

Multi-Time Euler-Lagrange Equations and Double Zeroes

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Lagrangian Multiform Theory and Pluri-Lagrangian Systems

Hangzhou, October 2023

Contents

- 1 Two routes from variational principle to equations
 - Stepped curves/surfaces
 - Variations of $d\mathcal{L} \iff$ double zeroes
- 2 Constructing examples using the double zero property
- 3 Semi-discrete Lagrangian multiforms
- 4 A curious connection
- 5 Summary

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Variational principle in multi-time

A **simultaneous solution** is a function

$$q : \mathbb{R}^N \rightarrow Q \quad (\text{multi-time to configuration space})$$

such that $\frac{\partial q}{\partial t_1}$ generates the dynamical system and $\frac{\partial q}{\partial t_i}$ its symmetries.

Pluri-Lagrangian principle

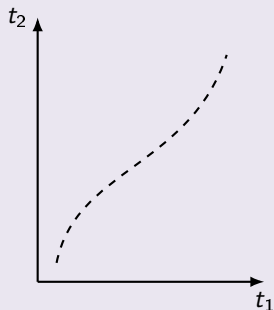
Combine the L_i into a **1-form**

$$\mathcal{L}[q] = \sum_{i=1}^N L_i[q] dt_i.$$

Look for dynamical variables $q(t_1, \dots, t_N)$ such that the action

$$\mathcal{I}_S = \int_S \mathcal{L}[q]$$

is critical w.r.t. **variations of q** , simultaneously over **every curve S** in multi-time \mathbb{R}^N



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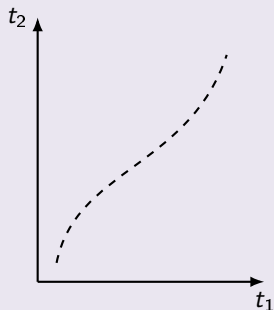
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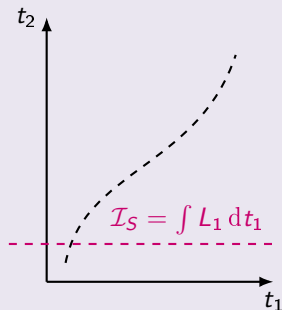
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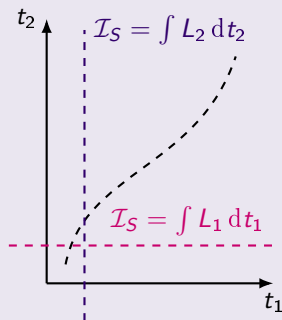
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Multi-time Euler-Lagrange equations

q_I : mixed partial derivative of q defined by a string $I = t_{i_1} \dots t_{i_k}$.

If I is empty then $q_I = q$.

Denote by $\frac{\delta_i}{\delta q_I}$ the variational derivative in the direction of t_i wrt q_I :

$$\frac{\delta_i f}{\delta q_I} = \sum_{\alpha=0}^{\infty} (-1)^\alpha \frac{d^\alpha}{dt_i^\alpha} \frac{\partial f}{\partial q_{I t_i^\alpha}} = \frac{\partial f}{\partial q_I} - \frac{d}{dt_i} \frac{\partial f}{\partial q_{I t_i}} + \frac{d^2}{dt_i^2} \frac{\partial f}{\partial q_{I t_i^2}} - \dots$$

Consider $\mathcal{L}[q] = \sum_i L_i[q] dt_i$ with $L_i[q] = L_i(q, q_{t_i}, q_{t_i t_j}, \dots)$.

Multi-time Euler-Lagrange equations / Multi-time EL eqns

Usual Euler-Lagrange equations: $\frac{\delta_i L_i}{\delta q_I} = 0 \quad \forall I \not\ni t_i,$

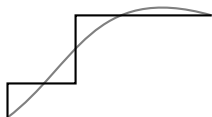
Additional conditions: $\frac{\delta_i L_i}{\delta q_{I t_i}} = \frac{\delta_j L_j}{\delta q_{I t_j}} \quad \forall I,$

Derivation of the multi-time Euler-Lagrange equations

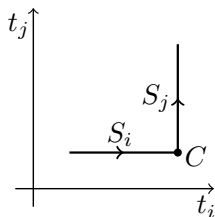
Consider a Lagrangian one-form $\mathcal{L} = \sum_i L_i[q] dt_i$.

Lemma

If the action $\int_S \mathcal{L}$ is critical on all **stepped curves** S in \mathbb{R}^N , then it is critical on all smooth curves.



Variations are local, so it is sufficient to look at an **L-shaped curve** $S = S_i \cup S_j$.

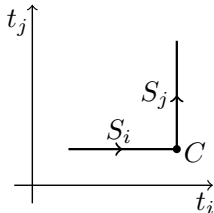


Suris, V. *On the Lagrangian structure of integrable hierarchies*. Advances in Discrete Differential Geometry, 2016.

Derivation of the multi-time Euler-Lagrange equations

On one of the straight pieces, S_i ($i \neq 1$), we get

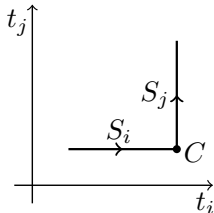
$$\begin{aligned}\delta \int_{S_i} L_i dt_i &= \int_{S_i} \sum_l \frac{\partial L_i}{\partial q_l} \delta q_l dt_i \\ &= \int_{S_i} \sum_{l \neq t_i} \sum_{\alpha=0}^{\infty} \frac{\partial L_i}{\partial q_l t_i^\alpha} \delta q_l t_i^\alpha dt_i\end{aligned}$$



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Integration by parts (wrt t_i only) yields

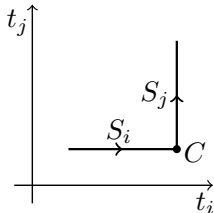
$$\delta \int_{S_i} L_i dt_i = \int_{S_i} \sum_{l \neq t_i} \frac{\delta_l L_i}{\delta q_l} \delta q_l dt_i + \sum_l \frac{\partial L_i}{\partial q_{l t_i}} \delta q_l \Big|_C$$

Since p is an interior point of the curve, we cannot set $\delta q(C) = 0$!

