The weak Arthur packets of real classical groups

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(joint with Dan Barbasch, Binyong Sun and Chengbo Zhu)

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(BIRS-IASM, Arthur packets)

Jordan-Chevalley decomposition

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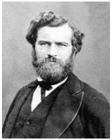

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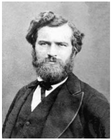

- Classification of conjugation classes in G:

$$G/\sim = \bigsqcup_{s \in G_{\text{s.s.}}/\sim} \{\, su \mid u \in G_s \,\}/\sim \;\xleftrightarrow{bij.}\; \bigsqcup_{s \in G_{\text{s.s.}}/\sim} \mathrm{unip}(G_s)$$

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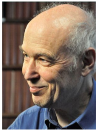

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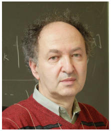

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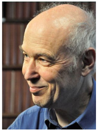

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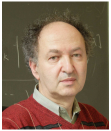

Lusztig's map to the unipotent packet.

$$\mathcal{E}(G, s) \xrightarrow[\mathcal{L}_s]{bij.} \mathcal{E}(\check{G}_s, 1)$$

Representations of Real Lie groups

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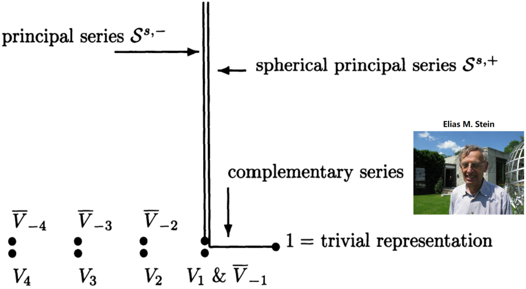

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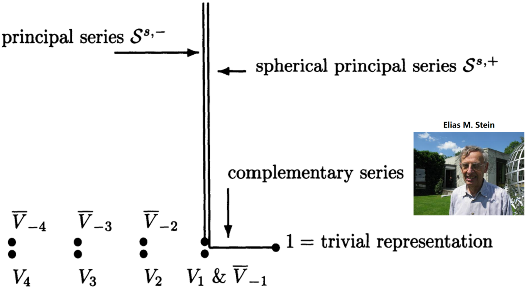

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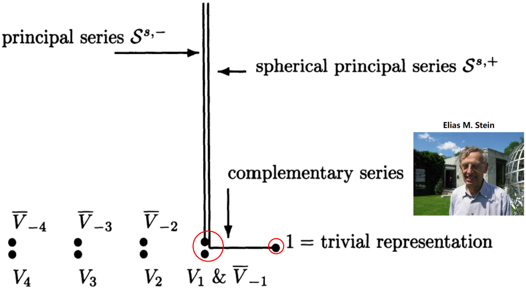

- *Open problem:* Structure of the unitary dual!

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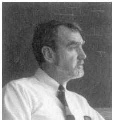

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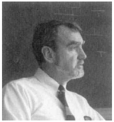

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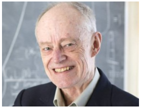
-  *Conjecture:* $\exists$ unipotent Arthur packets

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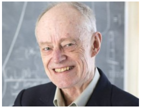
-  *Conjecture:* $\exists$ unipotent Arthur packets
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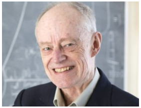
-  *Conjecture:* $\exists$ unipotent Arthur packets
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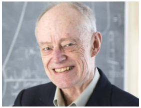
-  *Conjecture:* $\exists$ unipotent Arthur packets

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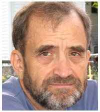
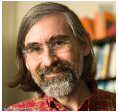
  "Sur Les paquets d'Arthur des groupes classiques réels" (2020 JEMS)

Barbasch-Vogan's definition of special unip. repn.

