Discrete Lagrangian multiforms on the difference variational bicomplex

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Based on joint works with Peter Hydon (Kent) and Frank Nijhoff (Leeds)

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Discrete integrable systems: closure relation

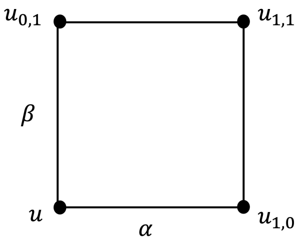
A review of the differential variational bicomplex

Construction of the difference variational bicomplex

Discrete Lagrangian multiforms

Closure relation of discrete integrable systems: H1 eq.

- Let m, n be two discrete independent variables and let $u = u(m, n)$ be the dependent variable.
- Shifts of u will be denoted by $u_{i,j} = u(m+i, n+j)$, e.g., $u_{1,0} = u(m+1, n)$, $u_{0,1} = u(m, n+1)$, etc.



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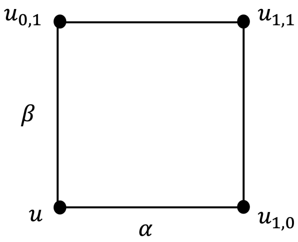

Example. H1 (lattice potential KdV, 3-leg form) equation

$$u_{1,0} - u_{0,1} - \frac{\alpha - \beta}{u - u_{1,1}} = 0$$

Closure relation of discrete integrable systems: H1 eq.

- (Discrete) Lagrangian [Capel–Nijhoff–Papageorgiou, 1991]:

$$L(u, u_{1,0}, u_{0,1}; \alpha, \beta) = (u_{1,0} - u_{0,1})u - (\alpha - \beta)\ln(u_{1,0} - u_{0,1})$$

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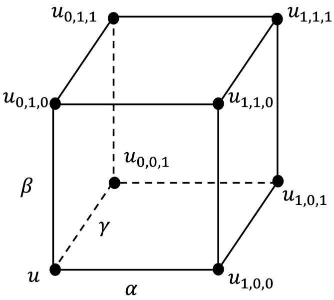
- **Closure relation** [Lobb–Nijhoff, 2009]:

$$(\mathrm{S}_3 - \mathrm{id})L_{12} + (\mathrm{S}_2 - \mathrm{id})L_{31} + (\mathrm{S}_1 - \mathrm{id})L_{23} = 0 \text{ on solutions}$$

where

$$L_{12} = L(u, u_{1,0,0}, u_{0,1,0}; \alpha, \beta)$$
$$L_{31} = L(u, u_{0,0,1}, u_{1,0,0}; \gamma, \alpha), L_{23} = L(u, u_{0,1,0}, u_{0,0,1}; \beta, \gamma)$$



A review of the differential variational bicomplex

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- Consider a trivial bundle $\pi : X \times U \to X$ with $\pi(\mathbf{x}, \mathbf{u}) = \mathbf{x}$:
  - $\mathbf{x} = (x^1, \ldots, x^p) \in X \subset \mathbb{R}^p$ (independent variables)
  - $\mathbf{u} = (u^1, \ldots, u^q) \in U \subset \mathbb{R}^q$ (dependent variables)
- Solution $\mathbf{u} = f(\mathbf{x})$ of a DE is interpreted as a local section $s(\mathbf{x}) = (\mathbf{x}, f(\mathbf{x}))$.

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- Solution $\mathbf{u} = f(\mathbf{x})$ of a DE is interpreted as a local section $s(\mathbf{x}) = (\mathbf{x}, f(\mathbf{x}))$.
- A DE defines a submanifold of prolonged jet bundles; in particular, the infinite jet bundle $J^\infty(X \times U)$ is coordinated by

$$(x^i, u^\alpha, u^\alpha_{\mathbf{1}_i}, \ldots, u^\alpha_{\mathbf{J}}, \ldots),$$

  where a section $s(\mathbf{x}) = (\mathbf{x}, f(\mathbf{x}))$ is prolonged to

$$(u^\alpha_i =) u^\alpha_{\mathbf{1}_i} = \frac{\partial f^\alpha(x)}{\partial x^i}, \quad \ldots, \quad u^\alpha_{\mathbf{J}} = \frac{\partial^{|\mathbf{J}|} f^\alpha(x)}{\partial(x^1)^{j_1} \partial(x^2)^{j_2} \ldots \partial(x^p)^{j_p}}, \quad \ldots$$

- Here $\mathbf{1}_i = (0, \ldots, 1, \ldots, 0)$, $\mathbf{J} = (j_1, j_2, \ldots, j_p)$ and $|\mathbf{J}| = j_1 + j_2 + \cdots + j_p$.

- Let $[\mathbf{u}]$ denote $\mathbf{u}$ and finitely many of their partial derivatives, e.g. $([\mathbf{u}]) = (u^\alpha, u^\alpha_{\mathbf{1}_i}, \dots, u^\alpha_{\mathbf{K}})$.
- The *differential* of a function $F(\mathbf{x}, [\mathbf{u}])$ on $J^\infty(X \times U)$ is

$$
\begin{aligned}
\mathrm{d}F(\mathbf{x}, [\mathbf{u}]) &= \frac{\partial F}{\partial x^i}\,\mathrm{d}x^i + \frac{\partial F}{\partial u^\alpha_{\mathbf{J}}}\,\mathrm{d}u^\alpha_{\mathbf{J}} \\
&= (\mathrm{D}_i F)\,\mathrm{d}x^i + \frac{\partial F}{\partial u^\alpha_{\mathbf{J}}}\left(\mathrm{d}u^\alpha_{\mathbf{J}} - u^\alpha_{\mathbf{J}+\mathbf{1}_i}\,\mathrm{d}x^i\right),
\end{aligned}
$$

  where the total derivative is

$$
\mathrm{D}_i = \frac{\partial}{\partial x^i} + u^\alpha_{\mathbf{1}_i}\frac{\partial}{\partial u^\alpha} + \cdots + u^\alpha_{\mathbf{J}+\mathbf{1}_i}\frac{\partial}{\partial u^\alpha_{\mathbf{J}}} + \cdots .
$$

