Applications of Elliptic Functions in Solving the Boussinesq equation

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Outline

1. Introduction on Bäcklund transformations and lattice equations
2. A potential Boussinesq system and its Bäcklund transformations
3. Elliptic multi-soliton solutions of the Boussinesq system
4. Elliptic multi-soliton solutions of the lattice Boussinesq system

1.1 Introduction on BTs and lattice equations

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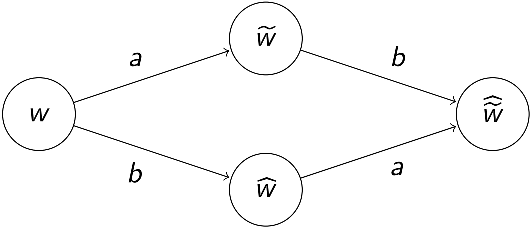

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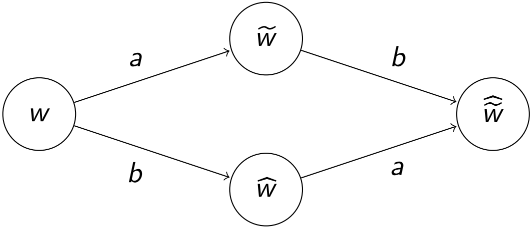

- e.g. the KdV equation: $u_t = u_{xxx} + 6uu_x$, $u = w_x$

$$\Longrightarrow w_t = w_{xxx} + 3w_x^2 \overset{BT}{\Longrightarrow} (\widetilde{w} - \widehat{w})(\widehat{\widetilde{w}} - w) = a - b$$

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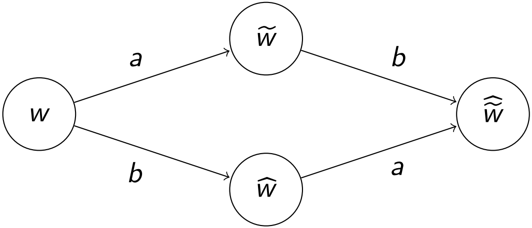

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$$\xRightarrow{w=w_{n,m}} (w_{n+1,m} - w_{n,m+1})(w_{n+1,m+1} - w_{n,m}) = a - b.$$

1.2 The Boussinesq system and its BT

- The lattice potential Boussinesq system [Tongas,Nijhoff-2005]

$$w_{n+1,m} = u_{n,m}u_{n+1,m} - v_{n,m} \ , \tag{1a}$$

$$w_{n,m+1} = u_{n,m}u_{n,m+1} - v_{n,m} \ , \tag{1b}$$

$$w_{n,m} - u_{n,m}u_{n+1,m+1} + v_{n+1,m+1} + \frac{p-q}{u_{n+1,m}-u_{n,m+1}} = 0 \ , \tag{1c}$$

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or 9-point equation:

$$\frac{p-q}{u_{n+1,m+1}-u_{n,m+2}} - \left(u_{n+1,m+2} - u_{n+2,m+1}\right)\left(u_{n,m+1} - u_{n+2,m+2}\right)$$
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- The BT for the Boussinesq equation was first given by Chen [1976] without Bäcklund parameter.
- The nonlinear superposition formula contains derivatives with respect to a continuous variable.

1.2 The Boussinesq system and its BT

The potential Boussinesq system [Rasin,Schiff-2017] :

$$u_t - (u_x + u^2 - 2v)_x = 0 \ , \qquad (4a)$$

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- deriving the Boussinesq equation (3) by eliminating v in (4)
- presenting a BT for the system (4) :

$$\widetilde{u} = u + s, \qquad \widetilde{v} = v - u_x + us.$$

Here $\widetilde{u}, \widetilde{v}$ are new solutions for the potential Boussinesq system, and s is a function of x and t.

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- deriving the Boussinesq equation (3) by eliminating v in (4)
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  Here $\widetilde{u}, \widetilde{v}$ are new solutions for the potential Boussinesq system, and s is a function of x and t.
- trying to identify the nonlinear superposition formula as (1).
- modifying this BT, we get an updated one connecting (1) with (3).

