Algebro-geometric solutions to the lattice potential modified Kadomtsev–Petviashvili equation

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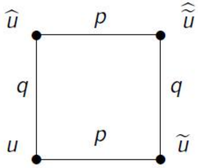
Outline

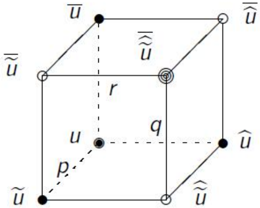
Outline

Outline

Multidimensional Consistency (MDC)[Nijhoff,Walker-2001]

With $u, \tilde{u}, \hat{u}, \bar{u}$ given we would solve for $\hat{\tilde{u}}, \bar{\tilde{u}}, \bar{\hat{u}}$ from the three equations on the left and then the three equations on the right should give the same value for $\bar{\hat{\tilde{u}}}$.





$$
\begin{aligned}
Q(u, \tilde{u}, \hat{u}, \hat{\tilde{u}}; p, q) = 0, &\quad Q(\bar{u}, \bar{\tilde{u}}, \bar{\hat{u}}, \bar{\hat{\tilde{u}}}; p, q) = 0, \\
Q(u, \tilde{u}, \bar{u}, \bar{\tilde{u}}; p, r) = 0, &\quad Q(\hat{u}, \hat{\tilde{u}}, \bar{\hat{u}}, \bar{\hat{\tilde{u}}}; p, r) = 0, \\
Q(u, \hat{u}, \bar{u}, \bar{\hat{u}}; q, r) = 0, &\quad Q(\tilde{u}, \hat{\tilde{u}}, \bar{\tilde{u}}, \bar{\hat{\tilde{u}}}; q, r) = 0.
\end{aligned}
$$

Consistency around the cube (CAC) [ABS-2003]

CAC: MDC with some conditions (Linearity, Symmetry, Tetrahedron Condition)

ABS List (9 equations):

- H-List:

$$H_1 : (u - \hat{\tilde{u}})(\hat{u} - \tilde{u}) = p^2 - q^2$$

$$H_2 : (u - \hat{\tilde{u}})(\tilde{u} - \hat{u}) = (p - q)(u + \tilde{u} + \hat{u} + \hat{\tilde{u}}) + p^2 - q^2$$

$$H_3 : p(u\tilde{u} + \hat{u}\hat{\tilde{u}}) - q(u\hat{u} + \tilde{u}\hat{\tilde{u}}) = \delta^2(p^2 - q^2)$$

- A-List:

$$A_1 : p(u + \hat{u})(\tilde{u} + \hat{\tilde{u}}) - q(u + \tilde{u})(\hat{u} + \hat{\tilde{u}}) = \delta^2 pq(p^2 - q^2)$$

$$A_2 : p(1 - q^2)(u\hat{u} + \tilde{u}\hat{\tilde{u}}) - q(1 - p^2)(u\tilde{u} + \hat{u}\hat{\tilde{u}}) + (p^2 - q^2)(1 + u\tilde{u}\hat{u}\hat{\tilde{u}}) = 0$$

Consistency around the cube (CAC) [ABS-2003]

- Q-List:

$$Q_1 : p(u-\hat{u})(\tilde{u}-\hat{\tilde{u}}) - q(u-\tilde{u})(\hat{u}-\hat{\tilde{u}}) = \delta^2 pq(q-p)$$

$$Q_2 : p(u-\hat{u})(\tilde{u}-\hat{\tilde{u}}) - q(u-\tilde{u})(\hat{u}-\hat{\tilde{u}}) + pq(p-q)(u+\tilde{u}+\hat{u}+\hat{\tilde{u}}) = \\ = pq(p-q)(p^2-pq+q^2)$$

$$Q_3 : p(1-q^2)(u\hat{u}+\tilde{u}\hat{\tilde{u}}) - q(1-p^2)(u\tilde{u}+\hat{u}\hat{\tilde{u}}) = \\ = (p^2-q^2)\left((\hat{u}\tilde{u}+u\hat{\tilde{u}}) + \delta^2\frac{(1-p^2)(1-q^2)}{4pq}\right)$$

$$Q_4 : p(u\tilde{u}+\hat{u}\hat{\tilde{u}}) - q(u\hat{u}+\tilde{u}\hat{\tilde{u}}) = \\ = \frac{pQ-qP}{1-p^2q^2}\left((\hat{u}\tilde{u}+u\hat{\tilde{u}}) - pq(1+u\tilde{u}\hat{u}\hat{\tilde{u}})\right)$$

where $P^2 = p^4 + \gamma p^2 + 1, \quad Q^2 = q^4 - \gamma q^2 + 1.$

Consistent on a 4D cube [ABS-2012, IMRN]

Octahedron-Type Lattice Equations:

- The bilinear lattice KP equation

$$\hat{\tilde{u}}\bar{u} - \hat{\bar{u}}\tilde{u} + \tilde{\bar{u}}\hat{u} = 0$$

- The lattice Schwarzian KP

$$\frac{(\hat{\tilde{u}} - \hat{u})(\hat{\bar{u}} - \bar{u})(\tilde{\bar{u}} - \tilde{u})}{(\hat{\tilde{u}} - \tilde{u})(\hat{\bar{u}} - \hat{u})(\tilde{\bar{u}} - \bar{u})} = 1$$