- Barbasch  Vogan  $\mathrm{Unip}(G) :=$ repn. with smallest "size"

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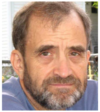
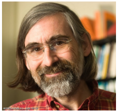
- Barbasch  Vogan  $\mathrm{Unip}(G) :=$ repn. with smallest "size"
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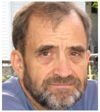
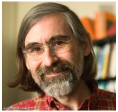
- Barbasch  Vogan  $\mathrm{Unip}(G) :=$ repn. with smallest "size"
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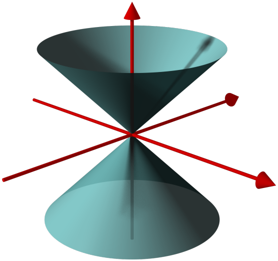

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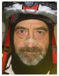

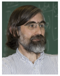

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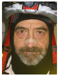
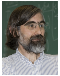
- $\mathrm{Nil}^{\mathrm{special}}(\mathbf{G}) \ni \mathcal{O} :=$ the  Spaltenstein    dual of $\check{\mathcal{O}}$.

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Special unip. repn. of simply conn. classical groups

Theorem (Barbasch-M.-Sun-Zhu)

Suppose G is a simply connected real classical group, i.e. one of the following groups

$$\mathrm{SU}(p,q), \mathrm{Spin}(p,q), \mathrm{Spin}(2n,\mathbb{H}), \mathrm{Sp}(2n,\mathbb{R}), \mathrm{Mp}(2n,\mathbb{R}), \mathrm{Sp}(p,q)$$

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Arthur-Barbasch-Vogan's unitarity conj. for special unipotent repn. holds:

All *special unipotent repn.* of G are *unitarizable*.

Answer for *exceptional groups*

J. Adams, S. Miller, M. van Leeuven, and D. A. Vogan

atlas  + other method.

Counting G-module with a specific ann. variety

- Fix inf. char. $\mu \in \mathfrak{h}^*/W$

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double cell $\mathcal{D} \subset \mathrm{Irr}(W(\mu)) \rightsquigarrow$ the special repn. τ_0

$\rightsquigarrow$ truncated ind. $J_{W(\mu)}^{W} \tau_0 \xrightarrow{\text{Springer corr.}} \mathcal{O}$.

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Lemma:

$$\#\{\, \pi \in \operatorname{Irr}_\mu(\mathfrak{g}, K)(G) \mid \operatorname{AV}_{\mathbb{C}}(\pi) = \overline{\mathcal{O}} \,\} = \sum_{\substack{\mathcal{D} \rightsquigarrow \mathcal{O} \\ \tau \in \mathcal{D}}} [\tau : 1_{W_\mu}] \cdot [\tau : \mathscr{G}_\mu(G)]$$

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Lemma: If $\mathfrak{g}$ has no E_8 factor, then

$$\begin{aligned}&\#\{\, \pi \in \mathrm{Irr}_\mu(\mathfrak{g},K)(G) \mid \mathrm{AV}_{\mathbb{C}}(\pi) = \overline{\mathcal{O}} \,\} \\ &= \sum_{\substack{\mathcal{D} \rightsquigarrow \mathcal{O} \\ \tau \in \mathcal{D}}} [\tau : 1_{W_\mu}] \cdot [\tau : \mathscr{G}_\mu(G)]\end{aligned}$$

Applications

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- $\mathrm{Unip}_{\check{\mathcal{O}}}(\mathrm{SU}_(p,q))$ = rest. of $\mathrm{Unip}_{\check{\mathcal{O}}}$(det double cover of $\mathrm{U}(p,q)$).
- genuine special unipotent repn. of $\mathrm{Spin}(p,q)$ are some obvious irreducibly parabolicaly induced module.

Nilpotent orbits with "good/bad parity"

- Bad parity (must occur with even multiplicity in $\check{\mathcal{O}}$):

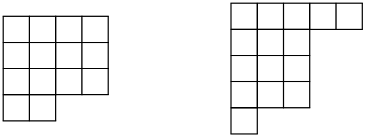
$$\begin{cases} \text{even number,} & \text{when } \check{G} \text{ is type } B \text{ or } D \\ \text{odd number,} & \text{when } \check{G} \text{ is type } C \end{cases}$$

- $\check{\mathcal{O}}$ has *"good parity"* if $\check{\mathcal{O}}$ only contains

$$\begin{cases} \text{odd rows,} & \text{when } \check{G} \text{ is type } B \text{ or } D \\ \text{even rows,} & \text{when } \check{G} \text{ is type } C \end{cases}$$

- $\lambda_{\check{\mathcal{O}}}$ is integral.