2.1 An updated BT

- The BT was [Rasin,Schiff-2017]

$$\widetilde{u} = u + s, \quad \widetilde{v} = v - u_x + us. \tag{5}$$

- We modify (5) and give an updated BT:

$$\widetilde{u} = u + s, \quad \widetilde{v} = v + \widetilde{u}_x + \widetilde{u}s, \tag{6}$$

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where s satisfies the equations:

$$s_{xx} = -p - 3ss_x - s^3 - 3u_x s + 3uu_x - 3v_x, \tag{7a}$$
$$s_t = p + ss_x + s^3 + 3u_x s - 2u_{xx} + 3v_x - 3uu_x, \tag{7b}$$

p is the BT parameter.

2.2 Nonlinear superpositon formula of the updated BT

Introduce a new variable w by setting $w = -v + u_x + u^2$, from the BT

$$\widetilde{u} = u + s, \quad \widetilde{v} = v + \widetilde{u}_x + \widetilde{u}s,$$

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And if we denote the solutions obtained from $\{u, v, w\}$ by using the BT with a parameter q by $\{\widehat{u}, \widehat{v}, \widehat{w}\}$, we get a similar relation

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Replacing s by $\widetilde{u} - u$ in the equation set for s, we get another superpositon formula

$$w - u\widehat{\widetilde{u}} + \widehat{\widetilde{v}} + \frac{p - q}{\widetilde{u} - \widehat{u}} = 0. \tag{10}$$

Identifying these three algebraic relations on the lattice leads to the lattice Boussinesq system (1).

2.3 Why do we take $\widetilde{v} = v + \widetilde{u}_x + \widetilde{u}s$?

- Weierstrass functions: $\sigma(z)$, $\zeta(z)$ and $\wp(z)$ functions.

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- $\wp(z)$ satisfies the following differential equations:

$$(\wp'(z))^2 = 4(\wp(z))^3 - g_2\wp(z) - g_3,$$
$$\wp''(z) = 6(\wp(z))^2 - \frac{g_2}{2}, \quad \wp'''(z) = 12\wp(z)\wp'(z),$$

- Addition formulas:

$$\wp(z_1) + \wp(z_2) + \wp(z_1 + z_2) = \frac{1}{4}\left(\frac{\wp'(z_1) - \wp'(z_2)}{\wp(z_1) - \wp(z_2)}\right)^2,$$
$$\zeta(z_1 + z_2) - \zeta(z_1) - \zeta(z_2) = \frac{1}{2}\frac{\wp'(z_1) - \wp'(z_2)}{\wp(z_1) - \wp(z_2)}.$$

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Recall the potential BSQ equation

$$u_{tt} + \frac{1}{3}u_{xxxx} + 4u_x u_{xx} = 0. \tag{3}$$

Its stationary solution is $u(t,x) = \zeta(x) + c_1$. c_1 is an arbitrary constant.
From the potential BSQ system

$$u_t - (u_x + u^2 - 2v)_x = 0, \tag{4a}$$

$$v_t - (\frac{2}{3}u_{xx} - v_x + \frac{2}{3}u^3 + 2uu_x)_x + 2uv_x = 0, \tag{4b}$$

we obtain that $v(t,x) = \frac{1}{2}(\zeta(x) + c_1)^2 - \frac{1}{2}\wp(x) + \frac{1}{12}g_2 t + c_2$. c_2 is also an arbitrary constant, and g_2 is the invariant for $\wp$-func.

2.3 Why do we take $\widetilde{v} = v + \widetilde{u}_x + \widetilde{u}s$?

The BT can provide the following solution:

$$\widetilde{u} = \zeta(x+\delta) - \zeta(\delta) + c_1,$$

and

$$\widetilde{v} = \frac{1}{2}(\zeta(x+\delta) - \zeta(\delta) + c_1)^2 - \frac{1}{2}(\wp(x+\delta) - \wp(\delta)) + \frac{1}{12}g_2 t + c_2.$$

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Recall the equation set for s and $s = \widetilde{u} - u$, we obtain $p = -\frac{1}{2}\wp'(\delta)$.
Similarly we can use the BT with the parameter $q = -\frac{1}{2}\wp'(\varepsilon)$.