- The lattice potential KP equation

$$(\hat{\tilde{u}} - \hat{\bar{u}})\hat{u} + (\hat{\bar{u}} - \tilde{\bar{u}})\bar{u} + (\tilde{\bar{u}} - \hat{\tilde{u}})\tilde{u} = 0$$

Octahedron-Type Lattice Equations [ABS-2012, IMRN]

- The lattice potential modified KP equation

$$\frac{\hat{\tilde{u}}-\hat{\bar{u}}}{\hat{u}}+\frac{\hat{\bar{u}}-\tilde{\bar{u}}}{\bar{u}}+\frac{\tilde{\bar{u}}-\hat{\tilde{u}}}{\tilde{u}}=0$$

- The asymmetric lattice modified KP

$$\frac{(\hat{\bar{u}}-\tilde{\bar{u}})}{\bar{u}}=\hat{\tilde{u}}(\frac{1}{\tilde{u}}-\frac{1}{\hat{u}})$$

All these equations already appeared in the literature. [Hirota R-1981, Nijhoff F W, Capel H W, Wiersma G L and Quispel G R W-1984, Dorfman I, Nijhoff F W-1991, Bogdanov L V, Konopelchenko B G-1998]

Progress: pluri-Lagrangian structure, soliton solutions, elliptic solutions, Darboux transformation, Bäcklund transformation, conservation law, symmetry, continuum limits etc.

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Aim: Algebro-geometric solutions

The finite-gap integration method was created for solving the Korteweg–de Vries (KdV) equation with periodic initial value problem by Novikov, Matveev and their collaborators Dubrovin, Its and Krichever in 1970s. The obtained periodic solutions are called finite-gap solutions or algebro-geometric solutions. After the original work, the theory has undergone a true development and had a strong impact on the evolution of modern mathematics and theoretical physics.

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Our recent work

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§2. Lattice potential modified KP equation

Spectral problems

- **Continuous type**

$$\partial_x \chi = U_1 \chi, \quad U_1 = \begin{pmatrix} \lambda^2/2 & \lambda u \\ \lambda v & -\lambda^2/2 \end{pmatrix}$$

$$\partial_y \chi = U_2 \chi, \quad U_2 = \lambda^2 U_1 + \begin{pmatrix} \lambda^2(-uv) & \lambda(u_x - 2u^2 v) \\ \lambda(-v_x - 2uv^2) & -\lambda^2(-uv) \end{pmatrix}$$

$$\partial_t \chi = U_3 \chi,$$

$$U_3 = \lambda^2 U_2 + \begin{pmatrix} \lambda^2(-u_x v + 6uv_x + 3u^2v^2 & \lambda(u_{xx} - 6uvu_x + 6u^3v^2 \\ \lambda(v_{xx} + 6uvv_x + 6u^2v^3 & -\lambda^2(-u_x v + 6uv_x + 3u^2v^2) \end{pmatrix}$$

- **Discrete type**

$$\tilde{\chi} = D^{(\beta)} \chi, \quad D^{(\beta)}(\lambda, a, b) = \begin{pmatrix} \lambda^2(ab+1) - \beta^2 & \lambda a \\ \lambda b & 1 \end{pmatrix}$$

Soliton equations

Continuous type

- (U_1, U_2), the derivative nonlinear Schrödinger (dNLS):

$$u_y - u_{xx} + (2u^2v)_x = 0,$$
$$v_y + v_{xx} + (2uv^2)_x = 0.$$

- (U_1, U_2, U_3), the potential modified Kadomtsev-Petviashvili (pMKP)($W_x = uv$):

$$\Xi^{(3,0)} \equiv \frac{1}{4}(W_{xxx} - 2W_x^3)_x - \frac{3}{2}W_{xx}W_y + \frac{3}{4}W_{yy} - W_{xt} = 0$$

Soliton equations

Discrete and semi-discrete type

- $(U_1, D^{(\beta_1)})$, the semi-discrete dNLS
  $(a = 2u/(1+\sqrt{1-4u\tilde{v}}), \quad b = 2\tilde{v}/(1+\sqrt{1-4u\tilde{v}}))$:

$$u_x + (\tilde{u}\tilde{v} - uv)u - \frac{1}{2}(1+\sqrt{1-4u\tilde{v}})(\tilde{u}+\beta_1^2 u) = 0,$$
$$\tilde{v}_x + (\tilde{u}\tilde{v} - uv)\tilde{v} + \frac{1}{2}(1+\sqrt{1-4u\tilde{v}})(v+\beta_1^2 \tilde{v}) = 0$$

- $(D^{(\beta_1)}, D^{(\beta_2)})$, the lattice dNLS:

$$\frac{1}{2}(1+\sqrt{1-4u\tilde{v}})(\tilde{u}+\beta_1^2 u) - \frac{1}{2}(1+\sqrt{1-4u\bar{v}})(\bar{u}+\beta_2^2 u) - (\tilde{u}\tilde{v} - \bar{u}\bar{v})u = 0,$$
$$\frac{1}{2}(1+\sqrt{1-4\tilde{u}\bar{\tilde{v}}})(\tilde{v}+\beta_2^2 \bar{\tilde{v}}) - \frac{1}{2}(1+\sqrt{1-4\bar{u}\bar{\tilde{v}}})(\bar{v}+\beta_1^2 \bar{\tilde{v}}) - (\tilde{u}\tilde{v} - \bar{u}\bar{v})\bar{\tilde{v}} = 0$$

