Subfactors and Fourier Duality in memory of Vaughan Jones

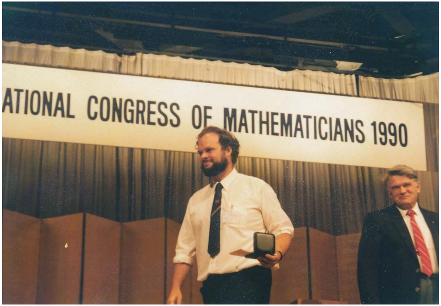
Zhengwei Liu

Tsinghua University
Beijing Institute of Mathematical Science and Applications

Sep 20, 2021, IAMS-BIRS workshop

Subfactor Theory

Subfactor theory has wide connections in mathematics and physics: Operator Algebras, Quantum Groups, Representation theory, Knot Theory, Lower Dimensional Topology, Category Theory, Statistical Physics, Quantum Field Theory etc.



Vaughan Jones won the Fields metal at the 1990 ICM at Kyoto.

von Neumann Algebras

A Hilbert space $\mathcal{H}$ is a complete inner product space over the field $\mathbb{C}$. (We assume that $\dim \mathcal{H}$ is countable.)
$B(\mathcal{H})$ is the set of all bounded operators on $\mathcal{H}$.

Von Neumann's double commutant theorem:

Theorem

*Let $\mathcal{M}$ be a unital *-algebra acting on a Hilbert space $\mathcal{H}$. Then*

$$\overline{\mathcal{M}}^{WOT} = \mathcal{M}''.$$

Moreover, $\mathcal{M}$ is called a **von Neumann algebra**, *if $\mathcal{M}'' = \mathcal{M}$.*

- $A_\lambda \to A$ in *Weak Operator Topology* (WOT) in $B(\mathcal{H})$, if

$$\langle v, A_\lambda w \rangle \to \langle v, Aw \rangle, \ \forall \ v, w \in \mathcal{H}.$$

- $M' = \{a \in B(H) : ab = ba, \ \forall \ b \in M\}$, and $M'' = \{M'\}'$.

Factors

A von Neumann algebra $\mathcal{M}$ is called a **factor**, if its center $\mathcal{M} \cap \mathcal{M}' = \mathbb{C}I$.

Murray and von Neumann classified factors into three types by comparing minimal projections P, Q in $\mathcal{M}$:
$P \sim Q$, if $\exists V \in \mathcal{M}$, s. t. $P = VV^*$ and $Q = V^*V$.
The equivalent classes of minimal projections are classified as follows:

- Type I_n: $\{0, 1, 2, \cdots, n\}$, $n \in \mathbb{N} \cup \{\infty\}$;
- Type II_1: $[0, 1]$;
- Type II_∞: $[0, \infty]$;
- Type III: $\{0, 1\}$.

$B(H)$ is of type I_n, $n = \dim \mathcal{H}$.
Factors appeared in Conformal Field Theory are of type III.

II_1 Factors

A II_1 factor is infinite dimensional and it has a unique tracial state τ.

Murray-von Neumann construction of II_1 factors:
For a countable group K, the left action of K on $L^2(K)$ generates a von Neumann algebra $\mathcal{L}(K)$, called the group von Neumann algebra.
Furthermore, $\mathcal{L}(K)$ is a II_1 factor, if K is a i.c.c. group, namely, the conjugacy class of any non-trivial element of K is infinite.

$$\mathcal{L}(\lim_{n\to\infty} S_n) \ncong \mathcal{L}(F_2).$$

Here, S_n is the permutation group on n elements and F_n is the free group with n generators.
Big Open Question: $\mathcal{L}(F_2) \cong \mathcal{L}(F_3)$?.

Hyperfinite II_1 Factors

A von Neumann algebra is called **hyperfinite** if it is an inductive limit of finite dimensional ones.
The hyperfinite II_1 factor $\mathcal{R}$ is unique, and is smallest among all II_1 factors.
The von Neumann algebra generated by $\bigotimes_{k=1}^{\infty} M_2(\mathbb{C})$ w.r.t. the trace is the hyperfinite II_1 factor.