Derivation of the multi-time Euler-Lagrange equations

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Multi-time Euler-Lagrange equations

$$\frac{\delta_l L_i}{\delta q_l} = 0 \quad \forall l \neq t_i, \quad \text{and} \quad \frac{\delta_l L_i}{\delta q_{l t_i}} = \frac{\delta_j L_j}{\delta q_{l t_j}}$$

Newtonian system with symmetries

Consider $\mathcal{L} = \sum_i L_i dt_i$ with

$$L_1 = \frac{1}{2}|q_1|^2 - V_i(q)$$

$$L_i = q_1 \cdot q_i - H_i(q, q_1)$$

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Multi-time Euler-Lagrange equations

$$\frac{\delta_1 L_1}{\delta q} = 0 \Rightarrow \frac{\partial L_1}{\partial q} - \frac{d}{dt_1} \frac{\partial L_1}{\partial q_1} = 0 \Rightarrow q_{11} = -V'(q)$$

$$\frac{\delta_i L_i}{\delta q} = 0 \Rightarrow \frac{\partial L_i}{\partial q} - \frac{d}{dt_i} \frac{\partial L_i}{\partial q_i} = 0 \Rightarrow q_{1i} = -\frac{\partial H_i}{\partial q}$$

$$\frac{\delta_i L_i}{\delta q_1} = 0 \Rightarrow \frac{\partial L_i}{\partial q_1} = 0, \Rightarrow q_i = \frac{\partial H_i}{\partial q_1}$$

$$\frac{\delta_i L_i}{\delta q_j} = \frac{\delta_j L_j}{\delta q_j} \Rightarrow \frac{\partial L_i}{\partial q_{t_j}} = \frac{\partial L_j}{\partial q_{t_j}} \quad \text{trivially satisfied}$$

Exterior derivative of \mathcal{L}

As before, consider $L_i = q_1 \cdot q_i - H_i(q, q_1)$

Multi-time Euler-Lagrange equations:

$$q_i = \frac{\partial H_i}{\partial q_1} \quad \text{and} \quad q_{1i} = -\frac{\partial H_i}{\partial q}$$

Coefficient of $d\mathcal{L}$

$$\frac{dL_j}{dt_j} - \frac{dL_i}{dt_i} = \left(q_{1i} + \frac{\partial H_i}{\partial q} \right) \left(q_j - \frac{\partial H_j}{\partial q_i} \right) - \left(q_{1j} + \frac{\partial H_j}{\partial q} \right) \left(q_i - \frac{\partial H_i}{\partial q_j} \right) - \{H_i, H_j\}$$

Observation (also for PDEs): $d\mathcal{L}$ has a “double zero” on solutions.

$d\mathcal{L} = 0^2$ is key to the **Lagrangian multiform** approach.

Relation between $d\mathcal{L}$, double zeroes, and Poisson brackets is emphasised in:

Caudrelier, Dell’Atti, Singh. **Lagrangian multiforms on coadjoint orbits for finite-dimensional integrable systems** arXiv:2307.07339

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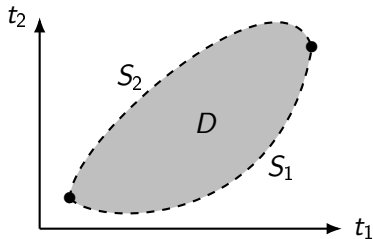
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Interpretation of closedness condition

If $d\mathcal{L} = 0$, the action is **invariant**
wrt variations in geometry

Deforming the curve (surface) of
integration leaves action invariant.



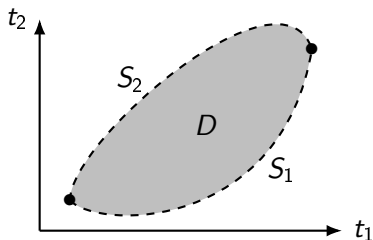
$$\int_{S_1} \mathcal{L} - \int_{S_2} \mathcal{L} = \int_D d\mathcal{L} = 0$$

Recall: pluri-Lagrangian principle
only considers variations of q .

Interpretation of closedness condition

If $d\mathcal{L} = 0$, the action is **invariant wrt variations in geometry**

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$$\int_{S_1} \mathcal{L} - \int_{S_2} \mathcal{L} = \int_D d\mathcal{L} = 0$$

Recall: pluri-Lagrangian principle only considers variations of q .

$d\mathcal{L}$ provides an **alternative derivation of the EL equations**:

We can restrict the variational principle to simple closed curves (surfaces) of integration, i.e. boundaries of a submanifold S .

Then

$$\delta \int_{\partial S} \mathcal{L} = - \int_S \delta d\mathcal{L},$$

hence the pluri-Lagrangian principle is equivalent to $\delta d\mathcal{L} = 0$.

Multi-time EL equations can be obtained from variations of $d\mathcal{L}$.

If $d\mathcal{L}$ has a double zero, then q is critical

Multi-time Euler-Lagrange equations from $\delta d\mathcal{L} = 0$

Coefficients of $d\mathcal{L}$: $P_{ij} = \frac{dL_j}{dt_i} - \frac{dL_i}{dt_j}$ (1-form case)

$$\frac{\delta_{ij}}{\delta q_l} P_{ij} = \frac{\delta_j L_j}{\delta q_{l \setminus t_i}} - \frac{\delta_i L_i}{\delta q_{l \setminus t_j}} \quad \text{where} \quad \frac{\delta}{\delta q_{l \setminus t_i}} = \begin{cases} 0 & \text{if } l \not\supseteq t_i \\ \frac{\delta}{\delta q_k} & \text{if } l = kt_i \end{cases}$$

where
$$\frac{\delta_{ij} L_{ij}}{\delta q_l} = \sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\infty} (-1)^{\alpha+\beta} \frac{d^\alpha}{dt_i^\alpha} \frac{d^\beta}{dt_j^\beta} \frac{\partial L_{ij}}{\partial q_{lt_i^\alpha t_j^\beta}}$$

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Generalises to k -forms for any k : $P_{i_0 \dots i_k} = \sum_{\alpha=0}^k (-1)^\alpha \frac{d}{dt_{i_\alpha}} L_{i_0 \dots \hat{i}_\alpha \dots i_k}$

$$\frac{\delta_{i_0 \dots i_k}}{\delta q_l} P_{i_0 \dots i_k} = \sum_{\alpha=0}^k (-1)^\alpha \frac{\delta_{i_0 \dots \hat{i}_\alpha \dots i_k}}{\delta q_{l \setminus i_\alpha}} L_{i_0 \dots \hat{i}_\alpha \dots i_k}$$

Sleigh, Nijhoff, Caudrelier. Lagrangian Multiforms for the KP and the GD Hierarchy. IMRN, 2021.