Soliton equations

The lattice potential modified KP (LpMKP) with 1, 2, 3 discrete arguments:

$(Z = ab + 1, \widetilde{W} - W = \ln Z^{(1)}, \overline{W} - W = \ln Z^{(2)}, \widehat{W} - W = \ln Z^{(3)})$

- $(U_1, U_2, D^{(\beta_1)})$:

$$\Xi^{(2,1)} \equiv (\widetilde{W} + W)_x x - (\widetilde{W}_x^2 - W_x^2) - 2\beta_1^2 (e^{-\widetilde{W}+W})_x - (\widetilde{W} - W)_y = 0$$

- $(U_1, D^{(\beta_1)}, D^{(\beta_2)})$:

$$\Xi^{(1,2)} \equiv (\widetilde{W} - \overline{W})_x + \beta_1^2 (e^{-\overline{\widetilde{W}}+\overline{W}} - e^{-\widetilde{W}+W}) - \beta_2^2 (e^{-\widetilde{\overline{W}}+\widetilde{W}} - e^{-\overline{W}+W}) = 0$$

- $(D^{(\beta_1)}, D^{(\beta_2)}, D^{(\beta_3)})$:

$$\Xi^{(0,3)} \equiv \beta_1^2 (e^{-\overline{\widetilde{W}}+\overline{W}} - e^{-\widehat{\widetilde{W}}+\widehat{W}}) + \beta_2^2 (e^{-\overline{\widehat{W}}+\widehat{W}} - e^{-\widetilde{\overline{W}}+\widetilde{W}}) +$$
$$+ \beta_3^2 (e^{-\widetilde{\widehat{W}}+\widetilde{W}} - e^{-\overline{\widehat{W}}+\widehat{W}}) = 0$$

[Nijhoff F W, Capel H W, Wiersma G L and Quispel G R W-1984]

Continuum limit

- In the neighborhood of $\varepsilon \sim 0$

$$\Xi^{(2,1)} = \Xi^{(3,0)} \frac{2c_1^2}{3} \varepsilon^2 + O(\varepsilon^3),$$
$$\Xi^{(1,2)} = \Xi^{(3,0)} \frac{c_1 c_2 (c_1 - c_2)}{3} \varepsilon^3 + O(\varepsilon^4),$$
$$\Xi^{(0,3)} = \Xi^{(3,0)} \frac{1}{3} [c_1 c_2 (c_1 - c_2) + c_2 c_3 (c_2 - c_3) + c_3 c_1 (c_3 - c_1)] \varepsilon^3 + O(\varepsilon^4).$$

where $\beta_k^{-2} = c_k \varepsilon, k = 1, 2, 3,$ with distinct non-zero constants c_1, c_2, c_3.

Integrable symplectic map

Consider the Lax matrix

$$L(\lambda; p, q) = \begin{pmatrix} \frac{1}{2} + Q_\lambda(A^2p, q) & -\lambda Q_\lambda(Ap, p) \\ \lambda Q_\lambda(Aq, q) & -\frac{1}{2} - Q_\lambda(A^2p, q) \end{pmatrix}, \quad (2.1)$$

where $Q_\lambda(\xi, \eta) =< (\lambda^2 - A^2)^{-1}\xi, \eta >$. It satisfies the r-matrix Ansatz

$$\{L(\lambda) \underset{,}{\otimes} L(\mu)\} = [r(\lambda, \mu), L(\lambda) \otimes I] + [r'(\lambda, \mu), I \otimes L(\lambda)],$$

with

$$r(\lambda, \mu) = \frac{2\lambda}{\lambda^2 - \mu^2} P_{\lambda\mu}, \quad r'(\lambda, \mu) = \frac{2\mu}{\lambda^2 - \mu^2} P_{\mu\lambda} = -r(\mu, \lambda),$$

$$P_{\lambda\mu} = \begin{pmatrix} \lambda & 0 & 0 & 0 \\ 0 & 0 & \mu & 0 \\ 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & \lambda \end{pmatrix}.$$

Integrable symplectic map

Suppose that the roots of $F(\lambda) = \det L(\lambda)$ are $\zeta_j = \lambda_j^2$, $j = 1, \ldots, N$, then we have the factorization

$$F(\lambda) = -\frac{\prod_{j=1}^{N}(\zeta - \lambda_j^2)}{4\alpha(\zeta)} = -\frac{R(\zeta)}{4\alpha^2(\zeta)},$$

where $\alpha(\zeta) = \prod_{j=1}^{N}(\zeta - \alpha_j^2), R(\zeta) = \alpha(\zeta)\prod_{j=1}^{N}(\zeta - \lambda_j^2)$. Thus a hyperelliptic curve

$$\mathcal{R}: \quad \xi^2 = R(\zeta) = \prod_{j=1}^{N}(\zeta - \alpha_j^2)(\zeta - \lambda_j^2),$$

with genus $g = N - 1$, is defined. The Riemann surface where ζ is consists of two sheets, and the curve $\mathcal{R}$ is of hyperelliptic involution in the sense that $\tau : (\zeta, \xi) \to (\zeta, -\xi)$ maps $\mathcal{R}$ to itself. For a non-branching point ζ on the Riemann surface, we have

$$\mathfrak{p}_+(\zeta) = \left(\zeta,\, \xi = \sqrt{R(\zeta)}\right), \quad \mathfrak{p}_-(\zeta) = \left(\zeta,\, \xi = -\sqrt{R(\zeta)}\right);$$

and in particular, for the infinity ∞ on the Riemann surface, we denote the two corresponding points on $\mathcal{R}$ by ∞_+, ∞_-.