Connes' deep result in 1973: For type II_1 factors,
Hyperfinite $\iff$ Amenable $\iff$ Injective $\iff$ $\cdots$

Subfactors

Observation: The hyperfinite II_1 factor $\mathcal{R}$ is universal for amenable groups, but it does not remember the group.

Question: How to recover group symmetries from $\mathcal{R}$?

Observation: The hyperfinite II_1 factor $\mathcal{R}$ is universal for amenable groups, but it does not remember the group.

Question: How to recover group symmetries from $\mathcal{R}$?

For any finite group G, $\mathcal{R} = \bigotimes_{k=1}^{\infty} M_2(\mathbb{C})''$ admits an outer action of G as permutations of the indices.

The crossed product $\mathcal{R} \rtimes G$ is a II_1 factor containing $\mathcal{R}$.

An inclusion of factors $\mathcal{N} \subseteq \mathcal{M}$ is called a **subfactor**.

Subfactors

Observation: The hyperfinite II_1 factor $\mathcal{R}$ is universal for amenable groups, but it does not remember the group.

Question: How to recover group symmetries from $\mathcal{R}$?

For any finite group G, $\mathcal{R} = \bigotimes_{k=1}^{\infty} M_2(\mathbb{C})''$ admits an outer action of G as permutations of the indices.

The crossed product $\mathcal{R} \rtimes G$ is a II_1 factor containing $\mathcal{R}$.

An inclusion of factors $\mathcal{N} \subseteq \mathcal{M}$ is called a **subfactor**.

Answer: The subfactor $\mathcal{R} \subseteq \mathcal{R} \rtimes G$ remembers the finite group G,

Subfactors

Observation: The hyperfinite II_1 factor $\mathcal{R}$ is universal for amenable groups, but it does not remember the group.

Question: How to recover group symmetries from $\mathcal{R}$?

For any finite group G, $\mathcal{R} = \bigotimes_{k=1}^{\infty} M_2(\mathbb{C})''$ admits an outer action of G as permutations of the indices.

The crossed product $\mathcal{R} \rtimes G$ is a II_1 factor containing $\mathcal{R}$.

An inclusion of factors $\mathcal{N} \subseteq \mathcal{M}$ is called a **subfactor**.

Answer: The subfactor $\mathcal{R} \subseteq \mathcal{R} \rtimes G$ remembers the finite group G, and its Fourier dual $\hat{G}$.

Modules of Factors

The module category of a factor $\mathcal{M}$ is captured by $\mathcal{M}'$ (acting on the universal representation).
Modules of factors are classified by "dimensions":

- Type I: $\mathbb{N}$;
- Type II: $[0, \infty]$;
- Type III: $\{1\}$.

Gelfund-Naimark-Segal construction: For a type II_1 factor $\mathcal{M}$, its unique trace τ defines an inner product on $\mathcal{M}$, so the closure is a Hilbert space $L^2(M)$, and it is a $\mathcal{M}$ module, denoted by ${}_{\mathcal{M}}\mathcal{M}$.
Define $\dim_{\mathcal{M}} \mathcal{M} = 1$.
In particular, $\dim_{\mathcal{R}} \mathcal{R} \rtimes G = |G|$.

Jones Index

For a subfactor $\mathcal{N} \subseteq \mathcal{M}$ of type II_1, the Jones index is their relative size:

$$[\mathcal{M} : \mathcal{N}] := \dim_{\mathcal{N}} \mathcal{M}$$

Theorem (Jones 1983)

The set of Jones indices of subfactors is

$$\left\{4\cos^2 \frac{\pi}{\ell+2}, \ell \in \mathbb{N}\right\} \cup [4, \infty].$$

The Jones index could be "quantum".
A subfactor can be regarded as $\mathcal{N} \subseteq \mathcal{N} \rtimes G$ for a "quantum group" G action, even though we do not see G and its action directly.
Indeed, the subfactor with Jones index $4\cos^2 \frac{\pi}{\ell+2}$ is close related to the theory of quantum $SU(2)$ at level ℓ.