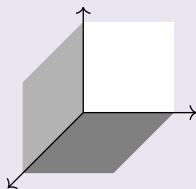
Multi-time EL equations for $\mathcal{L}[q] = \sum_{i,j} L_{ij}[q] dt_i \wedge dt_j$

For 2-forms we have three types of multi-time EL equation:

$$\frac{\delta_{ij} L_{ij}}{\delta q_l} = 0 \quad \forall l \neq t_i, t_j,$$

$$\frac{\delta_{ij} L_{ij}}{\delta q_{l t_j}} = \frac{\delta_{ik} L_{ik}}{\delta q_{l t_k}} \quad \forall l \neq t_i,$$

$$\frac{\delta_{ij} L_{ij}}{\delta q_{l t_i t_j}} + \frac{\delta_{jk} L_{jk}}{\delta q_{l t_j t_k}} + \frac{\delta_{ki} L_{ki}}{\delta q_{l t_k t_i}} = 0 \quad \forall l.$$



where

$$\frac{\delta_{ij} L_{ij}}{\delta q_l} = \sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\infty} (-1)^{\alpha+\beta} \frac{d^\alpha}{dt_i^\alpha} \frac{d^\beta}{dt_j^\beta} \frac{\partial L_{ij}}{\partial q_{l t_i^\alpha t_j^\beta}}$$

Generalisation of stepped surface approach to higher forms is possible, but trickier than using $\delta d\mathcal{L} = 0$.

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Potential KdV-type hierarchy

Consider a 2-form with the first row of coefficients of the form

$$L_{1i} = \frac{1}{2} v_1 v_i - H_i(\cancel{v}, v_1, v_{11}, \dots) \quad (q \rightarrow v)$$

Its Euler-Lagrange equation is $v_{1i} = -\frac{\delta_1 H_i}{\delta v} \Leftrightarrow v_i = \frac{\delta_1 H_i}{\delta v_1}$

We look for L_{ij} such that $\frac{dL_{1i}}{dt_j} - \frac{dL_{1j}}{dt_i} + \frac{dL_{ij}}{dt_1} = 0^2$:

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$$\begin{aligned} \frac{dL_{1i}}{dt_j} - \frac{dL_{1j}}{dt_i} &= \frac{1}{2} \left(v_{1j} + \frac{\delta_1 H_j}{\delta v} \right) \left(v_i - \frac{\delta_1 H_i}{\delta v_1} \right) - \frac{1}{2} \left(v_{1i} + \frac{\delta_1 H_i}{\delta v} \right) \left(v_j - \frac{\delta_1 H_j}{\delta v_1} \right) \\ &\quad + \frac{1}{2} \left(\frac{\delta_1 H_j}{\delta v} \frac{\delta_1 H_i}{\delta v_1} - \frac{\delta_1 H_i}{\delta v} \frac{\delta_1 H_j}{\delta v_1} \right) \\ &\quad + \frac{d}{dt_1} \left(\frac{1}{2} \frac{\delta_1 H_i}{\delta v} v_j - \frac{1}{2} \frac{\delta_1 H_j}{\delta v} v_i + \sum_{\alpha \geq 1} \left(\frac{\delta_1 H_i}{\delta v_{1\alpha}} v_{1\alpha j} - \frac{\delta_1 H_j}{\delta v_{1\alpha}} v_{1\alpha i} \right) \right) \end{aligned}$$

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Potential KdV-type hierarchy

The multi-time Euler-Lagrange equations of this 2-form are

$$\frac{\delta_{1i} L_{1i}}{\delta v} = 0 \quad \Rightarrow \quad v_{1i} = -\frac{\delta_1 H_i}{\delta v}$$
$$\frac{\delta_{1i} L_{1i}}{\delta v_1} = \frac{\delta_{ji} L_{ji}}{\delta v_j} \quad \Rightarrow \quad v_j = \frac{\delta_1 H_i}{\delta v_1}$$

Formulating the construction in terms of double zeroes of $d\mathcal{L}$ explains the “miracle” of finding the evolutionary equations as Euler-Lagrange equations.

Potential KdV-type hierarchy

The multi-time Euler-Lagrange equations of this 2-form are

$$\frac{\delta_{1i} L_{1i}}{\delta v} = 0 \quad \Rightarrow \quad v_{1i} = -\frac{\delta_1 H_i}{\delta v}$$
$$\frac{\delta_{1i} L_{1i}}{\delta v_1} = \frac{\delta_{jj} L_{jj}}{\delta v_j} \quad \Rightarrow \quad v_j = \frac{\delta_1 H_i}{\delta v_1}$$

Formulating the construction in terms of double zeroes of $d\mathcal{L}$ explains the “miracle” of finding the evolutionary equations as Euler-Lagrange equations.

Early works relied on $d\mathcal{L} = 0$ but did not explicitly use the double-zero property:

Suris, V. [On the Lagrangian structure of integrable hierarchies](#). 2016

V. [Hamiltonian structures for integrable hierarchies of Lagrangian PDEs](#). 2021

Double zeroes did appear in works on variational symmetries and 2-forms:

Sleigh, Nijhoff, Caudrelier. [Variational symmetries and Lagrangian multiforms](#). 2020

Petrera, V. [Variational symmetries and pluri-Lagrangian structures for integrable hierarchies of PDEs](#). 2021

More general kinetic term

Consider $L_{1i} = P(v, v_1, v_{11}, \dots)v_i - H_i(v, v_1, v_{11}, \dots)$

It induces the Poisson bracket $\{H_i, H_j\} = \frac{\delta_1 H_i}{\delta v} (D_P^* - D_P)^{-1} \frac{\delta_1 H_j}{\delta v}$,

where D_P is the frechet derivative of P and D_P^* its adjoint.