Integrable symplectic map

Based on the relation $L(\lambda;\tilde{p},\tilde{q})D^{(\beta)}(\lambda;a,b) = D^{(\beta)}(\lambda;a,b)L(\lambda;p,q)$, we assert that the map

$$S_\beta : \mathbb{R}^{2N} \to \mathbb{R}^{2N}, \quad (p,q) \mapsto (\tilde{p},\tilde{q}),$$
$$\begin{pmatrix}\tilde{p}_j\\ \tilde{q}_j\end{pmatrix} = \frac{1}{\sqrt{\alpha_j^2-\beta^2}} D^{(\beta)}(\alpha_j;a,b)\begin{pmatrix}p_j\\ q_j\end{pmatrix}, \quad 1\leq j\leq N$$

is an integrable symplectic map under the constraint

$$a = \frac{-<Ap,p>}{1+<Ap,p>b}, \quad b = \frac{-1}{\beta^2 Q_\beta(Ap,p)}\Big(\frac{1}{2}+Q_\beta(A^2p,q)\pm\mathcal{H}(\beta)\Big).$$

Algebro-geometric solutions

Using the integrable symplectic map, we define discrete phase flow $\big(p(m),\, q(m)\big) = S_\beta^m(p(0), q(0))$ with initial point $(p(0), q(0)) \in \mathbb{R}^{2N}$, then the discrete KN spectral problem and the discrete Lax equation along the S_β^m-flow are rewritten as

$$h(m+1,\lambda) = D_m(\lambda)h(m,\lambda) \tag{2.2}$$

and

$$L_{m+1}(\lambda)D_m(\lambda) = D_m(\lambda)L_m(\lambda), \tag{2.3}$$

where the Darboux matrix $D_m(\lambda)$ is

$$D_m(\lambda) = D^{(\beta)}(\lambda; a_m, b_m) = \begin{pmatrix} \lambda^2 Z_m - \beta^2 & \lambda Z_m u_m \\ \lambda Z_m v_{m+1} & 1 \end{pmatrix} \tag{2.4}$$

Let $M(m,\lambda) = \begin{pmatrix} M^{11} & M^{12} \\ M^{21} & M^{22} \end{pmatrix}$ be a fundamental solution matrix of (2.2) with $M(0,\lambda)$ being the unit matrix I.

Algebro-geometric solutions

Equation (2.3) indicates that the solution space of equation (2.2) is invariant under the action of the linear map $L_m(\lambda)$. From (2.1), the traceless $L_m(\lambda)$ allows two opposite eigenvalues, denoted as $\pm\mathcal{H}(\lambda) = \pm\sqrt{-F(\lambda)}$, which are independent of the discrete argument m. Denoting the corresponding eigenvectors by $h_\pm(m,\lambda) = (h_\pm^{(1)}, h_\pm^{(2)})^T$, we have

$$L_m(\lambda)h_\pm(m,\lambda) = \pm\mathcal{H}(\lambda)h_\pm(m,\lambda), \tag{2.5a}$$

and

$$h_\pm(m+1,\lambda) = D_m(\lambda)h_\pm(m,\lambda), \tag{2.5b}$$

simultaneously.

Algebro-geometric solutions

Noting that the rank of $L_m(\lambda) \mp \mathcal{H}(\lambda)$ is 1, which means in each case the common eigenvector is uniquely determined up to a constant factor, we select two eigenvectors $h_\pm(m,\lambda)$ defined through $M(m,\lambda)$, as the following,

$$h_\pm(m,\lambda) = \begin{pmatrix} h_\pm^{(1)} \\ h_\pm^{(2)} \end{pmatrix} = M(m,\lambda) \begin{pmatrix} c_\lambda^\pm \\ 1 \end{pmatrix}, \tag{2.6}$$

where the constants $c_\lambda^\pm$ are determined by

$$L_0(\lambda) \begin{pmatrix} c_\lambda^+ & c_\lambda^- \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} c_\lambda^+ & c_\lambda^- \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \mathcal{H}(\lambda) & 0 \\ 0 & -\mathcal{H}(\lambda) \end{pmatrix},$$

i.e. taking $m = 0$ in equation (2.5a). It turns out that

$$c_\lambda^\pm = \frac{L_0^{11}(\lambda) \pm \mathcal{H}(\lambda)}{L_0^{21}(\lambda)} = \frac{-L_0^{12}(\lambda)}{L_0^{11}(\lambda) \mp \mathcal{H}(\lambda)}. \tag{2.7}$$