Jones Projections and Jones Tower

Basic construction: For a subfactor $\mathcal{N} \subset \mathcal{M}$ with Jones index λ and a trace τ, let e_1 be the Jones projection from $L^2(\mathcal{M})$ onto the subspace $L^2(\mathcal{N})$, then $\mathcal{M}_1 := \{e_1, \mathcal{M}\}''$ is a factor acting on $L^2(M)$, and

$$[\mathcal{M}_1 : \mathcal{M}] = [\mathcal{M} : \mathcal{N}] = \lambda.$$

Jones tower: $\mathcal{N} \subset \mathcal{M} \overset{e_1}{\subset} \mathcal{M}_1 \overset{e_2}{\subset} \mathcal{M}_2 \overset{e_3}{\subset} \cdots$

Fourier Duality

Jones tower: $\mathcal{N} \subset \mathcal{M} \subset \mathcal{M}_1 \subset \mathcal{M}_2 \subset \cdots$
$\mathbb{Z}_2$ periodicity: $\mathcal{M}_1 = \mathcal{N} \otimes M_\lambda(\mathbb{C})$ and $\mathcal{M}_2 = \mathcal{M} \otimes M_\lambda(\mathbb{C})$,

$$\mathcal{N} \subset \mathcal{M} \cong \mathcal{M}_1 \subset \mathcal{M}_2.$$

Fourier Duality: $\mathcal{N} \subset \mathcal{M} \longleftrightarrow \mathcal{M} \subset \mathcal{M}_1$.
If $\mathcal{M} = \mathcal{N} \rtimes G$, then $\mathcal{M}_1 = \mathcal{M} \rtimes \hat{G}$.

Temperley-Lieb Algebras

Subfactor with index $\lambda \to$ **Temperley-Lieb algebra** $TL(\lambda)$:

$$e_i^2 = e_i = e_i^*;$$
$$e_i e_j = e_j e_i, \quad |i-j| \geqslant 2;$$
$$e_i e_{i\pm 1} e_i = \lambda^{-1} e_i.$$

It has a **Markov trace** τ,

$$\tau(xe_n) = \lambda^{-1}\tau(x), \ \forall x \in TL_n.$$

where TL_n is the subalgebra generated by $\{e_i : 1 \leqslant i \leqslant n-1\}$.

For a Jones index $\lambda = 4\cos^2 \frac{\pi}{\ell+2}$, τ is positive semi-definite, and $\{e_i : i \geqslant n\}$ generate a factor $\mathcal{R}_n$, (of hyperfinite type II_1), by GNS construction.

The subfactor $\mathcal{R}_{n+1} \subseteq \mathcal{R}_n$ has Jones index λ.

Jones Polynomial

By changing the variables from projections e_i to the braids σ_i,

$$\sigma_i\sigma_j = \sigma_j\sigma_i, \quad |i-j| \geqslant 2;$$
$$\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1},$$

the trace τ becomes a Markov trace on the braid group, leading to a knot invariant, well-known as the **Jones polynomial**.

Invariants

Question: How to detect different subfactors?

Invariants

Question: How to detect different subfactors?
Answer: Invariants.

- Scalar: Jones index.
- Graph: principal graph.
- Ring: fusion ring.
- Representation Category: standard invariant.

The following deep result of Popa is a quantum analogue of the Tannaka-Krein duality for amenable subfactors.

Theorem (Popa 95)

The standard invariant is a complete invariant of amenable subfactors.