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where D_P is the frechet derivative of P and D_P^* its adjoint.

We find

$$\begin{aligned} L_{ij} &= - \sum_{\alpha=0}^{\infty} \left(\frac{\delta_1 H_i}{\delta v_{1\alpha}} v_{1\alpha j} - \frac{\delta_1 H_j}{\delta v_{1\alpha}} v_{1\alpha i} \right) - \sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\alpha-1} (-1)^\beta v_{i1^{\alpha-\beta-1}} \partial_1^\beta \left(\frac{\partial P}{\partial v_{1\alpha}} v_j \right) \\ &\quad - \sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\alpha-1} (-1)^\beta \left(\partial_1^{\alpha-\beta-1} (D_P^* - D_P)^{-1} \frac{\delta_1 H_i}{\delta v} \right) \partial_1^\beta \left(\frac{\partial P}{\partial v_{1\alpha}} v_j \right) \\ &\quad - \sum_{\alpha=0}^{\infty} \sum_{\beta=0}^{\alpha-1} (-1)^\beta (\partial_1^{\alpha-\beta-1} v_j) \partial_1^\beta \left(\frac{\partial P}{\partial v_{1\alpha}} (D_P^* - D_P)^{-1} \frac{\delta_1 H_i}{\delta v} \right). \end{aligned}$$

Question. For which P and H_i is $(D_P^* - D_P)^{-1} \frac{\delta_1 H_i}{\delta v}$ well-defined?

Question. For which P and H_i do the flows commute?

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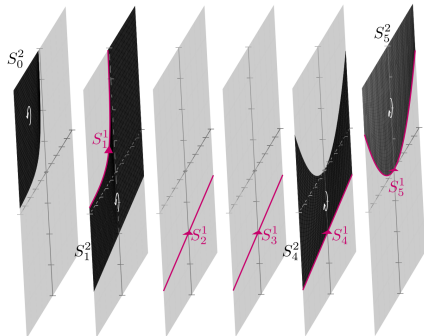
Joint work with Duncan Sleight

Semi-discrete geometry

Only 1 discrete direction: $\mathbb{Z} \times \mathbb{R}^N$

A **semi-discrete surface** is a collection of surfaces and curves in \mathbb{R}^N at a specified point in \mathbb{Z}

Intuition: curves where the surface jumps to a different value of \mathbb{Z}



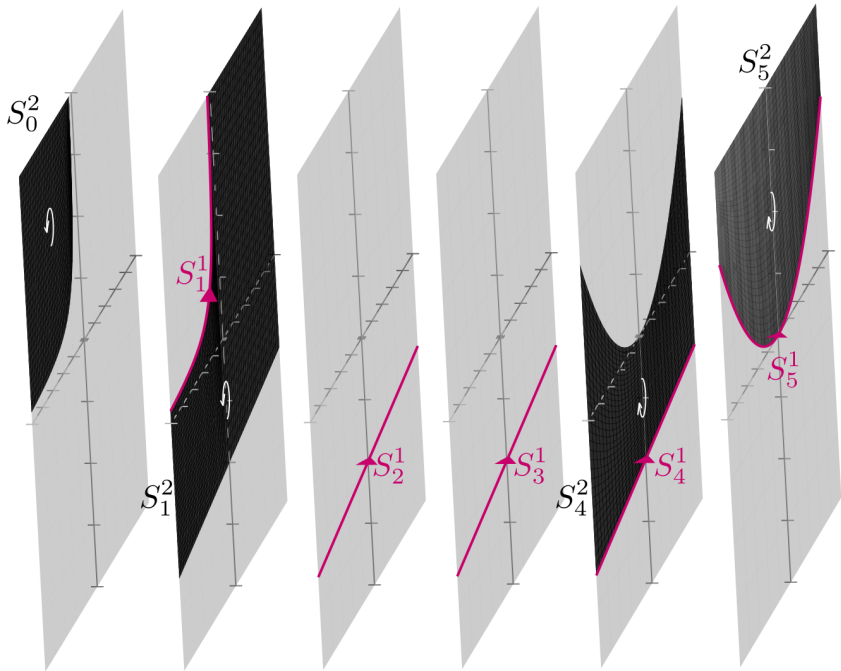
d -dimensional semi-discrete submanifold S

$$S = \left(\bigsqcup_{k \in \mathbb{Z}} S_k^{d-1}, \bigsqcup_{k \in \mathbb{Z}} S_k^d \right)$$

S_k^{d-1} : oriented $(d-1)$ -dimensional submanifold of \mathbb{R}^N

S_k^d : oriented d -dimensional submanifold of \mathbb{R}^N .

Sleigh, V. [Semi-discrete Lagrangian 2-forms and the Toda hierarchy](#). Journal of Physics A, 2022.



Semi-discrete geometry

- ▶ Consider (scalar) functions q of $\mathbb{Z} \times \mathbb{R}^N$.
Superscript to emphasise lattice position: $q^{[k]} = q(k, t_1, \dots, t_N)$
- ▶ **Semi-discrete d -form**

$$\mathcal{L}[q] = \left(\mathcal{L}^{d-1}[q], \mathcal{L}^d[q] \right)$$

consists of a $(d - 1)$ -form and a d -form.

- ▶ The **semi-discrete integral** over semi-discrete submanifold S

$$\int_S \mathcal{L}[q] = \sum_k \int_{S_k^{d-1}} \mathcal{L}^{d-1}[q^{[k]}] + \sum_k \int_{S_k^d} \mathcal{L}^d[q^{[k]}],$$

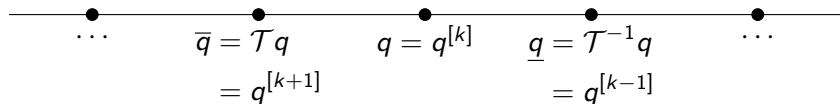
- ▶ We have semi-discrete versions of the **exterior derivative**, the **boundary**, and **Stokes theorem**

Variational principle

Look for $q(k, t_1, \dots, t_N)$ such that the action $\int_S \mathcal{L}[q]$ is critical w.r.t. **variations of q** , simultaneously over **every semi-discrete surface S** .