- This allows a **splitting of the exterior derivative** $\mathrm{d} = \mathrm{d_h} + \mathrm{d_v}$ with
  - *Horizontal operator:* $\mathrm{d_h} := \mathrm{d}x^i \wedge \mathrm{D}_i$
  - *Vertical operator:* $\mathrm{d_v} := \left(\mathrm{d}u^\alpha_{\mathbf{J}} - u^\alpha_{\mathbf{J}+\mathbf{1}_i}\,\mathrm{d}x^i\right) \wedge \frac{\partial}{\partial u^\alpha_{\mathbf{J}}}$

- A basis for one-forms on $J^\infty(X \times U)$ can then be chosen as

$$\{\mathrm{d}x^i\}$$

and the *contact forms*

$$\{\mathrm{d_v}\, u^\alpha = \mathrm{d}u^\alpha - u^\alpha_{\mathbf{1}_i}\,\mathrm{d}x^i, \quad \ldots, \quad \mathrm{d_v}\, u^\alpha_{\mathbf{J}} = \mathrm{d}u^\alpha_{\mathbf{J}} - u^\alpha_{\mathbf{J}+\mathbf{1}_i}\,\mathrm{d}x^i, \quad \ldots\}.$$

This basis extends to a basis for the set of all differential forms on $J^\infty(X \times U)$, denoted by Ω, using the wedge product.

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- From $\mathrm{d}^2 = 0$, direct calculations lead to

$$\mathrm{d}_\mathrm{h}^2 = 0, \quad \mathrm{d_h}\mathrm{d_v} = -\,\mathrm{d_v}\mathrm{d_h}, \quad \mathrm{d}_\mathrm{v}^2 = 0.$$

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- A $(k+l)$-form ω is said to be of type (k,l) if it can be written as

$$\omega = f^{\mathbf{J}_1,\ldots,\mathbf{J}_l}_{i_1,\ldots,i_k;\alpha_1,\ldots,\alpha_l}(\mathbf{x},[\mathbf{u}])\,\mathrm{d_h}\,x^{i_1} \wedge \cdots \wedge \mathrm{d_h}\,x^{i_k} \wedge \mathrm{d_v}\,u^{\alpha_1}_{\mathbf{J}_1} \wedge \cdots \mathrm{d_h}\,u^{\alpha_l}_{\mathbf{J}_l}.$$

Denote all (k,l)-forms over $J^\infty(X \times U)$ as $\Omega^{k,l}$ and

$$\mathrm{d_h} : \Omega^{k,l} \to \Omega^{k+1,l}, \quad \mathrm{d_v} : \Omega^{k,l} \to \Omega^{k,l+1}$$

yield a **double complex**.

The (differential) variational bicomplex

$$
\begin{array}{ccccccccccccc}
 & & & & \vdots & & \vdots & & & & \vdots & & \vdots & & \\
 & & & & \uparrow d_v & & \uparrow d_v & & & & \uparrow d_v & & \uparrow d_v & & \\
0 & \longrightarrow & & & \Omega^{0,2} & \xrightarrow{d_h} & \Omega^{1,2} & \xrightarrow{d_h} & \cdots & \xrightarrow{d_h} & \Omega^{p-1,2} & \xrightarrow{d_h} & \Omega^{p,2} & \xrightarrow{d_h} & 0 \\
 & & & & \uparrow d_v & & \uparrow d_v & & & & \uparrow d_v & & \uparrow d_v & & \\
0 & \longrightarrow & & & \Omega^{0,1} & \xrightarrow{d_h} & \Omega^{1,1} & \xrightarrow{d_h} & \cdots & \xrightarrow{d_h} & \Omega^{p-1,1} & \xrightarrow{d_h} & \Omega^{p,1} & \xrightarrow{d_h} & 0 \\
 & & & & \uparrow d_v & & \uparrow d_v & & & & \uparrow d_v & & \uparrow d_v & & \\
0 & \longrightarrow & \mathbb{R} & \longrightarrow & \Omega^{0,0} & \xrightarrow{d_h} & \Omega^{1,0} & \xrightarrow{d_h} & \cdots & \xrightarrow{d_h} & \Omega^{p-1,0} & \xrightarrow{d_h} & \Omega^{p,0} & \xrightarrow{d_h} & 0
\end{array}
$$

Cohomology of the variational bicomplex

Note: For a cochain complex

$$\cdots \to A_{i-1} \xrightarrow{\mathrm{d}_{i-1}} A_i \xrightarrow{\mathrm{d}_i} A_{i+1} \to \cdots ,$$

its cohomology groups are

$$H^i := \frac{\ker \mathrm{d}_i}{\operatorname{im} \mathrm{d}_{i-1}} .$$

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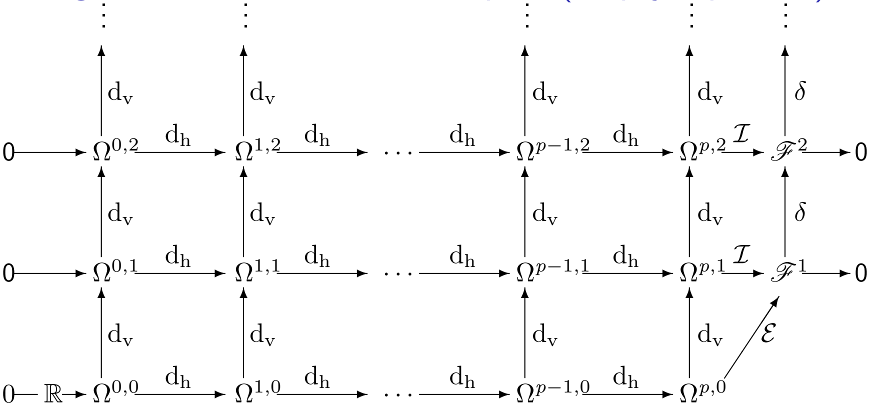
Theorems. [Vinogradov, 1984]