$\mathcal{O}$ $\check{\mathcal{O}}$



$Sp(14, \mathbb{C})$ $SO(15, \mathbb{C})$

Reduction to the “good parity”

- Consider $G = \mathrm{Sp}(2n, \mathbb{R})$.
- $\check{\mathcal{O}}$ decompose into two parts $\check{\mathcal{O}}_g$ (good parity) and $\check{\mathcal{O}}_b$ (bad parity).
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Theorem Let $\check{\mathcal{O}}_b' = \{\, r_1, \cdots, r_k \,\} \in \mathrm{Nil}_{\mathrm{GL}}$.

$$\begin{array}{ccc} \mathrm{Unip}_{\check{\mathcal{O}}_b'}(\mathrm{GL}) \times \mathrm{Unip}_{\check{\mathcal{O}}_g}(\mathrm{Sp}) & \xrightarrow{bij.} & \mathrm{Unip}_{\check{\mathcal{O}}}(\mathrm{Sp}) \\ (\pi', \pi_0) & \mapsto & \mathrm{Ind}_{\mathrm{GL}_{|\check{\mathcal{O}}_b'|} \times \mathrm{Sp}(2n_0, \mathbb{R}) \ltimes U}^{\mathrm{Sp}(2n, \mathbb{R})} \pi' \otimes \pi_0 \end{array}$$

$$\mathrm{Unip}_{\check{\mathcal{O}}_b'}(\mathrm{GL}) = \left\{ \mathrm{Ind} \bigotimes_{j=1}^{k} \mathrm{sgn}_{\mathrm{GL}(r_j, \mathbb{R})}^{\epsilon_j} \;\middle|\; \epsilon_j \in \mathbb{Z}/2\mathbb{Z} \right\}$$

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- Use *theta correspondence* to study $\mathrm{Unip}_{\check{\mathcal{O}}_g}(G)$.
- We assume $\check{\mathcal{O}}$ has good parity from now on.

Counting unipotent representations I

- Example: $G = \mathrm{Sp}(2n, \mathbb{R})$ and $\chi_{\check{\mathcal{O}}} \in \rho_G +$ weight lattice.

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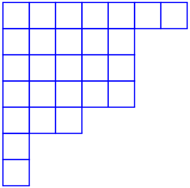
- $[\tau : \mathscr{G}_{\rho_G}(G)]$ is counted by painted bi-partitions $\mathrm{PBP}(\check{\mathcal{O}})$.

Example of PBP

$\check{\mathcal{O}} = [7,5,5,5,3,1,1] =$ 

$\mathrm{PBP}_{\check{\mathcal{O}}}(\mathrm{Sp}(2n,\mathbb{R}))$

•	•	•
•	c	
r	d	
d		

$\times$

•	•	•
•	s	

$\mapsto$

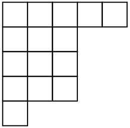
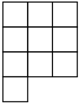
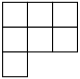
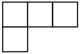
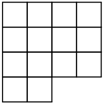
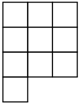
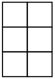
Associated character