Newtonian ($p = q_1$) lattice

Consider particles on a line:



One discrete dimension, many continuous times

Suppose we have **Hamiltonian densities** $H_i(q, q_1, \bar{q}, \bar{q}_1, \dots)$ such that

$$q_i = \frac{\delta_0 H_i}{\delta q_1} \quad \text{and} \quad q_{1i} = -\frac{\delta_0 H_i}{\delta q}$$

Then we consider the **Lagrangians**

$$L_{0i} = q_1 q_i - H_i$$

E.g. Toda lattice: $H_1 = \frac{1}{2} q_1^2 + \exp(\bar{q} - q)$

$$H_2 = \frac{1}{3} q_1^3 + (q_1 + \bar{q}_1) \exp(\bar{q} - q)$$

Finding L_{ij}

$$\begin{aligned}
 \partial_j L_{0i} - \partial_i L_{0j} &= \left(q_{1j} + \frac{\delta_0 H_j}{\delta q} \right) \left(q_i - \frac{\delta_0 H_i}{\delta q_1} \right) - \left(q_{1i} + \frac{\delta_0 H_i}{\delta q} \right) \left(q_j - \frac{\delta_0 H_j}{\delta q_1} \right) \\
 &+ \left(\frac{\delta_0 H_i}{\delta q_1} \frac{\delta_0 H_j}{\delta q} - \frac{\delta_0 H_i}{\delta q} \frac{\delta_0 H_j}{\delta q_1} \right) \\
 &- \Delta \left(\sum_{\alpha=0}^{\infty} \left(\frac{\delta_0 H_i}{\delta \mathcal{T}^{\alpha+1} q} \mathcal{T}^{\alpha+1} q_j + \frac{\delta_0 H_i}{\delta \mathcal{T}^{\alpha+1} q_1} \mathcal{T}^{\alpha+1} q_{1j} \right. \right. \\
 &\quad \left. \left. - \frac{\delta_0 H_j}{\delta \mathcal{T}^{\alpha+1} q} \mathcal{T}^{\alpha+1} q_i - \frac{\delta_0 H_j}{\delta \mathcal{T}^{\alpha+1} q_1} \mathcal{T}^{\alpha+1} q_{1i} \right) \right)
 \end{aligned}$$

Finding L_{ij}

$$\begin{aligned}
 \partial_j L_{0i} - \partial_i L_{0j} &= \left(q_{1j} + \frac{\delta_0 H_j}{\delta q} \right) \left(q_i - \frac{\delta_0 H_i}{\delta q_1} \right) - \left(q_{1i} + \frac{\delta_0 H_i}{\delta q} \right) \left(q_j - \frac{\delta_0 H_j}{\delta q_1} \right) \\
 &+ \left(\frac{\delta_0 H_i}{\delta q_1} \frac{\delta_0 H_j}{\delta q} - \frac{\delta_0 H_i}{\delta q} \frac{\delta_0 H_j}{\delta q_1} \right) = \Delta F_{ij} \text{ if } \{ \cdot, \cdot \} = 0 \\
 &- \Delta \left(\sum_{\alpha=0}^{\infty} \left(\frac{\delta_0 H_i}{\delta \mathcal{T}^{\alpha+1} q} \mathcal{T}^{\alpha+1} q_j + \frac{\delta_0 H_i}{\delta \mathcal{T}^{\alpha+1} q_1} \mathcal{T}^{\alpha+1} q_{1j} \right. \right. \\
 &\quad \left. \left. - \frac{\delta_0 H_j}{\delta \mathcal{T}^{\alpha+1} q} \mathcal{T}^{\alpha+1} q_i - \frac{\delta_0 H_j}{\delta \mathcal{T}^{\alpha+1} q_1} \mathcal{T}^{\alpha+1} q_{1i} \right) \right)
 \end{aligned}$$

so we achieve a **double zero of $\partial_j L_{0i} - \partial_i L_{0j} + \Delta L_{ij}$** if

$$\begin{aligned}
 L_{ij} &= \sum_{\alpha=0}^{\infty} \left(\frac{\delta_0 H_i}{\delta \mathcal{T}^{\alpha+1} q} \mathcal{T}^{\alpha+1} q_j + \frac{\delta_0 H_i}{\delta \mathcal{T}^{\alpha+1} q_1} \mathcal{T}^{\alpha+1} q_{1j} \right. \\
 &\quad \left. - \frac{\delta_0 H_j}{\delta \mathcal{T}^{\alpha+1} q} \mathcal{T}^{\alpha+1} q_i - \frac{\delta_0 H_j}{\delta \mathcal{T}^{\alpha+1} q_1} \mathcal{T}^{\alpha+1} q_{1i} \right) - \Delta F_{ij}
 \end{aligned}$$

Toda lattice

$$L_{01} = \frac{1}{2}q_1^2 - \exp(\bar{q} - q)$$

$$L_{02} = q_1q_2 - \frac{1}{3}q_1^3 - (q_1 + \bar{q}_1)\exp(\bar{q} - q)$$

$$L_{12} = -\exp(\bar{q} - q)^2 - \exp(\bar{q} - q)(\bar{q}_1^2 + \bar{q}_{11} - \bar{q}_2)$$

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$$\begin{aligned} L_{12} &= -\exp(\bar{q} - q)^2 - \exp(\bar{q} - q)(\bar{q}_1^2 + \bar{q}_{11} - \bar{q}_2) \\ &= \frac{1}{4}(\bar{q}_1^2 + \bar{q}_{11} - \bar{q}_2)^2 - \left(\exp(\bar{q} - q) + \frac{1}{2}(\bar{q}_1^2 + \bar{q}_{11} - \bar{q}_2) \right)^2 \end{aligned}$$

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Euler-Lagrange equations:

$$\frac{\delta_{01} L_{01}}{\delta q} = 0 \quad \Rightarrow \quad q_{11} = \exp(\bar{q} - q) - \exp(q - \underline{q}) \quad (1)$$

$$\frac{\delta_{02} L_{02}}{\delta q_1} = 0 \quad \Rightarrow \quad q_2 = q_1^2 + \exp(\bar{q} - q) + \exp(q - \underline{q}) \quad (2)$$

$$\frac{\delta_{12} L_{12}}{\delta q} = 0 \quad \Rightarrow \quad \frac{1}{2}q_{22} - q_{11}q_2 - 2q_{12}q_1 - \frac{1}{2}q_{1111} + 3q_1^2q_{11} = 0 \quad (3)$$

The PDE (3) is a consequence of the DDEs (1)–(2)

because exterior derivative has a double zero on solutions of (1)–(2)

The PDE

The equation

$$\frac{1}{2}q_{22} - q_{11}q_2 - 2q_{12}q_1 - \frac{1}{2}q_{1111} + 3q_1^2q_{11} = 0$$

is a Boussinesq-type integrable PDE.