Algebro-geometric solutions

Introduce the meromorphic functions on $\mathcal{R}$,

$$\begin{aligned} \mathfrak{h}^{(1)}(m, \mathfrak{p}_+(\lambda^2)) &= \lambda h_+^{(1)}(m,\lambda), \quad \mathfrak{h}^{(1)}(m, \mathfrak{p}_-(\lambda^2)) = \lambda h_-^{(1)}(m,\lambda), \\ \mathfrak{h}^{(2)}(m, \mathfrak{p}_+(\lambda^2)) &= h_+^{(2)}(m,\lambda), \quad \mathfrak{h}^{(2)}(m, \mathfrak{p}_-(\lambda^2)) = h_-^{(2)}(m,\lambda). \end{aligned} \tag{2.8}$$

To associate them with the Riemann theta function, we investigate their analytic behaviors and divisors. To this end, introduce elliptic variables μ_j, ν_j in L^{12} and L^{21} by

$$\lambda^{-1} L_m^{12}(\lambda) = -Q_\lambda(Ap(m), p(m)) = \frac{u_m}{\alpha(\zeta)} \prod_{j=1}^{N-1} \big(\zeta - \mu_j^2(m)\big), \tag{2.9a}$$

$$\lambda^{-1} L_m^{21}(\lambda) = Q_\lambda(Aq(m), q(m)) = \frac{v_m}{\alpha(\zeta)} \prod_{j=1}^{N-1} \big(\zeta - \nu_j^2(m)\big), \tag{2.9b}$$

Algebro-geometric solutions

It turns out that

$$M(m,\lambda) = D_{m-1}(\lambda)D_{m-2}(\lambda)\cdots D_0(\lambda), \tag{2.10a}$$

$$L_m(\lambda)M(m,\lambda) = M(m,\lambda)L_0(\lambda), \tag{2.10b}$$

and we then have $\det M(m,\lambda) = (\zeta-\beta^2)^m$ due to $\det D_m(\lambda) = \zeta - \beta^2$.
When $\zeta \sim \infty$, for $m \geq 2$ we have

$$M^{11}(m,\lambda) = Z_0 Z_1 \cdots Z_{m-1}\zeta^m + O(\zeta^{m-1}), \tag{2.11a}$$

$$\lambda M^{12}(m,\lambda) = u_0 Z_0 Z_1 \cdots Z_{m-1}\zeta^m + O(\zeta^{m-1}), \tag{2.11b}$$

$$\lambda M^{21}(m,\lambda) = v_m Z_0 Z_1 \cdots Z_{m-1}\zeta^m + O(\zeta^{m-1}), \tag{2.11c}$$

$$M^{22}(m,\lambda) = u_0 v_m Z_0 Z_1 \cdots Z_{m-1}\zeta^{m-1} + O(\zeta^{m-2}), \tag{2.11d}$$

and for $m=1$ they are still valid except $M^{22}(1,\lambda) = 1$.

Algebro-geometric solutions

From equation (2.6) we obtain

$$\mathfrak{h}^{(1)}(m,\mathfrak{p}_+(\lambda^2))\cdot\mathfrak{h}^{(1)}(m,\mathfrak{p}_-(\lambda^2))=\zeta(\zeta-\beta^2)^m\frac{-u_m}{v_0}\prod_{j=1}^{N-1}\frac{\zeta-\mu_j^2(m)}{\zeta-\nu_j^2(0)}, \tag{2.12a}$$

$$\mathfrak{h}^{(2)}(m,\mathfrak{p}_+(\lambda^2))\cdot\mathfrak{h}^{(2)}(m,\mathfrak{p}_-(\lambda^2))=(\zeta-\beta^2)^m\frac{v_m}{v_0}\prod_{j=1}^{N-1}\frac{\zeta-\nu_j^2(m)}{\zeta-\nu_j^2(0)}. \tag{2.12b}$$

and asymptotic behaviors ($\zeta=\lambda^2\sim\infty$),

$$\mathfrak{h}^{(1)}(m,\mathfrak{p}_+(\lambda^2))=\frac{1}{2v_0}Z_0Z_1\cdots Z_{m-1}\zeta^{m+1}(1+O(\zeta^{-1})), \tag{2.13a}$$

$$\mathfrak{h}^{(1)}(m,\mathfrak{p}_-(\lambda^2))=\frac{-2u_m}{Z_0Z_1\cdots Z_{m-1}}(1+O(\zeta^{-1})), \tag{2.13b}$$

$$\mathfrak{h}^{(2)}(m,\mathfrak{p}_+(\lambda^2))=\frac{v_m}{2v_0}Z_0Z_1\cdots Z_{m-1}\zeta^{m}(1+O(\zeta^{-1})), \tag{2.13c}$$

$$\mathfrak{h}^{(2)}(m,\mathfrak{p}_-(\lambda^2))=\frac{2}{Z_0Z_1\cdots Z_{m-1}}(1+O(\zeta^{-1})). \tag{2.13d}$$

