\infty,\infty}}\, dt \ \ \& \ \ \int_0^T \|\nabla\nabla a_{\leq Q(a)}(t)\|_{B^1_{\infty,\infty}}\, dt,$$

or

$$b \in L^s(0, T; L^r(\mathbb{T}^2)) \ \ \text{with} \ \ \frac{2}{s} + \frac{2}{r} \leq 1 \ \ \forall \ \ r \in (2, \infty],$$

Then $(a(t), b(t))$ *is regular on* $[0, T]$.

Hall equilibria

Examples of Hall equilibria

(i) shear type: $a^* = a^*(x)$ and $b^* = b^*(x)$;

(ii) shear type: $a^* = a^*(y)$ and $b^* = b^*(y)$;

(iii) radial type: $a^* = a^*(r)$ and $b^* = b^*(r)$ with $r = (x^2 + y^2)^{\frac{1}{2}}$.

Questions: stability/instability?

Theorem (MD, 2023)

Dissipation in one direction, Δb. Global existence of solution in Sobolev space near shear steady state $(a^*, b^*) = (y, 0)$.

Ongoing and future work

- Rigorous proof of singularity formation for coupled 1D MHD models
- Understand the nonlinearity BJ_x in the 1D electron MHD model
- Singularity formation scenarios for the $2\frac{1}{2}$D electron MHD
- Singularity formation for original MHD systems (with/without Hall effect)

THANK YOU!