Standard Invariant

Jones tower: $\mathcal{N} \subset \mathcal{M} \subset \mathcal{M}_1 \subset \mathcal{M}_2 \subset \cdots$

Standard invariant:

$$\begin{array}{ccccccccc} \mathcal{N}'\cap\mathcal{N} & \subset & \mathcal{N}'\cap\mathcal{M} & \subset & \mathcal{N}'\cap\mathcal{M}_1 & \subset & \mathcal{N}'\cap\mathcal{M}_2 & \subset & \cdots \\ & & \cup & & \cup & & \cup & & \\ & & \mathcal{M}'\cap\mathcal{M} & \subset & \mathcal{M}'\cap\mathcal{M}_1 & \subset & \mathcal{M}'\cap\mathcal{M}_2 & \subset & \cdots \end{array}$$

Example: For the subfactor $\mathcal{R} \subset \mathcal{R} \rtimes G$,
Jones tower: $\mathcal{R} \subset \mathcal{R} \rtimes G \subset \mathcal{R} \rtimes G \rtimes \hat{G} \subset \mathcal{R} \rtimes G \rtimes \hat{G} \rtimes G \subset \cdots$
Standard invariant:

$$\begin{array}{ccccccccc} \mathbb{C} & \subset & \mathbb{C} & \subset & L^\infty(G) & \subset & L^\infty(G) \rtimes \hat{G} & \subset & \cdots \\ & & \cup & & \cup & & \cup & & \\ & & \mathbb{C} & \subset & \mathbb{C} & \subset & \mathcal{L}G & \subset & \cdots \end{array}$$

Axiomatizations

There are various axiomatizations of the standard invariants.

- Ocneanu: Connections + Flatness, unitary Bimodule Categories, unitary 2+1 Turaev-Viro TQFT, (assuming finite depth)
- Popa: Standard λ-lattice.
- Jones: Subfactor Planar Algebras.
- A Frobenius algebra in a rigid C^*-tensor categories.

Bimodule Category

Ocneanu 94: A finite-index subfactor $\mathcal{N} \subset \mathcal{M}$ $\to$ bimodule category $\to$ 3D Turaev-Viro TQFT (assuming finite depth)

Bimodule Category:

- 0-morphism: $\mathcal{N}$ and $\mathcal{M}$
- 1-morphism: irreducible bimodules in $\lim_{k\to\infty} \mathcal{M}_k$
- 2-morphism: bimodule maps

The tensor functor $\otimes$ is Connes' fusion for bimodules.
Example: When $\mathcal{M} = \mathcal{N} \rtimes G$, the $\mathcal{N} - \mathcal{N}$ bimodule category is $Rep(G)$ and the $\mathcal{M} - \mathcal{M}$ bimodule category is $Vec(G)$, and they are Morita equivalent.

Bisch 97: The bimodule category of a finite index subfactor $\mathcal{N} \subset \mathcal{M}$ is isomorphic to the standard invariant:

$$\begin{array}{ccccccccc}
\hom(1_{\mathcal{N}}) & \subset & \hom(X) & \subset & \hom(X \otimes \overline{X}) & \subset & \hom(X \otimes \overline{X} \otimes X) & \subset & \cdots \\
 & & \cup & & \cup & & \cup & & \\
 & & \hom(1_{\mathcal{M}}) & \subset & \hom(\overline{X}) & \subset & \hom(\overline{X} \otimes X) & \subset & \cdots
\end{array}$$

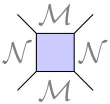
Here $X =_{\mathcal{N}} \mathcal{M}_{\mathcal{M}}$, $\overline{X} =_{\mathcal{M}} \mathcal{M}_{\mathcal{N}}$, $1_{\mathcal{N}} =_{\mathcal{N}} \mathcal{N}_{\mathcal{N}}$, $1_{\mathcal{M}} =_{\mathcal{M}} \mathcal{M}_{\mathcal{M}}$.

The factor $\mathcal{M}$, corresponding to $X \otimes \overline{X}$, defines a Frobenius algebra in the $\mathcal{N} - \mathcal{N}$ bimodule category $\mathscr{D}$.

The bimodule category could be recovered from the Frobenius algebra $X \otimes \overline{X}$ and $\mathscr{D}$.

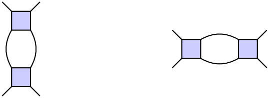
Pictorial Interpretations

2D pictorial representation in planar algebras:
A morphism in $\mathcal{A} := \hom(X \otimes \overline{X}) \cong \mathcal{N}' \cap \mathcal{M}_1 =$ "$L^\infty(G)$" is

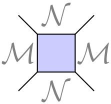


.