- *Empty equation/free case*: One-line theorem
  † Only horizontal cohomologies at the last column are nontrivial.
- *ℓ-normal equations*: Two-line theorem (e.g. Kovalevskaya type of equations)
  † Symmetries are in the kernel of the linearization operator, while conservation laws (co-symmetries) are in the kernel of its adjoint.
  † Euler–Lagrange equations are self-adjoint $\longrightarrow$ Noether's theorem
- *Non ℓ-normal equations:* Three-line theorem (e.g. Maxwell, Yang–Mills, Einstein equations)

The augmented variational bicomplex (empty equation)



- The *interior Euler operator* is

$$\mathcal{I}(\omega) := \frac{1}{l}\, d_v u^\alpha \wedge (-D)_{\mathbf{J}} \left(\frac{\partial}{\partial u^\alpha_{\mathbf{J}}} \lrcorner\, \omega \right), \quad \forall \omega \in \Omega^{p,l}, \quad l \geq 1,$$

where $(-D)_{\mathbf{J}} = (-1)^{|\mathbf{J}|} D_{\mathbf{J}}$ is adjoint to $D_{\mathbf{J}} = D_1^{j_1} D_2^{j_2} \cdots D_p^{j_p}$ for $\mathbf{J} = (j_1, j_2, \ldots, j_p)$.

- The interior Euler operator is a projection, namely $\mathcal{I}^2 = \mathcal{I}$, and $\mathscr{F}^l = \mathcal{I}(\Omega^{p,l}) \subset \Omega^{p,l}$.
- The *Euler–Lagrange operator* is given by $\mathcal{E} := \mathcal{I}\,\mathrm{d}_{\mathrm{v}}$ and define $\delta := \mathcal{I}\,\mathrm{d}_{\mathrm{v}}$.

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Theorem. The following properties hold that

$$\mathcal{I}\,\mathrm{d}_\mathrm{h} = 0, \quad \mathcal{E}\,\mathrm{d}_\mathrm{h} = 0, \quad \delta\mathcal{E} = 0, \quad \delta^2 = 0.$$

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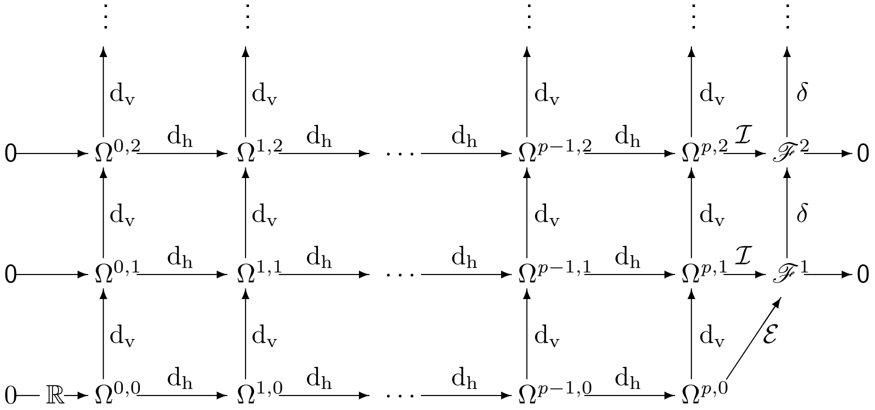
Remark. The boundary complex is called the *Euler–Lagrange complex* or the *variational complex*. When $p = 3$, it is

$$0 \longrightarrow \mathbb{R} \longrightarrow \Omega^0 \longrightarrow \Omega^1 \longrightarrow \Omega^2 \longrightarrow \Omega^3 \longrightarrow \mathscr{F}^1 \longrightarrow \mathscr{F}^2 \longrightarrow \cdots$$

Grad Curl Div Euler Helmholtz

The augmented variational bicomplex



- Lagrangian forms: $\Omega^{p,0}$ & Euler–Lagrange equations: $\mathscr{F}^1$
- Conservation Laws: $\Omega^{p-1,0} \longleftrightarrow$ Symmetries
- Helmholtz conditions: $\mathscr{F}^2 \longleftrightarrow$ Inverse problems
- Lagrangian k-forms: $\Omega^{k,0} \longleftarrow$ Integrability

The difference variational bicomplex

LP, From Differential to Difference: The Variational Bicomplex and Invariant Noether's Theorems, Ph.D. Thesis, University of Surrey, 2013.

LP–Hydon, The difference variational bicomplex and multisymplectic systems, arXiv:2307.13935, 2023.

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Some challenges:

- Discrete counterpart of jet spaces (differentiable manifolds) ✓
- Arrange differential and difference forms into horizontal and vertical forms ✓
- Cohomology
  - One-line theorem: variational calculus, inverse problem, Noether's theorem ✓
  - Two-line theorem: conservation laws (cosymmetries) of normal equations ([Mikhailov–Wang–Xenitidis, 2011] on cosymmetries)
  - Three-line theorem

The total prolongation space (discrete counterpart of jets)

[Mansfield–Rojo-Echeburúa–Hydon–LP, 2019], [LP–Hydon, 2023]

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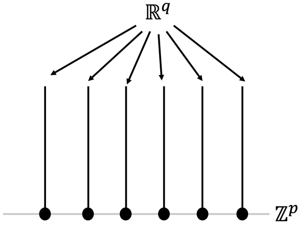
- Consider a P∆E with p independent variables $\mathbf{n} = (n^1, \ldots, n^p) \in \mathbb{Z}^p$, and q dependent variables $\mathbf{u} = (u^1, \ldots, u^q) \in \mathbb{R}^q$. They form a *total space* $\mathbb{Z}^p \times \mathbb{R}^q$.