(Equivalent forms have appeared in classification results.)

DDEs as Bäcklund transformations

The first two Toda equations can be written as

$$\exp(\bar{q} - q) = \frac{1}{2}(q_2 + q_{11} - q_1^2),$$

$$\exp(q - \underline{q}) = \frac{1}{2}(q_2 - q_{11} - q_1^2).$$

which give an auto-Bäcklund transformation for (3)

Mikhailov, Novikov, Wang. [On classification of integrable nonevolutionary equations](#). Studies in Applied Mathematics, 2007.

Potential Volterra hierarchy

The equations

$$q_{11} = -q_1 (\bar{q}_1 - \underline{q}_1)$$

$$q_2 = q_1 (\bar{q}_1 + q_1 + \underline{q}_1)$$

Provide a double zero for the semi-discrete Lagrangian 2-form

$$L_{01} = q_1 \log(q_1) - q\bar{q}_1 - q_1$$

$$L_{02} = q_2 \log(q_1) - q\bar{q}_2 - \frac{1}{2}q_1^2 - q_1\bar{q}_1$$

$$L_{12} = -q_1^2\bar{q}_1 - q_1\bar{q}_1^2 + q_1\bar{q}_{11} + q_1\bar{q}_2$$

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$$L_{12} = -q_1^2\bar{q}_1 - q_1\bar{q}_1^2 + q_1\bar{q}_{11} + q_1\bar{q}_2$$

$$= \frac{1}{4}\bar{q}_1^3 - \frac{1}{2}\bar{q}_1\bar{q}_{11} - \frac{1}{2}\bar{q}_1\bar{q}_2 + \frac{\bar{q}_{11}^2 + 2\bar{q}_{11}\bar{q}_2 + \bar{q}_2^2}{4\bar{q}_1} - \frac{(\bar{q}_2 + \bar{q}_{11} - 2q_1\bar{q}_1 - \bar{q}_1^2)^2}{4\bar{q}_1}$$

Potential Volterra hierarchy

The equations

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$$= \frac{1}{4}\bar{q}_1^3 - \frac{1}{2}\bar{q}_1\bar{q}_{11} - \frac{1}{2}\bar{q}_1\bar{q}_2 + \frac{\bar{q}_{11}^2 + 2\bar{q}_{11}\bar{q}_2 + \bar{q}_2^2}{4\bar{q}_1} - \frac{(\bar{q}_2 + \bar{q}_{11} - 2q_1\bar{q}_1 - \bar{q}_1^2)^2}{4\bar{q}_1}$$

The first form of L_{12} yields the **Levi system**

$$2q_{11}\bar{q}_1 + 2q_1\bar{q}_{11} + 2\bar{q}_1\bar{q}_{11} - \bar{q}_{111} - \bar{q}_{12} = 0$$

$$2q_{11}\bar{q}_1 + 2q_1\bar{q}_{11} + 2q_1q_{11} - q_{111} - q_{12} = 0$$

The second form of L_{12} yields a **scalar PDE** equivalent to it.

PDEs for any integrable differential-difference equation?

Probably not...

- ▶ Given L_{0i} , the double-zero property provides an algorithm to find L_{ij} .
- ▶ The L_{ij} generally depend on several lattice shifts.
Eliminating these has been done by trial and error.

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Joint work with Evgeny Ferapontov

The generating PDE, stripped down

The generating PDE for KdV has a Lagrangian 2-form with coefficients

$$L_{ij} = \frac{1}{2}(t_i - t_j) \frac{u_{ij}^2}{u_i u_j} + \frac{1}{2(t_i - t_j)} \left(n_j^2 \frac{u_i}{u_j} + n_i^2 \frac{u_j}{u_i} \right)$$

Set $n_i = n_j = 0$ and replace t_i by constants.

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Set $n_i = n_j = 0$ and replace t_i by constants.

Then

$$d\mathcal{L} = A_{ijk} B_{ijk} dt_i \wedge dt_j \wedge dt_k$$

with

$$A_{ijk} = (c_i - c_j) \frac{u_{ij}}{u_i u_j} + (c_j - c_k) \frac{u_{jk}}{u_j u_k} + (c_k - c_i) \frac{u_{ik}}{u_i u_k}$$

$$B_{ijk} = u_{ijk} - \frac{u_{ij} u_{ik}}{u_i} + \frac{u_{ij} u_{jk}}{u_j} + \frac{u_{ik} u_{jk}}{u_k}$$

Nijhoff, Hone, Joshi. [On a Schwarzian PDE associated with the KdV hierarchy](#). Physics Letters A, 2000

Lobb Nijhoff. [Lagrangian multiforms and multidimensional consistency](#). Journal of Physics A, 2009.

Sleigh. [The Lagrangian multiform approach to integrable systems](#). 2021.