2.4 Elliptic seed solution for the lattice BSQ equation

If we identify these seed solutions on the lattice, we could derive the elliptic seed solution for the lattice BSQ equation

$$\begin{aligned} u_0 &= \zeta(\xi) - n\zeta(\delta) - m\zeta(\varepsilon) - \zeta(\xi_0), \\ v_0 &= \frac{1}{2}u_0^2 - \frac{1}{2}\wp(\xi) + \frac{1}{2}(n\wp(\delta) + m\wp(\varepsilon) + \wp(\xi_0)), \\ w_0 &= \frac{1}{2}u_0^2 - \frac{1}{2}\wp(\xi) - \frac{1}{2}(n\wp(\delta) + m\wp(\varepsilon) + \wp(\xi_0)), \end{aligned}$$

where $\xi = \xi_{n,m} = n\delta + m\varepsilon + \xi_0$, and ξ_0 is an arbitrary constant.

2.5 What do we have now?

- BT: $\widetilde{u} = u + s, \quad \widetilde{v} = v + \widetilde{u}_x + \widetilde{u}s,$ where s satisfies:

$$s_{xx} = -p - 3ss_x - s^3 - 3u_x s + 3uu_x - 3v_x,$$
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- Seed solution:

$$u = \zeta(x) + c_1, \quad v = \frac{1}{2}\left(\zeta(x) + c_1\right)^2 - \frac{1}{2}\wp(x) + \frac{1}{12}g_2 t + c_2\,.$$

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c_1, c_2 are arbitrary constants. g_2 is the invariant for $\wp$-func.

- Seed solution for the lattice Boussinesq system: $(p = -\frac{1}{2}\wp'(\delta))$

$$u_0 = \zeta(\xi) - n\zeta(\delta) - m\zeta(\varepsilon) - \zeta(\xi_0),$$
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$$w_0 = \frac{1}{2}u_0^2 - \frac{1}{2}\wp(\xi) - \frac{1}{2}(n\wp(\delta) + m\wp(\varepsilon) + \wp(\xi_0)),$$

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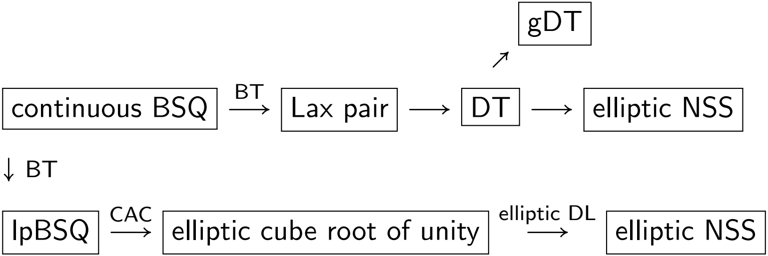
2.6 What can we do next?



lpBSQ: lattice potential Boussinesq system

gDT: generalised Darboux transformation

NSS: N soliton solutions

CAC: consistency around the cube

DL: direct linearisation scheme

3.1 Lax pair for the Boussinesq system

Setting $s = \frac{\psi_x}{\psi}$ in the equation set for s, we obtain the Lax pair for the Boussinesq system

$$\psi_{xxx} = \left(3uu_x - 3v_x - p\right)\psi - 3u_x\psi_x, \tag{11a}$$

$$\psi_t = -\psi_{xx} - 2u_x\psi. \tag{11b}$$

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The system (11) is covariant with respect to the transformation $\psi[1] = \psi_x - \frac{\psi_{1x}}{\psi_1}\psi$, i.e., $\psi[1]$ satisfies

$$\psi[1]_{xxx} = (3u[1]u[1]_x - 3v[1]_x - p)\psi[1] - 3u[1]_x\psi[1]_x \ ,$$
$$\psi[1]_t = -\psi[1]_{xx} - 2u[1]_x\psi[1], \quad \big(u[1] = \widetilde{u},\, v[1] = \widetilde{v}\,\big)\ .$$

This is known as the one-step DT.