Algebro-geometric solutions

Now we are able to write down divisors of $\mathfrak{h}^{(1)}(m,\mathfrak{p})$, $\mathfrak{h}^{(2)}(m,\mathfrak{p})$ on $\mathcal{R}$, which are, respectively,

$$\mathcal{D}(\mathfrak{h}^{(1)}(m,\mathfrak{p})) = \sum_{j=1}^{g}\Big(\mathfrak{p}\big(\mu_j^2(m)\big) - \mathfrak{p}\big(\nu_j^2(0)\big)\Big) + \{\mathfrak{o}_-\} + m\{\mathfrak{p}(\beta^2)\} - (m+1)\{\infty_+\}, \tag{2.14a}$$

$$\mathcal{D}(\mathfrak{h}^{(2)}(m,\mathfrak{p})) = \sum_{j=1}^{g}\Big(\mathfrak{p}\big(\nu_j^2(m)\big) - \mathfrak{p}\big(\nu_j^2(0)\big)\Big) + m\{\mathfrak{p}(\beta^2)\} - m\{\infty_+\}, \tag{2.14b}$$

where $\mathfrak{o}_- = (\zeta=0,\ \xi=-\sqrt{R(0)})$, $g=N-1$.

Algebro-geometric solutions

Next, introduce the Abel–Jacobi variables

$$\vec{\psi}(m) = \mathcal{A}\Big(\sum_{j=1}^{g} \mathfrak{p}\big(\mu_j^2(m)\big)\Big), \quad \vec{\phi}(m) = \mathcal{A}\Big(\sum_{j=1}^{g} \mathfrak{p}\big(\nu_j^2(m)\big)\Big), \tag{2.15}$$

by using the Abel map $\mathcal{A}$. Employing Toda's dipole technique, from (2.15) and (2.14) we have

$$\vec{\psi}(m) \equiv \vec{\phi}(0) + m\vec{\Omega}_\beta + \vec{\Omega}_0, \quad (\mathrm{mod}\, \mathcal{T}), \tag{2.16a}$$

$$\vec{\phi}(m) \equiv \vec{\phi}(0) + m\vec{\Omega}_\beta, \quad (\mathrm{mod}\, \mathcal{T}), \tag{2.16b}$$

$$\vec{\Omega}_\beta = \int_{\mathfrak{p}(\beta^2)}^{\infty_+} \vec{\omega}, \quad \vec{\Omega}_0 = \int_{\mathfrak{o}_-}^{\infty_+} \vec{\omega}. \tag{2.16c}$$

Algebro-geometric solutions

Then, by comparing divisors we obtain the meromorphic functions in terms of the Riemann theta function:

$$\mathfrak{h}^{(1)}(m,\mathfrak{p}) = C_m^{(1)} \frac{\theta(-\mathcal{A}(\mathfrak{p}) + \vec{\psi}(m) + \vec{K}; B)}{\theta(-\mathcal{A}(\mathfrak{p}) + \vec{\phi}(0) + \vec{K}; B)} \exp \int_{\mathfrak{p}_0}^{\mathfrak{p}} (m\,\omega[\mathfrak{p}(\beta^2), \infty_+] + \omega[\mathfrak{o}_-, \infty_+]), \tag{2.17a}$$

$$\mathfrak{h}^{(2)}(m,\mathfrak{p}) = C_m^{(2)} \frac{\theta(-\mathcal{A}(\mathfrak{p}) + \vec{\phi}(m) + \vec{K}; B)}{\theta(-\mathcal{A}(\mathfrak{p}) + \vec{\phi}(0) + \vec{K}; B)} \exp \int_{\mathfrak{p}_0}^{\mathfrak{p}} m\,\omega[\mathfrak{p}(\beta^2), \infty_+], \tag{2.17b}$$

where $C_m^{(1)}$ and $C_m^{(2)}$ are constant factors and the Riemann constant vector $\vec{K}$ is defined as

$$\vec{K} = -\sum_{k=1}^{g} \left[\int_{a_k} \mathcal{A}\, \omega_k - \left(\frac{B_{kk}}{2} + \mathcal{A}_k(\mathfrak{q}_k) \right) \vec{\delta}_k \right], \tag{2.18}$$

$$\vec{K}_j = \frac{1 + B_{jj}}{2} - \sum_{\substack{k=1 \\ k \neq j}}^{g} \int_{a_k} \mathcal{A}_j(\mathfrak{p})\, \omega_k, \quad j = 1, \ldots, g, \tag{2.19}$$

Here, $\omega[\mathfrak{p}, \mathfrak{q}]$ is the dipole, a meromorphical differential that has only simple poles at $\mathfrak{p}$ and $\mathfrak{q}$ with residues $+1$ and -1, respectively.

Algebro-geometric solutions

Our purpose is to derive explicit expression of Z_m in terms of the Riemann theta function. To achieve that, first, we take $\mathfrak{p} \to \infty_-$ in equation (2.17b). This gives rise to

$$C_m^{(2)} = \frac{2}{Z_0 Z_1 \cdots Z_{m-1}} \frac{\theta[\vec{\phi}(0) + \vec{K} + \vec{\eta}_{\infty_-}]}{\theta[\vec{\phi}(m) + \vec{K} + \vec{\eta}_{\infty_-}]} \exp \int_{\infty_-}^{\mathfrak{p}_0} m\,\omega[\mathfrak{p}(\beta^2), \infty_+], \quad (2.20)$$

where $\vec{\eta}_{\infty_-} = -\mathcal{A}(\infty_-)$. Next, we consider the second row in equation (2.5b), i.e.