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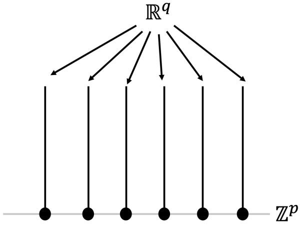

- Fibres are mapped to one another by translations ($\mathbf{J} \in \mathbb{Z}^p$)

$$\begin{aligned} \mathrm{T}_{\mathbf{J}} : \mathbb{Z}^p \times \mathbb{R}^q &\to \mathbb{Z}^p \times \mathbb{R}^q \\ (\mathbf{n}, \mathbf{u}) &\mapsto (\mathbf{n} + \mathbf{J}, \mathbf{u}). \end{aligned}$$

- As the total space is disconnected, it is necessary to construct a connected representative over each base point. We prolong each fibre to include values over all other fibres in a Cartesian product by pulling back each $\mathbf{u}$ using $\mathrm{T}_{\mathbf{J}}$:

$$u^{\alpha}_{\mathbf{J}} = \mathrm{T}^{*}_{\mathbf{J}}(u^{\alpha}|_{\mathbf{n}+\mathbf{J}}).$$

  This gives the (connected) **total prolongation space** $P(\mathbb{R}^q)$ over an arbitrary base point with local coordinates $(\ldots, u^{\alpha}_{\mathbf{J}}, \ldots)$.

- Let f be a function on $\mathbb{Z}^p \times P(\mathbb{R}^q)$. Its restriction to each total prolongation space $P_{\mathbf{n}}(\mathbb{R}^q)$ is denoted by

$$f_{\mathbf{n}}(\ldots, u^{\alpha}_{\mathbf{J}}, \ldots) = f(\mathbf{n}, \ldots, u^{\alpha}_{\mathbf{J}}, \ldots).$$

  The pullback of $f_{\mathbf{n}+\mathbf{K}}(\ldots, u^{\alpha}_{\mathbf{J}}, \ldots)$ defined in $P_{\mathbf{n}+\mathbf{K}}(\mathbb{R}^q)$ with respect to $\mathrm{T}_{\mathbf{K}}$ is the function

$$\mathrm{T}^{*}_{\mathbf{K}} f_{\mathbf{n}+\mathbf{K}}(\ldots, u^{\alpha}_{\mathbf{J}}, \ldots) = f(\mathbf{n}+\mathbf{K}, \ldots, u^{\alpha}_{\mathbf{J}+\mathbf{K}}, \ldots)$$

  on $P_{\mathbf{n}}(\mathbb{R}^q)$.

Shift operators

The **shift operator** $\mathrm{S}_{\mathbf{K}}$ is defined by $\mathrm{S}_{\mathbf{K}}\, f_{\mathbf{n}} = \mathrm{T}^{*}_{\mathbf{K}}\, f_{\mathbf{n}+\mathbf{K}}$:

$$\mathrm{S}_{\mathbf{K}} : f(\mathbf{n}, \ldots, u^{\alpha}_{\mathbf{J}}, \ldots) \mapsto f(\mathbf{n}+\mathbf{K}, \ldots, u^{\alpha}_{\mathbf{J}+\mathbf{K}}, \ldots),$$

where both $f_{\mathbf{n}}$ and $\mathrm{S}_{\mathbf{K}}\, f_{\mathbf{n}}$ are functions in $P_{\mathbf{n}}(\mathbb{R}^{q})$.

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where both $f_{\mathbf{n}}$ and $\mathrm{S}_{\mathbf{K}} f_{\mathbf{n}}$ are functions in $P_{\mathbf{n}}(\mathbb{R}^q)$.

- For any $\mathbf{K} = (k_1, \ldots, k_p)$, $\mathrm{S}_{\mathbf{K}} = \mathrm{S}_1^{k_1} \cdots \mathrm{S}_p^{k_p}$ where $\mathrm{S}_i = \mathrm{S}_{\mathbf{1}_i}$
- The **forward difference** in the n^i-direction is represented on $P_{\mathbf{n}}(\mathbb{R}^q)$ by the operator

$$\mathrm{D}_{n^i} = \mathrm{S}_i - \mathrm{id}$$

- Adjoint operators:

$$\mathrm{S}^{\dagger}_{\mathbf{K}} = \mathrm{S}_{-\mathbf{K}}, \quad \mathrm{D}^{\dagger}_{n^i} = -\,\mathrm{S}_i^{-1}\,\mathrm{D}_{n^i}$$

Differential forms

- Let ω be a differential form on $\mathbb{Z}^p \times P(\mathbb{R}^q)$ whose restriction to $P_{\mathbf{n}}(\mathbb{R}^q)$ is $\omega_{\mathbf{n}}$. The action of $\mathrm{S}_{\mathbf{K}}$ on $\omega_{\mathbf{n}}$ is represented by

$$\mathrm{S}_{\mathbf{K}}\,\omega_{\mathbf{n}} = \mathrm{T}_{\mathbf{K}}^{*}\,\omega_{\mathbf{n}+\mathbf{K}}.$$

- $\mathrm{S}_{\mathbf{K}}$ commutes with the wedge product and with the exterior derivative, denoted by d_{v}:

$$\mathrm{S}_{\mathbf{K}}(\omega_1 \wedge \omega_2) = (\mathrm{S}_{\mathbf{K}}\,\omega_1) \wedge (\mathrm{S}_{\mathbf{K}}\,\omega_2), \quad \mathrm{S}_{\mathbf{K}}(\mathrm{d}_{\mathrm{v}}\omega) = \mathrm{d}_{\mathrm{v}}(\mathrm{S}_{\mathbf{K}}\,\omega).$$

Difference forms

Exterior algebra of p symbols, $\Delta^1, \ldots, \Delta^p$ ([Kupershmidt, 1985]; [Hydon–Mansfield, 2004]).