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$$\psi[1]_t = -\psi[1]_{xx} - 2u[1]_x\psi[1], \quad (u[1] = \widetilde{u}, \ v[1] = \widetilde{v}\,) \ .$$

This is known as the one-step DT. N-step DT is given as follows.

3.2 DT for the Boussinesq system

Theorem

Denote ψ_j $(j = 1, 2, \cdots, N)$ *are fixed solutions of* (11) *at* $p = p_j$. *N-times repeated DT of the Boussinesq system is given by*

$$u[N] = u + (\ln W[\psi_1, \psi_2, ..., \psi_N])_x,$$

$$v[N] = v + \sum_{j=1}^{N} u[j]_x + \sum_{j=1}^{N} u[j]\big(u[j] - u[j-1]\big), \ (u[0] = u, v[0] = v),$$

$$\psi[N] = \frac{W[\psi_1, \psi_2, ..., \psi_N, \psi]}{W[\psi_1, \psi_2, ..., \psi_N]},$$

where $\{u[j], v[j]\}$ *satisfies the Boussinesq system and* $\psi[N]$ *satisfies*

$$\psi[N]_{xxx} = (-p - 3v[N]_x + 3u[N]u[N]_x)\psi[N] - 3u[N]_x\psi[N]_x,$$
$$\psi[N]_t = -\psi[N]_{xx} - 2u[N]_x\psi[N].$$

3.3 Elliptic soliton solutions for the Boussinesq system

Theorem

Through N-times repeated DT, elliptic N-soliton solutions of the Boussinesq equation are given by

$$u[N] = \zeta(x) + (\ln W[\psi_{p_1}, \psi_{p_2}, \dots, \psi_{p_N}])_x,$$

where p_l *(l=1,2,...,N) corresponds to the l-th spectral parameter of Lax pair,* α_{lj} $(j = 1, 2, 3)$ *are the elliptic cube roots satisfying* $p_l = -\frac{1}{2}\wp'(\alpha_{lj})$ *and*

$$\psi_{p_l} = \sum_{j=1}^{3} C_{lj}\Phi_{\alpha_{lj}}(x)\mathrm{e}^{-\zeta(\alpha_{lj})x-\wp(\alpha_{lj})t}.$$

$\Phi_x(y) := \frac{\sigma(x+y)}{\sigma(x)\sigma(y)}$

4.1 Elliptic seed solution of the lpBSQ system

Now we come to the lattice potential Boussinesq system

$$\widetilde{w} - u\widetilde{u} + v = 0, \tag{12a}$$

$$\widehat{w} - u\widehat{u} + v = 0, \tag{12b}$$

$$w - u\widehat{\widetilde{u}} + \widehat{\widetilde{v}} - \frac{p-q}{\widehat{u} - \widetilde{u}} = 0, \tag{12c}$$

with $p - q = -\frac{1}{2}\wp'(\delta) + \frac{1}{2}\wp'(\varepsilon)$ and the seed solution

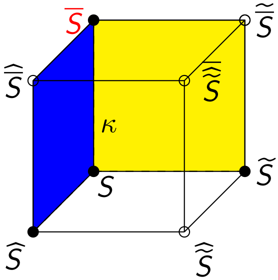
$$u_0 = \zeta(\xi) - n\zeta(\delta) - m\zeta(\varepsilon) - \zeta(\xi_0),$$

$$v_0 = \frac{1}{2}u_0^2 - \frac{1}{2}\wp(\xi) + \frac{1}{2}(n\wp(\delta) + m\wp(\varepsilon) + \wp(\xi_0)),$$

$$w_0 = \frac{1}{2}u_0^2 - \frac{1}{2}\wp(\xi) - \frac{1}{2}(n\wp(\delta) + m\wp(\varepsilon) + \wp(\xi_0)).$$

4.2 Consistensy around the cube

The property of consistensy around the cube $\Longleftrightarrow$ auto-BT.



Here S stands for the three components (u, v, w). κ is a lattice paremeter of the new lattice direction.