$$\mathfrak{h}^{(2)}(m+1, \mathfrak{p}) = b_m \mathfrak{h}^{(1)}(m, \mathfrak{p}) + \mathfrak{h}^{(2)}(m, \mathfrak{p}), \quad (2.21)$$

which reads

$$\mathfrak{h}^{(2)}(m+1, \mathfrak{o}_-) = \mathfrak{h}^{(2)}(m, \mathfrak{o}_-) \quad (2.22)$$

at the point $\mathfrak{o}_-$ since $\mathfrak{h}^{(1)}(m, \mathfrak{o}_-) = 0$. Substituting (2.17b) with $\mathfrak{p} = \mathfrak{o}_-$ into (2.22) immediately yields

$$\frac{C_m^{(2)}}{C_{m+1}^{(2)}} = \frac{\theta(\vec{\phi}(m+1) + \vec{K} + \vec{\eta}_{\mathfrak{o}_-}; B)}{\theta(\vec{\phi}(m) + \vec{K} + \vec{\eta}_{\mathfrak{o}_-}; B)} \exp \int_{\mathfrak{p}_0}^{\mathfrak{o}_-} \omega[\mathfrak{p}(\beta^2), \infty_+], \quad (2.23)$$

where $\vec{\eta}_{\mathfrak{o}_-} = -\mathcal{A}(\mathfrak{o}_-)$.

Algebro-geometric solutions

Now, substituting (2.20) into the above equation, we arrive at an explicit expression of Z_m in terms of theta function, i.e.

$$Z_m = \frac{\theta(\vec{\phi}(m+1) + \vec{K} + \vec{\eta}_{\mathfrak{o}_-}; B) \cdot \theta(\vec{\phi}(m) + \vec{K} + \vec{\eta}_{\infty_-}; B)}{\theta(\vec{\phi}(m+1) + \vec{K} + \vec{\eta}_{\infty_-}; B) \cdot \theta(\vec{\phi}(m) + \vec{K} + \vec{\eta}_{\mathfrak{o}_-}; B)} \exp \int_{\infty_-}^{\mathfrak{o}_-} \omega[\mathfrak{p}(\beta^2), \infty_+]. \tag{2.24}$$

With Z_m in hand, for a function W_m that obeys equation $W_{m+1} - W_m = \ln Z_m$ where Z_m is given in (2.24), one can obtain an explicit solution by "integration",

$$W_m = W_0 + \ln \frac{\theta[\vec{\phi}(m) + \vec{K} + \vec{\eta}_{\mathfrak{o}_-}] \cdot \theta[\vec{\phi}(0) + \vec{K} + \vec{\eta}_{\infty_-}]}{\theta[\vec{\phi}(m) + \vec{K} + \vec{\eta}_{\infty_-}] \cdot \theta[\vec{\phi}(0) + \vec{K} + \vec{\eta}_{\mathfrak{o}_-}]} + m \int_{\infty_-}^{\mathfrak{o}_-} \omega[\mathfrak{p}(\beta^2), \infty_+]. \tag{2.25}$$

Algebro-geometric solutions

The above discussions and results are valid for $(m, \beta) = (m_i, \beta_i)$, $i = 1, 2, 3$. Thus, we have three integrable symplectic maps S_{β_1}, S_{β_2} and S_{β_3}. This enables us to derive algebro-geometric solutions to lpmKP equation, namely

$$
\begin{aligned}
W(m_1, m_2, m_3) = \ln & \frac{\theta(\sum_{k=1}^{3} m_k \vec{\Omega}_{\beta_k} + \vec{\phi}(0,0,0) + \vec{K} + \vec{\eta}_{\mathfrak{o}_-}; B)}{\theta(\sum_{k=1}^{3} m_k \vec{\Omega}_{\beta_k} + \vec{\phi}(0,0,0) + \vec{K} + \vec{\eta}_{\infty_-}; B)} \cdot \\
& \cdot \frac{\theta(\vec{\phi}(0,0,0) + \vec{K} + \vec{\eta}_{\infty_-}; B)}{\theta(\vec{\phi}(0,0,0) + \vec{K} + \vec{\eta}_{\mathfrak{o}_-}; B)} \\
& + \sum_{k=1}^{3} m_k \int_{\infty_-}^{\mathfrak{o}_-} \omega[\mathfrak{p}(\beta_k^2), \infty_+] + W(0,0,0),
\end{aligned} \tag{2.26}
$$

where the dipole differential $\omega[\mathfrak{p}(\beta_k^2), \infty_+]$ is defined as

$$
\omega[\mathfrak{p}(\beta_k^2), \infty_+] = \left(\zeta + \frac{\xi + \sqrt{R(\beta_k^2)}}{\zeta - \beta_k^2} \right) \frac{\mathrm{d}\zeta}{2\sqrt{R(\zeta)}}. \tag{2.27}
$$