*	=	*	=	*	=
*	=	*	=	*	=
=	*	=	*		
*	=	*	=		
*	=	*	=		
−	+				

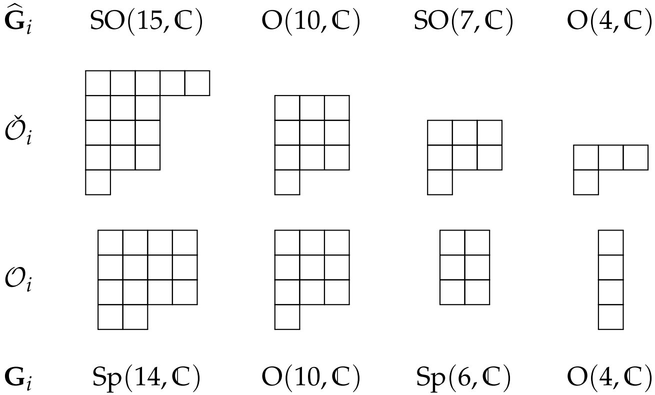
$\cup$

*	=	*	=	*	=
*	=	*	=	*	=
=	*	=	*		
=	*	=	*		
*	=	*	=		
+	−				

Inductive structure of nilpotent orbits

$\widehat{\mathbf{G}}_i$	$SO(15,\mathbb{C})$	$O(10,\mathbb{C})$	$SO(7,\mathbb{C})$	$O(4,\mathbb{C})$
$\check{\mathcal{O}}_i$				
$\mathcal{O}_i$				
$\mathbf{G}_i$	$Sp(14,\mathbb{C})$	$O(10,\mathbb{C})$	$Sp(6,\mathbb{C})$	$O(4,\mathbb{C})$

Inductive structure of nilpotent orbits



Kraft

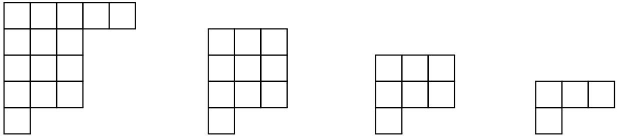
Procesi

resolution of singularities of nilpotent orbit closures.

Example of descent sequences

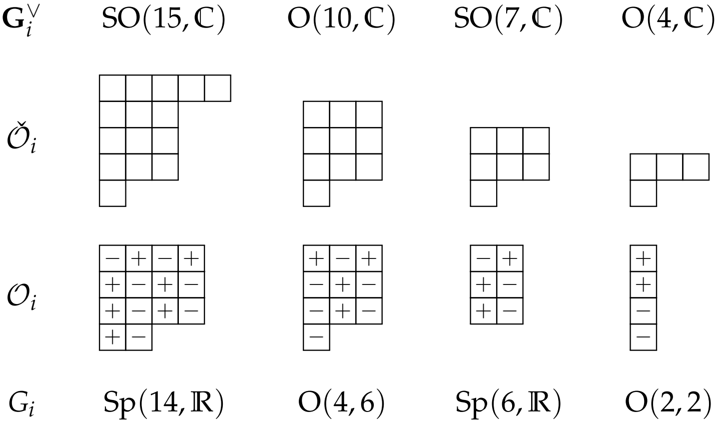
$\mathbf{G}_i^\vee$	$\mathrm{SO}(15,\mathbb{C})$	$\mathrm{O}(10,\mathbb{C})$	$\mathrm{SO}(7,\mathbb{C})$	$\mathrm{O}(4,\mathbb{C})$

$\check{\mathcal{O}}_i$



$\mathcal{O}_i$	$-+-+$ $+-+-$ $+-+-$ $+-$	$+-+$ $-+-$ $-+-$ $-$	$-+$ $+-$ $+-$	$+$ $+$ $-$ $-$
G_i	$\mathrm{Sp}(14,\mathbb{R})$	$\mathrm{O}(4,6)$	$\mathrm{Sp}(6,\mathbb{R})$	$\mathrm{O}(2,2)$

Example of descent sequences



Ohta's resolution of singularities of a nilpotent orbit closure in symmetric pairs.

Construction of elements in $\mathrm{Unip}_{\check{\mathcal{O}}}(G)$

- $\chi = \bigotimes_{j=0}^{a} \chi_j$, a character of $\prod_{j=0}^{a} G_j$.
- $\chi_j \in \{\mathbf{1}, \mathrm{sgn}^{+,-}, \mathrm{sgn}^{-,+}, \det\}$ when G_j is an orthogonal group.
- Define a smooth repn. of $G = G_a$

$$\pi_\chi := (\omega_{G_a,G_{a-1}} \widehat{\otimes} \omega_{G_{a-1},G_{a-2}} \widehat{\otimes} \cdots \widehat{\otimes} \omega_{G_1,G_0} \otimes \chi)_{G_{a-1}\times G_{a-2}\times\cdots\times G_0}$$

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Theorem (Barbasch-M.-Sun-Zhu)

Suppose $\check{\mathcal{O}}$ is an orbit with good parity. Then

- *either $\pi_\chi = 0$ or*
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- *either* $\pi_\chi = 0$ *or*
- $\pi_\chi \in \mathrm{Unip}_{\check{\mathcal{O}}}(G)$ *and unitarizable.*
- *Moreover,*

$$\mathrm{Unip}_{\check{\mathcal{O}}}(G) = \{\, \pi_\chi \mid \pi_\chi \neq 0 \,\}.$$

Example: Coincidences of theta liftings

Lift to $G = \mathrm{Sp}(6, \mathbb{R})$ from real forms of $\mathbf{G} = \mathrm{O}(4, \mathbb{C})$.
$\check{\mathcal{O}} = 3^2 1^1$ and $\mathcal{O} = 2^3$.