- Invariance with respect to shifts: $\Delta^i|_{\mathbf{n}} = \mathrm{T}^*_{\mathbf{K}}(\Delta^i|_{\mathbf{n}+\mathbf{K}}) =: \mathrm{S}_{\mathbf{K}}(\Delta^i|_{\mathbf{n}})$
- **Exterior difference operator** is defined by

$$\boldsymbol{\Delta}\omega = \Delta^i \wedge \mathrm{D}_{n^i}\omega$$

for a difference k-form over $\mathbb{Z}^p \times P(\mathbb{R}^q)$

$$\omega = f_{i_1,\ldots,i_k}(\mathbf{n},\ldots,u^\alpha_{\mathbf{J}},\ldots)\Delta^{i_1} \wedge \cdots \wedge \Delta^{i_k}.$$

Differential-difference forms

Using $[\mathbf{u}]$ to denote a finite subset of $(\dots, u^{\alpha}_{\mathbf{J}}, \dots)$, a (k,l)-form over $\mathbb{Z}^p \times P(\mathbb{R}^q)$ is a $(k+l)$-form, $\omega \in \Omega^{k,l}$, that can be written as

$$\omega = f^{\mathbf{J}_1,\dots,\mathbf{J}_l}_{i_1,\dots,i_k;\alpha_1,\dots,\alpha_l}(\mathbf{n},[\mathbf{u}])\boldsymbol{\Delta}^{i_1} \wedge \dots \wedge \boldsymbol{\Delta}^{i_k} \wedge \mathrm{d_v} u^{\alpha_1}_{\mathbf{J}_1} \wedge \dots \wedge \mathrm{d_v} u^{\alpha_l}_{\mathbf{J}_l}.$$

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- (Vertical) exterior derivative $\mathrm{d_v} : \Omega^{k,l} \to \Omega^{k,l+1}$:

$$\mathrm{d_v}\omega = \frac{\partial f^{\mathbf{J}_1,\ldots,\mathbf{J}_l}_{i_1,\ldots,i_k;\alpha_1,\ldots,\alpha_l}}{\partial u^{\beta}_{\mathbf{K}}} \mathrm{d_v} u^{\beta}_{\mathbf{K}} \wedge \Delta^{i_1} \wedge \cdots \wedge \Delta^{i_k} \wedge \mathrm{d_v} u^{\alpha_1}_{\mathbf{J}_1} \wedge \cdots \wedge \mathrm{d_v} u^{\alpha_l}_{\mathbf{J}_l}$$

- (Horizontal) exterior difference $\mathrm{d_h^{\Delta}} : \Omega^{k,l} \to \Omega^{k+1,l}$:

$$\mathrm{d_h^{\Delta}}\omega = \Delta^i \wedge \mathrm{D}_{n^i}\omega$$

where

$$\mathrm{S}_{\mathbf{K}}\,\omega = \mathrm{S}_{\mathbf{K}}\left(f^{\mathbf{J}_1,\ldots,\mathbf{J}_l}_{i_1,\ldots,i_k;\alpha_1,\ldots,\alpha_l}\right) \Delta^{i_1} \wedge \cdots \wedge \Delta^{i_k} \wedge \mathrm{d_v} u^{\alpha_1}_{\mathbf{J}_1+\mathbf{K}} \wedge \cdots \wedge \mathrm{d_v} u^{\alpha_l}_{\mathbf{J}_l+\mathbf{K}}.$$

Proposition. The exterior derivative and difference satisfy

$$\left(\mathrm{d}_{\mathrm{h}}^{\triangle}\right)^2 = 0, \quad \mathrm{d}_{\mathrm{h}}^{\triangle}\mathrm{d}_{\mathrm{v}} = -\,\mathrm{d}_{\mathrm{v}}\mathrm{d}_{\mathrm{h}}^{\triangle}, \quad \mathrm{d}_{\mathrm{v}}^2 = 0.$$

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Definition. Define $\mathrm{d}^{\triangle} = \mathrm{d}_{\mathrm{h}}^{\triangle} + \mathrm{d}_{\mathrm{v}}$. It satisfies $\left(\mathrm{d}^{\triangle}\right)^2 = 0$.

- For a function f defined over $\mathbb{Z}^p \times P(\mathbb{R}^q)$:

$$\mathrm{d}^{\triangle} f(\mathbf{n}, [\mathbf{u}]) := (\mathrm{D}_{n^i} f)\Delta^i + \frac{\partial f}{\partial u^{\alpha}_{\mathbf{J}}}\,\mathrm{d}_{\mathrm{v}}\, u^{\alpha}_{\mathbf{J}},$$

Lie difference

Remark. The operator D_{n^i} is the *Lie difference* [Crampin–Pirani, 1987] with respect to the translation $\mathrm{T}_{\mathbf{1}_i}$:

$$(\mathrm{D}_{n^i}\omega)\,|_{\mathbf{n}} = \mathrm{T}^*_{\mathbf{1}_i}(\omega_{\mathbf{n}+\mathbf{1}_i}) - \omega_{\mathbf{n}}.$$