$$L_{ij} = \frac{1}{2}(c_i - c_j) \frac{u_{ij}^2}{u_i u_j}$$

The multi-time Euler-Lagrange equations

$$u_{ijjj} - \frac{u_{ij} u_{ijj}}{u_j} - \frac{u_{ijj} u_{ij}}{u_i} + \frac{u_{ii} u_{ij}^2}{u_i^2} - \frac{u_{ijj} u_{ij}}{u_i} - \frac{u_{jj} u_{ijj}}{u_j} + \frac{u_{jj} u_{ij}^2}{u_j^2} + \frac{u_{ii} u_{ij} u_{jj}}{u_i u_j} = 0$$

$$\frac{c_i - c_j}{u_i u_j} \left(\frac{u_{ij}^2}{2u_j} - u_{ijj} + \frac{u_{ii} u_{ij}}{u_i} \right) - \frac{c_i - c_k}{u_i u_k} \left(\frac{u_{ik}^2}{2u_k} - u_{iik} + \frac{u_{ii} u_{ik}}{u_i} \right) = 0$$

$$(c_i - c_j) \frac{u_{ij}}{u_i u_j} + (c_j - c_k) \frac{u_{jk}}{u_j u_k} + (c_k - c_i) \frac{u_{ik}}{u_i u_k} = 0$$

are consequences of $A_{ijk} = 0$ and $B_{ijk} = 0$:

$$(c_i - c_j) \frac{u_{ij}}{u_i u_j} + (c_j - c_k) \frac{u_{jk}}{u_j u_k} + (c_k - c_i) \frac{u_{ik}}{u_i u_k} = 0$$

$$u_{ijk} - \frac{u_{ij} u_{ik}}{u_i} + \frac{u_{ij} u_{jk}}{u_j} + \frac{u_{ik} u_{jk}}{u_k} = 0$$

Both of these equations have geometric meaning.

$$A_{ijk} = 0$$

If u solves the **Veronese Web Equation**

$$(c_i - c_j) \frac{u_{ij}}{u_i u_j} + (c_j - c_k) \frac{u_{jk}}{u_j u_k} + (c_k - c_i) \frac{u_{ik}}{u_i u_k} = 0$$

then

$$g^{(1)} = (c_2 - c_3)^2 \frac{u_1}{u_2 u_3} dt_1^2 + (c_3 - c_1)^2 \frac{u_2}{u_1 u_3} dt_2^2 + (c_1 - c_2)^2 \frac{u_3}{u_1 u_2} dt_3^2,$$

$$\omega = -\frac{u_{11}}{u_1} dt_1 - \frac{u_{22}}{u_2} dt_2 - \frac{u_{33}}{u_3} dt_3$$

is an **Einstein-Weyl** structure, i.e. there is a torsion-free connection D s.t.

$$D_i g_{jk}^{(1)} = \omega_i g_{jk}^{(1)} \quad (\text{Weyl})$$

$$(\text{Ric}_D)_{(ij)} = \Lambda g_{ij} \quad (\text{Einstein-Weyl})$$

Calderbank, Pedersen. **Einstein-Weyl geometry**. Surveys in Differential Geometry, 2001.

Dunajski, Kryński. **Einstein-Weyl geometry, dispersionless Hirota equation and Veronese webs**. Math. Proc. Cambridge Philos. Soc, 2014

$$B_{ijk} = 0$$

If u solves

$$u_{ijk} - \frac{u_{ij}u_{ik}}{u_i} + \frac{u_{ij}u_{jk}}{u_j} + \frac{u_{ik}u_{jk}}{u_k} = 0$$

then the diagonal **potential metric** (Egorov metric)

$$g^{(2)} = u_1 dt_1^2 + u_2 dt_2^2 + u_3 dt_3^2$$

has diagonal curvature: only nonzero components of curvature are R^i_{jij}

$$B_{ijk} = 0$$

If u solves

$$u_{ijk} - \frac{u_{ij}u_{ik}}{u_i} + \frac{u_{ij}u_{jk}}{u_j} + \frac{u_{ik}u_{jk}}{u_k} = 0$$

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Lagrangian multiform connects these two geometric equations.

Do we believe in coincidence?

Dubrovin. **Differential geometry of strongly integrable systems of hydrodynamic type**. Functional Analysis and Its Applications, 1990.

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Summary

- ▶ Different derivations of multi-time Euler-Lagrange equations (stepped curves/surfaces vs $\delta d\mathcal{L}$): matter of personal taste.
- ▶ Factorisation of $d\mathcal{L}$ is key to constructing examples.
- ▶ Comparing factorisation of $d\mathcal{L}$ and multi-time Euler-Lagrange equations leads to nontrivial connections!
 - ▶ Differential-difference equations and PDEs.
 - ▶ PDEs and very different PDEs.

Summary

- ▶ Different derivations of multi-time Euler-Lagrange equations (stepped curves/surfaces vs $\delta d\mathcal{L}$): matter of personal taste.
- ▶ Factorisation of $d\mathcal{L}$ is key to constructing examples.
- ▶ Comparing factorisation of $d\mathcal{L}$ and multi-time Euler-Lagrange equations leads to nontrivial connections!
 - ▶ Differential-difference equations and PDEs.
 - ▶ PDEs and very different PDEs.

Thank you for your attention!

More semi-discrete geometry

- ▶ The **exterior derivative**:

$$\mathbb{d}\mathcal{L} = \left(\Delta(\mathcal{L}^d) - \mathbb{d}\mathcal{L}^{d-1}, \mathbb{d}\mathcal{L}^d \right),$$

where $\Delta = \text{id} - \mathcal{T}^{-1}$ is the backward difference operator,
 \mathcal{T} denotes shift operator: $\mathcal{T}q^{[k]} = q^{[k+1]}$

- ▶ **Boundary** of $S = \left(\bigsqcup_{k \in \mathbb{Z}} S_k^{d-1}, \bigsqcup_{k \in \mathbb{Z}} S_k^d \right)$ is given by

$$\partial S = \left(\bigsqcup_{k \in \mathbb{Z}} -\partial S_k^{d-1}, \bigsqcup_{k \in \mathbb{Z}} \left(\partial S_k^d \cup S_k^{d-1} \cup -S_{k+1}^{d-1} \right) \right),$$

where the minus sign denotes a change of orientation.

- ▶ **Stokes theorem**:

$$\int_S \mathbb{d}\mathcal{L} = \int_{\partial S} \mathcal{L}.$$

Semi-discrete variational derivatives

$$\frac{\delta_0 L}{\delta q_l} := \frac{\partial}{\partial q_l} \sum_{n \in \mathbb{N}} \mathcal{T}^{-n} L,$$

$$\frac{\delta_{0i} L}{\delta q_l} := \frac{\delta_i}{\delta q_l} \sum_{n \in \mathbb{N}} \mathcal{T}^{-n} L.$$

Traditional discrete EL eqn:

$$\sum_{n \in \mathbb{Z}} \mathcal{T}^{-n} \frac{\partial L}{\partial q^{[n]}} = 0$$

Traditional semi-discrete EL eqn:

$$\sum_{n \in \mathbb{Z}} \mathcal{T}^{-n} \frac{\delta_i L}{\delta q^{[n]}} = 0$$

Same if L does not contain negative shifts of q .