4.3 Elliptic one soliton solution of the lpBSQ system

The BT with κ as the BT parameter related to the bar direction reads

$$\overline{w} = u\overline{u} - v,$$

together with

$$\begin{cases} \widetilde{\overline{u}} = \frac{\widetilde{v}-\overline{v}}{\widetilde{u}-\overline{u}}, \\ \widetilde{\overline{v}} = u\widetilde{\overline{u}} - w - \frac{1}{2}\frac{\wp'(\delta)-\wp'(\kappa)}{\widetilde{u}-\overline{u}}, \end{cases} \qquad \begin{cases} \widehat{\overline{u}} = \frac{\overline{v}-\widehat{v}}{\overline{u}-\widehat{u}}, \\ \widehat{\overline{v}} = u\widehat{\overline{u}} - w - \frac{1}{2}\frac{\wp'(\kappa)-\wp'(\varepsilon)}{\widehat{u}-\overline{u}}, \end{cases}$$

where we have used the equations

$$\widetilde{w} - u\widetilde{u} + v = 0,$$
$$\widehat{w} - u\widehat{u} + v = 0.$$

4.3 Elliptic one soliton solution of the lpBSQ system

Consider $(u, v, w) = (u_0, v_0, w_0)$.The elliptic one soliton solution is

$$(\overline{u}, \overline{v}, \overline{w}) = (\overline{u}_0 + x, \overline{v}_0 + y, \overline{w}_0 + z),$$

where

$$u_0 = \zeta(\xi) - n\zeta(\delta) - m\zeta(\varepsilon) - h\zeta(\kappa) - \zeta(\xi_0), \quad \xi = n\delta + m\varepsilon + h\kappa + \xi_0,$$

$$v_0 = \frac{1}{2}u_0^2 - \frac{1}{2}\wp(\xi) + \frac{1}{2}(n\wp(\delta) + m\wp(\varepsilon) + h\wp(\kappa) + \wp(\xi_0)),$$

$$w_0 = \frac{1}{2}u_0^2 - \frac{1}{2}\wp(\xi) - \frac{1}{2}(n\wp(\delta) + m\wp(\varepsilon) + h\wp(\kappa) + \wp(\xi_0)),$$

and $(\overline{u}_0, \overline{v}_0, \overline{w}_0)$ is the bar-shifted of (u_0, v_0, w_0), i.e.,

$$\overline{u}_0 = u_0 + \eta_\kappa^\xi, \quad [\text{notation: } \eta_\kappa^\xi = \zeta(\xi + \kappa) - \zeta(\xi) - \zeta(\kappa)]$$

$$\overline{v}_0 = v_0 + \frac{1}{2}(\eta_\kappa^\xi)^2 + u_0\eta_\kappa^\xi - \frac{1}{2}(\wp(\xi + \kappa) - \wp(\kappa) - \wp(\xi)),$$

$$\overline{w}_0 = w_0 + \frac{1}{2}(\eta_\kappa^\xi)^2 + u_0\eta_\kappa^\xi - \frac{1}{2}(\wp(\xi + \kappa) + \wp(\kappa) - \wp(\xi)).$$

4.3 Elliptic one soliton solution of the lpBSQ system

With these definitions we find that $z = u_0 x$. Thus we only need to solve equations for x, y

$$\begin{cases} \widetilde{x} = \frac{-\widetilde{\overline{u}}_0\, x+y}{x-h(\xi+\delta,-\delta,\kappa)}, \\ \widehat{x} = \frac{-\widehat{\overline{u}}_0\, x+y}{x-h(\xi+\varepsilon,-\varepsilon,\kappa)}, \end{cases} \qquad \begin{cases} \widetilde{y} = \frac{-(\widetilde{\overline{v}}_0+w_0)\, x+u_0\, y}{x-h(\xi+\delta,-\delta,\kappa)}, \\ \widehat{y} = \frac{-(\widehat{\overline{v}}_0+w_0)\, x+u_0\, y}{x-h(\xi+\varepsilon,-\varepsilon,\kappa)}. \end{cases}$$

Notation: $h(\alpha,\beta,\gamma) = \zeta(\alpha)+\zeta(\beta)+\zeta(\gamma)-\zeta(\alpha+\beta+\gamma)$

Next we will present how we solve the equations for x, y.