- It satisfies the *Cartan formula*

$$\mathrm{D}_{n^i}\omega = \partial_{n^i} \lrcorner\, \mathrm{d}^{\Delta}\omega + \mathrm{d}^{\Delta}(\partial_{n^i} \lrcorner\, \omega)\,,$$

  where $\{\partial_{n^1}, \dots, \partial_{n^p}\}$ are the duals to the 1-forms $\{\Delta^1, \dots, \Delta^p\}$:

$$\partial_{n^i} \lrcorner \Delta^j = \delta^j_i, \quad \partial_{n^i} \lrcorner \mathrm{d_v} u^\alpha_{\mathbf{J}} = 0, \quad \frac{\partial}{\partial u^\alpha_{\mathbf{J}}} \lrcorner \Delta^j = 0, \quad \frac{\partial}{\partial u^\alpha_{\mathbf{J}}} \lrcorner \mathrm{d_v} u^\beta_{\mathbf{K}} = \delta^\beta_\alpha \delta^{\mathbf{K}}_{\mathbf{J}}$$

The augmented difference variational bicomplex

$$
\begin{array}{ccccccccccccc}
 & & \vdots & & \vdots & & & & \vdots & & \vdots & & \vdots & \\
 & & \uparrow \mathrm{d_v} & & \uparrow \mathrm{d_v} & & & & \uparrow \mathrm{d_v} & & \uparrow \mathrm{d_v} & & \uparrow \delta^{\Delta} & \\
0 & \longrightarrow & \Omega^{0,2} & \xrightarrow{\mathrm{d_h^{\Delta}}} & \Omega^{1,2} & \xrightarrow{\mathrm{d_h^{\Delta}}} & \cdots & \xrightarrow{\mathrm{d_h^{\Delta}}} & \Omega^{p-1,2} & \xrightarrow{\mathrm{d_h^{\Delta}}} & \Omega^{p,2} & \xrightarrow{\mathcal{I}^{\Delta}} & \mathscr{F}^{2} & \longrightarrow 0 \\
 & & \uparrow \mathrm{d_v} & & \uparrow \mathrm{d_v} & & & & \uparrow \mathrm{d_v} & & \uparrow \mathrm{d_v} & & \uparrow \delta^{\Delta} & \\
0 & \longrightarrow & \Omega^{0,1} & \xrightarrow{\mathrm{d_h^{\Delta}}} & \Omega^{1,1} & \xrightarrow{\mathrm{d_h^{\Delta}}} & \cdots & \xrightarrow{\mathrm{d_h^{\Delta}}} & \Omega^{p-1,1} & \xrightarrow{\mathrm{d_h^{\Delta}}} & \Omega^{p,1} & \xrightarrow{\mathcal{I}^{\Delta}} & \mathscr{F}^{1} & \longrightarrow 0 \\
 & & \uparrow \mathrm{d_v} & & \uparrow \mathrm{d_v} & & & & \uparrow \mathrm{d_v} & & \uparrow \mathrm{d_v} & \nearrow \mathcal{E}^{\Delta} & & \\
0 \longrightarrow \mathbb{R} & \longrightarrow & \Omega^{0,0} & \xrightarrow{\mathrm{d_h^{\Delta}}} & \Omega^{1,0} & \xrightarrow{\mathrm{d_h^{\Delta}}} & \cdots & \xrightarrow{\mathrm{d_h^{\Delta}}} & \Omega^{p-1,0} & \xrightarrow{\mathrm{d_h^{\Delta}}} & \Omega^{p,0} & & &
\end{array}
$$

- The difference interior Euler operator is defined as

$$\mathcal{I}^{\Delta}(\omega) := \frac{1}{l}\,\mathrm{d_v} u^{\alpha} \wedge \mathrm{S}_{-\mathbf{J}}\left(\frac{\partial}{\partial u^{\alpha}_{\mathbf{J}}} \lrcorner\, \omega\right), \quad \forall \omega \in \Omega^{p,l}, \quad l \geq 1.$$

- Define $\delta^{\Delta} := \mathcal{I}^{\Delta}\,\mathrm{d_v}$ and the difference Euler–Lagrange operator is $\mathcal{E}^{\Delta} := \mathcal{I}^{\Delta}\,\mathrm{d_v}$.

Cohomology of the difference variational bicomplex

Proposition. Analogous to the differential case, we have

$$\mathcal{I}^{\Delta}\,\mathrm{d}_{\mathrm{h}}^{\Delta} = 0, \quad \mathcal{E}^{\Delta}\,\mathrm{d}_{\mathrm{h}}^{\Delta} = 0, \quad \delta^{\Delta}\mathcal{E}^{\Delta} = 0, \quad (\delta^{\Delta})^2 = 0.$$

[**One-line theorem.**] The augmented difference variational bicomplex (empty equation) is exact:

$$\omega = h(\mathrm{d}^{\Delta}\omega) + \mathrm{d}^{\Delta}(h(\omega))$$

with h the homotopy operators.
(Note. Exactness of the EL complex was proved in [Hydon–Mansfield, 2004]; the exactness around $\mathcal{E}^{\Delta}$ was proved in [Kupershmidt, 1985].)

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- Lagrangian forms: $\Omega^{p,0}$ & Euler–Lagrange equations: $\mathscr{F}^1$
- Conservation Laws: $\Omega^{p-1,0} \longleftrightarrow$ Symmetries
- Helmholtz conditions: $\mathscr{F}^2 \longleftrightarrow$ Inverse problems

Discrete variational problems

The H1 equation.

$$u_{1,0} - u_{0,1} - \frac{\alpha - \beta}{u - u_{1,1}} = 0$$

- Lagrangian form in $\Omega^{2,0}$:

$$\mathscr{L} = L\Delta^1 \wedge \Delta^2, \quad L = (u_{1,0} - u_{0,1})u - (\alpha - \beta)\ln(u_{1,0} - u_{0,1})$$