Denote $\bar{q} = \mathcal{T}q$ and $\underline{q} = \mathcal{T}^{-1}q$.

Examples:

$$\frac{\delta_{0i} q_{t_i}^2}{\delta q} = \frac{\delta_i q_{t_i}^2}{\delta q} = -2 \frac{d}{dt_i} q_{t_i} = -2 q_{t_i; t_i},$$

$$\frac{\delta_{0i} \bar{q}_{t_i}^2}{\delta q} = -2 \frac{d}{dt_i} q_{t_i} = -2 q_{t_i; t_i},$$

$$\frac{\delta_{0i} q \bar{q}}{\delta q} = \frac{\delta_0 q \bar{q}}{\delta q} = \bar{q} + \underline{q},$$

$$\frac{\delta_{0i} q q}{\delta q} = \frac{\delta_0 q q}{\delta q} = \underline{q}.$$

Semi-discrete multi-time Euler-Lagrange equations

A field is critical if and only if the following **multi-time Euler-Lagrange equations** hold for all $n \in \mathbb{Z}$:

$$\frac{\delta_{ij} L_{ij}}{\delta q_i^{[n]}} = 0 \quad \forall I \not\ni t_i, t_j,$$

$$\frac{\delta_{ij} L_{ij}}{\delta q_{I t_j}^{[n]}} - \frac{\delta_{ik} L_{ik}}{\delta q_{I t_k}^{[n]}} = 0 \quad \forall I \not\ni t_i,$$

$$\frac{\delta_{ij} L_{ij}}{\delta q_{I t_i t_j}^{[n]}} + \frac{\delta_{jk} L_{jk}}{\delta q_{I t_j t_k}^{[n]}} + \frac{\delta_{ki} L_{ki}}{\delta q_{I t_k t_i}^{[n]}} = 0 \quad \forall I,$$

$$\frac{\delta_{ij} L_{ij}}{\delta q_{I t_j}^{[n]}} + \frac{\delta_{0i} L_{0i}}{\delta q_I^{[n]}} = 0 \quad \forall I \not\ni t_i,$$

$$\frac{\delta_{ij} L_{ij}}{\delta q_{I t_i t_j}^{[n]}} - \frac{\delta_{0j} L_{0j}}{\delta q_{I t_j}^{[n]}} + \frac{\delta_{0i} L_{0i}}{\delta q_{I t_i}^{[n]}} = 0 \quad \forall I,$$

If n is such that L_{ij} does not depend on $q_I^{[n]}$ for any I , then

$$\frac{\delta_{0i} L_{0i}}{\delta q_I^{[n]}} = 0 \quad \forall I \not\ni t_i,$$

$$\frac{\delta_{0j} L_{0j}}{\delta q_{I t_j}^{[n]}} - \frac{\delta_{0i} L_{0i}}{\delta q_{I t_i}^{[n]}} = 0 \quad \forall I.$$

Newtonian ($p = q_1$) lattice

Suppose that the H_i do not contain any negative shifts and that the Hamiltonians $\mathcal{H}_i = \sum_{\alpha} \mathcal{T}^{\alpha} H_i$ are in involution:

$$\begin{aligned} 0 = \{\mathcal{H}_i, \mathcal{H}_j\} &= \sum_{\gamma} \frac{\partial \mathcal{H}_i}{\partial \mathcal{T}^{\gamma} q_1} \frac{\partial \mathcal{H}_j}{\partial \mathcal{T}^{\gamma} q} - \frac{\partial \mathcal{H}_i}{\partial \mathcal{T}^{\gamma} q} \frac{\partial \mathcal{H}_j}{\partial \mathcal{T}^{\gamma} q_1} \\ &= \sum_{\gamma} \mathcal{T}^{\gamma} \left(\frac{\delta_0 H_i}{\delta q_1} \frac{\delta_0 H_j}{\delta q} - \frac{\delta_0 H_i}{\delta q} \frac{\delta_0 H_j}{\delta q_1} \right) \end{aligned}$$

So there exists a potential F_{ij} such that

$$\frac{\delta_0 H_i}{\delta q_1} \frac{\delta_0 H_j}{\delta q} - \frac{\delta_0 H_i}{\delta q} \frac{\delta_0 H_j}{\delta q_1} = \Delta F_{ij}$$

The Hamiltonians are in involution



The local Poisson expression of the Hamiltonian densities is a difference.

Third Toda equation

At the next level we find

$$L_{03} = q_1 q_3 - \frac{1}{4} q_1^4 - a(q_1^2 + \bar{q}_1^2 + q_1 \bar{q}_1) - a\bar{a} + \frac{1}{2} a^2,$$

$$L_{13} = -a(\bar{q}_1^3 + 2a\bar{q}_1 + \bar{a}\bar{q}_1 + 2\bar{q}_1 \bar{q}_{11} + q_1 \bar{q}_{11} - \bar{q}_3 + aq_1 - \bar{a}q_1),$$

$$L_{23} = -a(\bar{q}_2(q_1^2 + \bar{q}_1^2 + q_1 \bar{q}_1 + \underline{a} + a) + \bar{q}_2 \bar{a} + 2\bar{q}_1 \bar{q}_{12} + q_1 \bar{q}_{12} - \bar{q}_{13} \\ - \bar{q}_3(q_1 + \bar{q}_1) - q_1^2 \bar{q}_1^2 - \underline{a} \bar{q}_1^2 + 2aq_1 \bar{q}_1 - \bar{a}q_1^2 - a\bar{a} - \underline{a}\bar{a} - \underline{a}a - a^2)$$

where $a = \exp(\bar{q} - q)$

Again we can use the multi-time Euler-Lagrange equations to obtain a PDE at a single lattice site:

$$\frac{\delta_{13} L_{13}}{\delta q} = 0 \quad \Rightarrow \quad q_1^3 - 3q_1 q_{11} + 6q_1 a + q_{111} - q_3 = 0,$$

which can be simplified to

$$q_3 = -2q_1^3 + 3q_1 q_2 + q_{111}.$$