An example: $g = 1$ case

The algebro-geometric solution (2.26) in the case of $g = 1$ can be expressed as

$$W(m_1, m_2, m_3) = \ln \frac{\vartheta_3(\sum_{k=1}^{3} m_k \Omega_{\beta_k} + \phi(0,0,0) + K_1 + \eta_{\mathfrak{o}_-} | B_{11})}{\vartheta_3(\sum_{k=1}^{3} m_k \Omega_{\beta_k} + \phi(0,0,0) + K_1 + \eta_{\infty_-} | B_{11})} \cdot \frac{\vartheta_3(\phi(0,0,0) + K_1 + \eta_{\infty_-} | B_{11})}{\vartheta_3(\phi(0,0,0) + K_1 + \eta_{\mathfrak{o}_-} | B_{11})} + \sum_{k=1}^{3} m_k \int_{\infty_-}^{\mathfrak{o}_-} \omega[\mathfrak{p}(\beta_k^2), \infty_+] + W(0,0,0), \quad (2.28)$$

where

$$\Omega_{\beta_k} = \int_{\mathfrak{p}(\beta_k^2)}^{\infty_+} \omega_1, \quad K_1 = \frac{1 + B_{11}}{2}, \quad (2.29a)$$

$$\eta_{\mathfrak{o}_-} = -\int_{\mathfrak{p}_0}^{\mathfrak{o}_-} \omega_1, \quad \eta_{\infty_-} = -\int_{\mathfrak{p}_0}^{\infty_-} \omega_1, \quad (2.29b)$$

$$\omega[\mathfrak{p}(\beta_k^2), \infty_+] = \frac{1}{C_{11}} \left(\zeta + \frac{\xi + \sqrt{R(\beta_k^2)}}{\zeta - \beta_k^2} \right) \omega_1. \quad (2.29c)$$

An example: $g = 1$ case

Note that due to the arbitrariness of $\phi(0,0,0)$ we can always vanish $\phi(0,0,0) + K_1$ and thus we come to

$$W(m_1, m_2, m_3) = W_2(m_1, m_2, m_3) + W_1(m_1, m_2, m_3) \tag{2.30a}$$

with

$$W_2(m_1, m_2, m_3) = \ln \frac{\vartheta_3(\sum_{k=1}^{3} m_k \Omega_{\beta_k} + \eta_{\mathfrak{o}_-} | B_{11}) \cdot \vartheta_3(\eta_{\infty_-} | B_{11})}{\vartheta_3(\sum_{k=1}^{3} m_k \Omega_{\beta_k} + \eta_{\infty_-} | B_{11}) \cdot \vartheta_3(\eta_{\mathfrak{o}_-} | B_{11})}, \tag{2.30b}$$

$$W_1(m_1, m_2, m_3) = \sum_{k=1}^{3} m_k \int_{\infty_-}^{\mathfrak{o}_-} \omega[\mathfrak{p}(\beta_k^2), \infty_+] + W(0,0,0), \tag{2.30c}$$

where $\Omega_{\beta_k}, \eta_{\mathfrak{o}_-}, \eta_{\infty_-}$ and $\omega[\mathfrak{p}(\beta_k^2), \infty_+]$ are computed from (2.29b), and $W_1(m_1, m_2, m_3)$ acts as a linear background of $W(m_1, m_2, m_3)$.

An example: $g = 1$ case

The quasi-periodic evolution of $W_2(m_1, m_2, m_3)$ is shown in Figure.

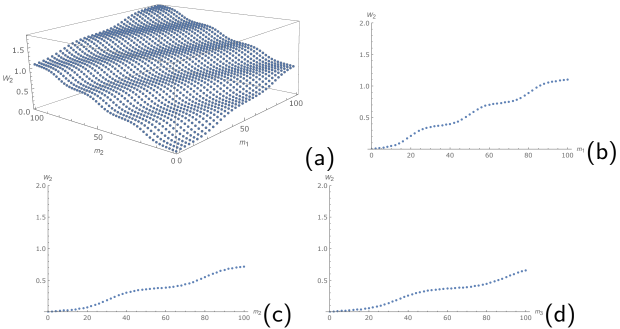

Figure: Shape and motion of $W_2(m_1, m_2, m_3)$ given in (2.30b) for $\mathfrak{p}_0 = (-3.0, 45.9565)$. (a) 3D plot of $W_2(m_1, m_2, 0)$. (b) 2D plot of $W_2(m_1, 0, 0)$. (c) 2D plot of $W_2(0, m_2, 0)$. (d) 2D plot of $W_2(0, 0, m_3)$.

An example: $g = 1$ case

One can see a periodic wave coupled with an apparent linear background that is different from $W_1(m_1, m_2, m_3)$. This is because in our example all $\{\Omega_k\}$ and B_{11} are pure imaginary and Jacobi's function $\vartheta_3(z|B_{11})$ has a z-dependent periodic multiplier $e^{-\pi i B_{11}} e^{-2\pi i z}$ with respect to B_{11}, i.e.

$$\vartheta_3(z + B_{11} \mid B_{11}) = e^{-\pi i B_{11}} e^{-2\pi i z} \vartheta_3(z \mid B_{11}).$$

It is the periodic multiplier to give rise to the linear background when $W_2(m_1, m_2, m_3)$ evolves with respect to $\{m_k\}$ via the formula (2.30b).

Concluding remarks

- Extending solutions to full space
- Constructing algebro-geometric solutions containing two soliton parameters for 3D lattice equations
- Applying the scheme to other ABS equations and 3D lattice equations that are 4D consistent
- Finite-gap integration based on theory of trigonal curves for discrete integrable systems

Thanks for your attention !