	$\mathrm{Sp}(6, \mathbb{R})$		
$\mathrm{O}(4,0)$		$\theta(\mathrm{sgn}^{+,-})$	
$\mathrm{O}(3,1)$	$\theta(\mathbf{1})$	$\theta(\mathrm{sgn}^{+,-})$	$\theta(\mathrm{sgn}^{-,+})$
$\mathrm{O}(2,2)$	$\theta(\mathbf{1})$	$\theta(\mathrm{sgn}^{+,-})$	$\theta(\mathrm{sgn}^{-,+})$
$\mathrm{O}(1,3)$	$\theta(\mathbf{1})$	$\theta(\mathrm{sgn}^{+,-})$	$\theta(\mathrm{sgn}^{-,+})$
$\mathrm{O}(0,4)$			$\theta(\mathrm{sgn}^{-,+})$

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For $\tau \in \mathrm{PBP}^{\mathrm{ext}}(\check{\mathcal{O}})$, define

$$\pi_\tau := \Theta(\pi_{\nabla(\tau)} \otimes \chi'_\tau) \otimes \chi_\tau$$

Lifting of Associated characters I

- Example $(G, G') = (\mathrm{Sp}(2n, \mathbb{R}), \mathrm{O}(p, q))$

$$
\begin{array}{ccccc}
 & & M_{p,n} \oplus M_{q,n} & & \\
 & \overset{\varphi'}{\swarrow} & & \overset{\varphi}{\searrow} & \\
M_{p,q} \cong \mathfrak{p}' & & w = (A, B) & & \mathfrak{p} \cong \mathrm{Sym}_n \oplus \mathrm{Sym}_n \\
 & \swarrow & & \searrow & \\
X' = AB^T & & & & X = (A^T A, B^T B)
\end{array}
$$

- $\overline{\mathcal{O}} \cap \mathfrak{p} \supset \varphi(\varphi'^{-1}(\mathfrak{p}' \cap \mathcal{O}'))$ where $\mathcal{O}$ is a cplx. nil. **G**-orbit.
- Upper bound of associated cycle: we can define

$$\vartheta^{\mathrm{geo}} \colon \mathcal{K}_{\mathcal{O}'}(G') \longrightarrow \mathcal{K}_{\mathcal{O}}(G)$$

such that

$$\mathrm{AC}(\Theta(\pi')) \preceq \vartheta^{\mathrm{geo}}(\mathrm{AC}(\pi')),$$

for every π' with $\mathrm{AV}(\pi') \subset \overline{\mathcal{O}'}$

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- Support of $\vartheta(\mathscr{L}')$ could be reducible.

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  use the structure of degenerate principle series.
- Estimation of matrix coeff. integrals $\rightsquigarrow$ *unitarity*
- Sharp formula of Asso. Char.: lower bound=upper bound
  - *double θ-lift* $\rightsquigarrow$ lower bound.
  - *double fibration of moment maps* $\rightsquigarrow$ upper bound
- Exhaustion: lower bound=upper bound (Combinatorics)
  - Character theory (Kazhdan-Lusztig-Vogan theory)
    $\rightsquigarrow$ upper bound by counting painted bipartitions.
  - Asso. Char.+Injectivity of θ $\rightsquigarrow$ lower bound.

Relevant papers

- *Definition for metaplectic groups*
  On the notion of metaplectic Barbasch-Vogan duality
  https://arxiv.org/abs/2010.16089
- *Counting and reduction to good parity*
  https://arxiv.org/abs/2205.05266
- *Construction and unitarity using θ-lifting*
  https://arxiv.org/abs/1712.05552

Thank you for your attention!