Gluing two pictures vertically and horizontally correspond to multiplication and convolution on "$L^\infty(G)$".



A morphism in $\mathcal{B} := \hom(\overline{X} \otimes X) \cong \mathcal{M}' \cap \mathcal{M}_2 =$ "$\mathcal{L}(G)$" is

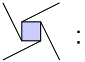


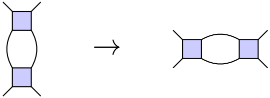
.

Pictorial Fourier Duality

The 90° rotation $\mathfrak{F}_s$, called the string Fourier transform (SFT), is a map from $\mathcal{A}$ to $\mathcal{B}$, with periodicity four.
It intertwines the multiplication and the convolution as illustrated.

Fourier Transform   Multiplication   Convolution





In general, $\mathscr{P}_{n,+} = \mathcal{N}' \cap \mathcal{M}_{n-1}$ and $\mathscr{P}_{n,-} = \mathcal{M}' \cap \mathcal{M}_n$ are represented by diagrams with $2n$ boundary points, the SFT $\mathfrak{F}_s : \mathscr{P}_{n,\pm} \to \mathscr{P}_{n,\mp}$ has periodicity $2n$.

Quantum Double Construction

For a unitary fusion category $\mathscr{C}$ with a set irreducible objects Irr, $\mathscr{D} = \mathscr{C} \otimes \mathscr{C}^{op}$ has a canonical Frobenius algebra $\gamma = \bigoplus_{Y \in Irr} Y \otimes Y^{op}$. Then the $\gamma - \gamma$ bimodule category $\mathscr{E}$ over $\mathscr{C} \otimes \mathscr{C}^{op}$ is ismorphic to the Drinfeld center of $\mathscr{C}$.
Take $\mathcal{A} = \hom_{\mathscr{D}}(\gamma)$ and $\mathcal{B} = \hom_{\mathscr{E}}(\gamma \otimes \gamma)$. Then $\mathcal{A} = L^\infty(G)$ and $\mathcal{B} = \mathcal{L}(G)$, where G is the *probability group* associated with the Grothendieck ring of $\mathscr{C}$.
In particular, $\mathscr{C}$ is $VecG$, $\mathcal{A} = L^\infty(G)$ and $\mathcal{B} = \mathcal{L}(G)$.

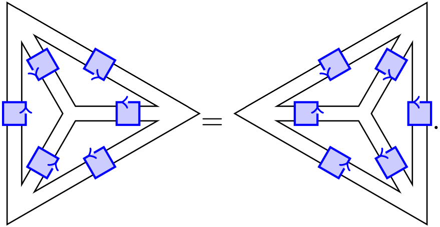
Furthermore, if $\mathscr{C}$ is a modular tensor category, then the $\mathcal{A} \cong \mathcal{B}$ and the SFT $\mathfrak{F}_s$ is the modular S-matrix of $\mathscr{C}$, see [L-Xu 2019]

6j-Symbol Self-Duality

Theorem (L 2019)

For any modular tensor category $\mathscr{C}$, *and any* $\vec{X} \in Irr^6$,

$$\left|\begin{pmatrix} \frac{X_6}{X_3} & \frac{X_5}{X_2} & \frac{X_4}{X_1} \end{pmatrix}\right|^2 = \sum_{\vec{Y} \in Irr^6} \left(\prod_{k=1}^{6} S_{X_k}^{Y_k} \right) \left| \begin{pmatrix} Y_1 & Y_2 & Y_3 \\ Y_4 & Y_5 & Y_6 \end{pmatrix} \right|^2 .$$



John Barrett proved the 6j-symbol self-duality for quantum $SU(2)$ in 2003. A general case for MTCs was conjectured by Shamil Shakirov in 2015 at Harvard, which we answer positively here.

Quantum Fourier Analysis

Question: Do we have a quantum analogue of Fourier analysis on subfactors?