4.3 Elliptic one soliton solution of the lpBSQ system

Setting $(x, y) = (\frac{G}{F}, \frac{H}{F})$ and $\Psi = \begin{bmatrix} G \\ H \\ F \end{bmatrix}$ turns the equations for x, y into equations for Ψ:

$$\widetilde{\Psi} = N\Psi, \quad \widehat{\Psi} = M\Psi,$$

where

$$N = \begin{bmatrix} \widetilde{\widetilde{u}}_0 & -1 & 0 \\ \widetilde{\widetilde{v}}_0 + w_0 & -u_0 & 0 \\ -1 & 0 & h(\xi + \delta, -\delta, \kappa) \end{bmatrix},$$

and M is the $\{\widehat{\ }, \varepsilon\}$ counterpart of N.

The matrices N, M satisfy the integrability condition $\widehat{N}M = \widetilde{M}N$.

4.3 Elliptic one soliton solution of the lpBSQ system

Solving the equation set for Ψ, namely, $\{F, G, H\}$, we obtain their explicit expressions

$$
\begin{aligned}
F &= F_0 + F_1 + F_2 \\
G &= -h(\xi+\kappa, -\kappa, \omega_1(\kappa))F_1 - h(\xi+\kappa, -\kappa, \omega_2(\kappa))F_2, \\
H &= u_0 G + (\wp(\omega_1(\kappa)) - \wp(\kappa))F_1 + (\wp(\omega_2(\kappa)) - \wp(\kappa))F_2,
\end{aligned}
$$

where $\rho_0^0, \rho_1^0, \rho_2^0$ are arbitrary constants and

$$
\begin{aligned}
F_0 &= \Phi_\kappa^n(-\delta)\Phi_\kappa^m(-\varepsilon)\Phi_\kappa(\xi)\rho_0^0, \\
F_1 &= \Phi_{\omega_1(\kappa)}^n(-\delta)\Phi_{\omega_1(\kappa)}^m(-\varepsilon)\Phi_{\omega_1(\kappa)}(\xi)\rho_1^0, \\
F_2 &= \Phi_{\omega_2(\kappa)}^n(-\delta)\Phi_{\omega_2(\kappa)}^m(-\varepsilon)\Phi_{\omega_2(\kappa)}(\xi)\rho_2^0.
\end{aligned}
$$

4.3 Elliptic one soliton solution of the lpBSQ system

Then we obtain the elliptic one soliton solution for the lattice BSQ system

$$u^{1SS}_{n,m} = u_0 + \frac{\eta^{\xi}_{\kappa}\Phi_{\kappa}(\xi) + \eta^{\xi}_{\omega_1(\kappa)}\Phi_{\omega_1(\kappa)}(\xi)\rho_1 + \eta^{\xi}_{\omega_2(\kappa)}\Phi_{\omega_2(\kappa)}(\xi)\rho_2}{\Phi_{\kappa}(\xi) + \Phi_{\omega_1(\kappa)}(\xi)\rho_1 + \Phi_{\omega_2(\kappa)}(\xi)\rho_2},$$

$$v^{1SS}_{n,m} = \overline{v}_0 + \frac{H}{F}, \qquad w^{1SS}_{n,m} = \overline{w}_0 + u_0\frac{G}{F},$$

where

$$\rho_i(n,m;\kappa) = \left(\frac{\Phi_{\omega_i(\kappa)}(-\delta)}{\Phi_{\kappa}(-\delta)}\right)^n \left(\frac{\Phi_{\omega_i(\kappa)}(-\varepsilon)}{\Phi_{\kappa}(-\varepsilon)}\right)^m \cdot \frac{\rho^0_i}{\rho^0_0}, \quad i = 1, 2.$$

$\kappa,\ \omega_1(\kappa),\ \omega_2(\kappa)$ are solutions (elliptic cube roots of unity) of $\wp'(x) - \wp'(\kappa) = 0$.

4.4 Elliptic direct linearisation scheme

Elliptic DL scheme for the lattice KP equation

↓ elliptic cube roots of unity

Elliptic DL scheme for the lpBSQ equation

↓

Elliptic multi-soliton solution for the lpBSQ equation

Main references

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Thank you!