- Discrete Euler–Lagrange equation (two copies of H1):

$$\mathscr{F}^1 \ni \mathcal{E}^{\triangle}(\mathscr{L}) = 0$$

  where

$$\mathcal{E}^{\triangle}(\mathscr{L}) = \mathbf{E}(L)\,\mathrm{d_v} u \wedge \Delta^1 \wedge \Delta^2$$

Note. *Euler operators*:

$$\mathbf{E}_\alpha := \mathrm{S}_{-\mathbf{K}} \frac{\partial}{\partial u^\alpha_{-\mathbf{K}}}$$

Discrete Noether's theorem

- Define the difference *divergence* as $\mathrm{Div}\,\mathbf{F} := \mathrm{D}_{n^i} F^i(\mathbf{n}, [\mathbf{u}])$. A *conservation law* $\mathrm{Div}\,\mathbf{F} = 0$ can be interpreted as

$$\mathrm{d}_\mathrm{h}^\triangle \omega = 0, \quad \text{where } \omega = F^i \partial_{n^i} \lrcorner \left(\Delta^1 \wedge \cdots \wedge \Delta^p\right) \in \Omega^{p-1,0}$$

- A *variational symmetry* satisfies

$$\mathbf{v}(L) = \mathrm{Div}\,\mathbf{P}, \quad \text{where } \mathbf{v} = (\mathrm{S}_\mathbf{J}\, Q^\alpha(\mathbf{n}, [\mathbf{u}]))\, \frac{\partial}{\partial u^\alpha_\mathbf{J}}.$$

$\Longleftrightarrow$

$$\mathbf{v} \lrcorner \mathrm{d}_\mathrm{v} \mathscr{L} = \mathrm{d}_\mathrm{h}^\triangle \sigma$$

Lemma. 1. There exists $\eta \in \Omega^{p-1,1}$ such that

$$\mathrm{d_v}\mathscr{L} - \mathcal{E}^{\triangle}(\mathscr{L}) = \mathrm{d_h^{\triangle}}\eta.$$

2. For an evolutionary vector field $\mathbf{v} = (\mathrm{S_J}\, Q^{\alpha}(\mathbf{n}, [\mathbf{u}])) \frac{\partial}{\partial u^{\alpha}_{\mathbf{J}}}$, the following identity holds

$$\mathbf{v} \lrcorner\, \mathrm{d_h^{\triangle}}\omega + \mathrm{d_h^{\triangle}}(\mathbf{v} \lrcorner\, \omega) = 0, \quad \forall \omega \in \Omega^{k,l}.$$

Noether's Theorem.

$$\begin{aligned} 0 &= \mathbf{v} \lrcorner \left(\mathrm{d_v}\mathscr{L} - \mathcal{E}^{\triangle}(\mathscr{L}) - \mathrm{d_h^{\triangle}}\eta\right) \\ &= \mathrm{d_h^{\triangle}}\sigma - Q^{\alpha}\mathbf{E}_{\alpha}(L)\Delta^1 \wedge \cdots \wedge \Delta^p - \mathbf{v} \lrcorner\, \mathrm{d_h^{\triangle}}\eta \\ &= \mathrm{d_h^{\triangle}}(\sigma + \mathbf{v} \lrcorner\, \eta) - Q^{\alpha}\mathbf{E}_{\alpha}(L)\Delta^1 \wedge \cdots \wedge \Delta^p \end{aligned}$$

Discrete Lagrangian k-forms $\Omega^{k,0}$

[Hydon–Nijhoff–LP, draft]

Discrete Lagrangian k-forms $\Omega^{k,0}$

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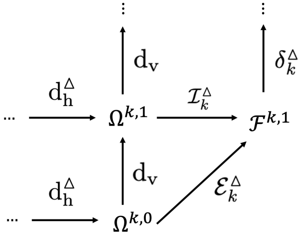

- Lagrangian k-forms:

$$\Omega^{k,0} \ni \mathscr{L}_k = \sum_{i_1<\cdots<i_k} L_{i_1\ldots i_k}(\mathbf{n},[\mathbf{u}])\Delta^{i_1} \wedge \cdots \wedge \Delta^{i_k}$$

- $\mathcal{I}_k^\triangle = \mathcal{I}^\triangle|_{\Omega^{k,l}}$. Again $\mathcal{I}_k^\triangle \, \mathrm{d}_\mathrm{h}^\triangle \equiv 0$.
- Multi Euler–Lagrange equations:

$$\mathcal{E}_k^\triangle(\mathscr{L}_k) = 0 \ \& \ \text{BEs} = 0$$

where

$$\mathrm{d}_\mathrm{v}\mathscr{L}_k - \mathcal{E}_k^\triangle(\mathscr{L}_k) - \text{BEs} = \mathrm{d}_\mathrm{h}^\triangle \eta \text{ for some } \eta \in \Omega^{k-1,1}$$

The closure relation in discrete integrable systems can then be interpreted as

$$\mathrm{d}_\mathrm{h}^\triangle \mathscr{L}_k = 0 \text{ for } k = p - 1,$$

on solutions of the multi EL equations.

The closure relation in discrete integrable systems can then be interpreted as

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on solutions of the multi EL equations.

- Recall closure relation of the H1 equation:

$$(\mathrm{S}_3 - \mathrm{id})L_{12} + (\mathrm{S}_2 - \mathrm{id})L_{31} + (\mathrm{S}_1 - \mathrm{id})L_{23} = 0 \text{ on solutions}$$

  where

$$\begin{aligned} L_{12} &= L(u, u_{1,0,0}, u_{0,1,0}; \alpha, \beta), \\ L_{31} &= L(u, u_{0,0,1}, u_{1,0,0}; \gamma, \alpha), \\ L_{23} &= L(u, u_{0,1,0}, u_{0,0,1}; \beta, \gamma). \end{aligned}$$

- The Lagrangian 2-form ($p = 3$) is simply

$$\mathscr{L}_2 = L_{12}\Delta^1 \wedge \Delta^2 + L_{31}\Delta^3 \wedge \Delta^1 + L_{23}\Delta^2 \wedge \Delta^3$$

  and each of the multi EL equations is H1.