Quantum Fourier Analysis

Question: Do we have a quantum analogue of Fourier analysis on subfactors? Yes! Quantum Fourier Analysis!

Papers: L 2016, Jiang-L-Wu 2016, Jaffe-Jiang-L-Ren-Wu 2020, L-Palcoux-Wu 2021, Huang-L-Wu 2021+ etc.

- Schur-product theorem
- Hausdorff-Young inequality
- Young's inequality
- Hirschman-Beckner uncertainty principle
- Donoho-Stark uncertainty principle
- Sum set estimate
- The characterization of operators which attain the equality of the above inequalities
- Hardy uncertainty principle
- Rényi entropic uncertainty principle
- Block maps, 2D central limit theorem (new for $\mathbb{Z}_2$)
- ...

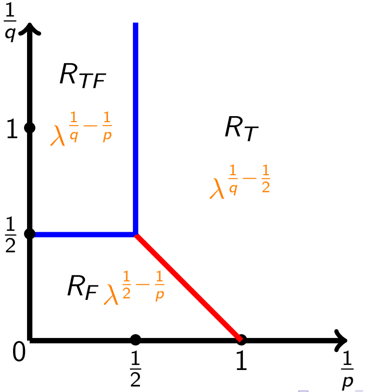
p,q-Norm of SFT

Theorem (L-Wu 2019)

Let $x \in \mathcal{A}$ be such that $\|x\|_2 = 1$. Then for any $p, q > 0$,

$$\|\mathfrak{F}_s(x)\|_q \leqslant K(1/p, 1/q)\|x\|_p.$$

K(1/p,1/q):=



Rényi entropic uncertainty principles

For $p \in (0,1) \cup (1,\infty)$, we define the Rényi entropy of order p of x in $\mathcal{A}$ by

$$h_p(x) = \frac{p}{1-p} \log \|x\|_p.$$

$$h_1(x) = H(x) = tr_2(-\|x\| \log \|x\|).$$

Theorem (L-Wu 2019)

Let $x \in \mathcal{A}$ be such that $\|x\|_2 = 1$. Then for any $p, q > 0$,

$$(1/p - 1/2)h_{p/2}(|x|^2) + (1/2 - 1/q)h_{q/2}(|\mathfrak{F}_s(x)|^2) \geqslant -\log K(1/p, 1/q).$$

When $1/p, 1/q \to 1/2$, we obtain the Hirschman-Beckner uncertainty principle.

Theorem (Jiang-L-Wu 2016)

For any nonzero $x \in \mathcal{A}$,

$$H(|x|^2) + H(|\mathcal{F}(x)|^2) \geqslant \|x\|_2 (2 \log \delta - 4 \log \|x\|_2),$$

where $H(|x|^2) = -tr_2(|x|^2 \log |x|^2)$ is the von Neumann entropy of $|x|^2$.

The equality holds $\iff$ x is a bi-shift of a biprojection.

When $1/p, 1/q \to \infty$, we obtain the Donoho-Stark uncertainty principle.

Theorem (Jiang-L-Wu 2016)

For any nonzero $x \in \mathcal{A}$,

$$\mathcal{S}(x)\mathcal{S}(\mathcal{F}(x)) \geqslant \delta^2,$$

where $\mathcal{S}(x)$ is the trace of range projection of x.

Thank you!

- L-Xu 2019, Jones Wassermann Subfactors for Modular Tensor Categories, Adv. Math. 355 (2019) 106775
- L 2019, Quon Language: Surface Algebras and Fourier Duality, CMP, 366 (2019) 865-894
- Jaffe-Jiang-L-Ren-Wu 20: Quantum Fourier Analysis, PNAS, 117(20) (2020) 10715-10720
- L-Palcoux-Wu 21: Fusion Bialgebras and Fourier Analysis: Analytic obstructions for unitary categorification, Adv. Math. 390(29) (2021), 107905
- Huang-L-Wu 21: Quantum Smooth Uncertainty Principles for von Neumann bi-Algebras, https://arxiv.org/abs/2107.09057
  and further references therein