Noether's theorem when the BEs vanish

This is the case for most integrable systems with the closure relation.

- Assume an evolutionary vector field $\mathbf{v} = (\mathrm{S}_{\mathbf{J}}\, Q^{\alpha}(\mathbf{n}, [\mathbf{u}])) \frac{\partial}{\partial u^{\alpha}_{\mathbf{J}}}$ generates a variational symmetry for each $L_{i_1,\ldots,i_k}$ as

$$\mathbf{v}\left(L_{i_1,\ldots,i_k}\right) = 0$$

  and hence

$$\mathbf{v} \lrcorner \mathrm{d_v} \mathscr{L}_k = \mathrm{d}^{\triangle}_{\mathrm{h}} \eta.$$

- A similar proof to the usual Noether's theorem follows when all BEs vanish. These conservation laws are $(k-1, 0)$-forms.

When the BEs do not vanish:

$$\mathbf{v} \lrcorner \mathcal{E}^{\triangle}_{k}(\mathscr{L}_k) + \mathbf{v} \lrcorner \text{BEs} = \mathrm{d}^{\triangle}_{\mathrm{h}} \omega$$

and $\mathbf{v} \lrcorner$ BEs is not in characteristic form.

H1 (7-point equation): no BEs

- The Lagrangian 2-form with three independent variables (m, n, l):

$$\mathscr{L}_2 = L_{12}\Delta^1 \wedge \Delta^2 + L_{31}\Delta^3 \wedge \Delta^1 + L_{23}\Delta^2 \wedge \Delta^3$$

- Multi EL equations:

$$\begin{aligned}\mathcal{E}_2^{\triangle}(\mathscr{L}_2) = {}& \mathbf{E}(L_{12})\,\mathrm{d_v}u \wedge \Delta^1 \wedge \Delta^2 \\ &+ \mathbf{E}(L_{31})\,\mathrm{d_v}u \wedge \Delta^3 \wedge \Delta^1 + \mathbf{E}(L_{23})\,\mathrm{d_v}u \wedge \Delta^2 \wedge \Delta^3\end{aligned}$$

- Consider a variational symmetry $\mathbf{v} = (\mathrm{S}_\mathbf{J}\, Q^\alpha(\mathbf{n}, [\mathbf{u}]))\,\frac{\partial}{\partial u^\alpha_\mathbf{J}}$ of $\mathscr{L}_2$, e.g., $Q = (-1)^{m+n+l}u$, such that

$$\mathbf{v} \lrcorner\, \mathrm{d_v}\mathscr{L}_2 = \mathrm{d}_\mathrm{h}^{\triangle}\eta \text{ for some } \eta \in \Omega^{1,0}$$

- Noether's theorem gives a conservation law/form $\omega \in \Omega^{1,0}$ satisfying

$$\mathbf{v} \lrcorner\, \mathcal{E}_2^{\triangle}(\mathscr{L}_2) = \mathrm{d}_\mathrm{h}^{\triangle}\omega$$

- Assume $\omega = a_1\Delta^1 + a_2\Delta^2 + a_3\Delta^3$, and the corresponding conservation laws are

$$
\begin{aligned}
Q\mathbf{E}(L_{12}) &= \mathrm{Div}(a_2, -a_1, 0) \\
Q\mathbf{E}(L_{31}) &= \mathrm{Div}(-a_3, 0, a_1) \\
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\end{aligned}
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Remark. If BEs do not vanish, then the conservation laws will look like, for instance,

$$Q\mathbf{E}(L_{12}) + (\mathrm{S}_{\mathbf{J}}\,Q) \times \text{BEs} = \mathrm{Div}\,\mathbf{F}, \quad \dots$$

meaning that summation by parts must be applied to achieve the characteristic form:

$$(\mathrm{S}_{\mathbf{J}}\,Q) \times \text{BEs} = Q \times \mathrm{S}_{-\mathbf{J}}(\text{BEs}) + \mathrm{Div}\,\mathbf{F}_0, \quad \dots$$

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Question. Can it be achieved without using local coordinates?

Questions. 1. Cohomology/exactness of the multi Euler–Lagrange complex and its relation with the canonical Euler–Lagrange complex.

$$
\begin{array}{ccccccccc}
\cdots \xrightarrow{\mathrm{d}_{\mathrm{h}}^{\Delta}} \Omega^{p-2,0} & \xrightarrow{\mathrm{d}_{\mathrm{h}}^{\Delta}} & \Omega^{p-1,0} & \xrightarrow{\mathcal{E}_{p-1}^{\Delta}} & \mathcal{F}^{p-1,1} & \xrightarrow{\delta_{p-1}^{\Delta}} & \cdots & & \\
 & & & \searrow^{\mathrm{d}_{\mathrm{h}}^{\Delta}} & & \searrow & & & \\
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 & & & & \mathrm{d}_\mathrm{h}^\Delta \searrow & & \searrow\!\!\!\dashrightarrow & & \\
 & & & & & \Omega^{p,0} \xrightarrow{\mathcal{E}^\Delta} & \mathscr{F}^1 & \xrightarrow{\delta^\Delta} & \cdots
\end{array}
$$

2. To determine the integrability of a PΔE or to classify integrable systems with the closure relation (double zeroes?) feature may be related to the two-line theorem?

Summary

- Structure of the total prolongation space
- Construction of the difference variational bicomplex
- Discrete integrable systems with closure relation

- Ongoing
  - Two-line and three-line theorems for PΔEs
  - Further analysis on the BEs
  - ...

Thanks!

Return!