Tensor networks, commuting squares and higher relative commutants of subfactors

Yasu Kawahigashi

the University of Tokyo/Kavli IPMU/RIKEN

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Bi-unitary connections, subfactors and tensor networks

Physicists in condensed matter physics are recently interested in a certain family $(a_{ijkl})_{ijkl}$ of complex numbers labeled with 4 indices, called a 4-tensor, in connection to two-dimensional topological order. They construct certain finite dimensional projections out of this and make physical studies of their ranges.

We first show that their 4-tensor corresponds to a bi-unitary connection giving a finite dimensional commuting square, labeled with 4 edges from the 4 Bratteli diagrams. Then our main result identifies the range of their projections with the higher relative commutants of the subfactor arising from such a commuting square.

A commuting square

Consider the following inclusions of four finite dimensional C^*-algebras,

$$\begin{array}{ccc} A & \subset & B \\ \cap & & \cap \\ C & \subset & D, \end{array}$$

with a normalized trace tr on D and $A = B \cap C$.

When the orthogonal projections onto subalgebras B, C with respect to the L^2-norm arising from the trace commute on D, we say that the above is a commuting square. If we have $\mathbf{span}\, BC = D$, then we say that the commuting square is non-degenerate. Finite dimensional non-degenerate commuting squares have been important and well-studied in subfactor theory of Jones over many years.

Repeated basic constructions

Starting with a finite dimensional non-degenerate commuting square, we can repeat basic constructions of Jones and get increasing sequences of finite dimensional algebras.

$$\begin{array}{ccccccccc} A & \subset & B & \subset & B_1 & \subset & B_2 & \subset & \cdots \\ \cap & & \cap & & \cap & & \cap & & \\ C & \subset & D & \subset & D_1 & \subset & D_2 & \subset & \cdots \end{array}$$

We take the GNS-completions of the unions $\bigcup_{n=1}^{\infty} B_n \subset \bigcup_{n=1}^{\infty} D_n$ with respect to the trace to get $N \subset M$. Both N and M are hyperfinite II_1 factors and we get a subfactor of the finite Jones index. We can also repeat the basic construction vertically and get another subfactor $P \subset Q$.

An old question of Jones

When we have a subfactor $N \subset M$ with finite Jones index, we have the Jones tower $N \subset M \subset M_1 \subset M_2 \subset \cdots$ arising from the basic constructions. When we have only finitely many irreducible bimodules arising from ${}_N M_{k N}$, we say that $N \subset M$ has a finite depth. This is an important finiteness condition in connection to 3-dimensional topology and mathematical physics.

In 1995, Jones asked the following question.

When one of the two subfactors of $N \subset M$ and $P \subset Q$ has a finite depth, so does the other?

Sato gave a positive answer and a more detailed characterization of the relation between the two.

Strongly amenable subfactors and Popa's classification

From a subfactor $N \subset M$ with finite Jones index, we get the following sequence of commuting squares.

$$\begin{array}{ccccccc} M' \cap M & \subset & M' \cap M_1 & \subset & M' \cap M_2 & \subset & \cdots \\ \cap & & \cap & & \cap & & \\ N' \cap M & \subset & N' \cap M_1 & \subset & N' \cap M_2 & \subset & \cdots \end{array}$$

Popa proved that the subfactor $N \subset M$ is completely recovered from the above commuting squares if the subfactor satisfies a nice analytic property called strong amenability. If we have a finite depth and M is hyperfinite, a single commuting square

$$\begin{array}{ccc} M' \cap M_k & \subset & M' \cap M_{k+1} \\ \cap & & \cap \\ N' \cap M_k & \subset & N' \cap M_{k+1} \end{array}$$

for a large k suffices.

A bi-unitary connection

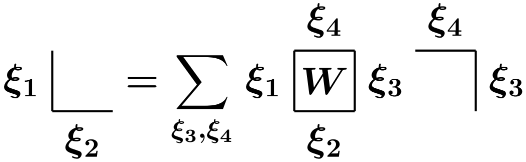
We choose one edge each from the four Bratteli diagrams of a commuting square. Then we get an assignment W of a complex number to each such square with the following. This is called a bi-unitary connection.

$$\sum_{z,\xi_1,\xi_2} \begin{array}{ccc} x & \xi_4 & y \\ \xi_1 & \boxed{W} & \xi_3 \\ z & \xi_2 & w \end{array} \; \overline{\begin{array}{ccc} x & \xi_4' & y' \\ \xi_1 & \boxed{W} & \xi_3' \\ z & \xi_2 & w \end{array}} = \delta_{\xi_3,\xi_3'}\delta_{\xi_4,\xi_4'}$$

$$\begin{array}{ccc} y & \tilde{\xi}_4 & x \\ \xi_3 & \boxed{W'} & \xi_1 \\ w & \tilde{\xi}_2 & z \end{array} = \sqrt{\frac{\mu_x\mu_w}{\mu_y\mu_z}} \; \overline{\begin{array}{ccc} x & \xi_4 & y \\ \xi_1 & \boxed{W} & \xi_3 \\ z & \xi_2 & w \end{array}}$$

Basis change with a bi-unitary connection

Paths of length 2 on two Bratteli diagrams give an orthonormal basis $|\xi_1\xi_2\rangle$ of a (finite dimensional) Hilbert space. Those on the other two Bratteli diagrams give another basis $|\xi_4\xi_3\rangle$ of the same space, and a bi-unitary connection gives a basis change as follows.



Namely, the bi-unitary connection W gives a unitary matrix $\langle\xi_1\xi_2|\xi_4\xi_3\rangle$ on this Hilbert space. This unitarity is a "half" of bi-unitarity.

The string algebra construction

Suppose we have a series of Bratteli diagrams for inclusions $\mathbb{C} = A_0 \subset A_1 \subset A_2 \subset A_3 \subset A_4 \subset \cdots$. We have a model for these inclusions as follows. Let (ξ_1, ξ_2) be a pair of path of the same length on this Bratteli diagram with a common starting vertex and a common ending vertex at some stage. We call such a pair a string and they span a finite dimensional $\mathbb{C}$-vector space. A string (ξ, η) really means an operator $|\xi\rangle\langle\eta|$ in the bracket notation, and this gives an algebra structure among strings of the same length.
We make an embedding of a string (ξ_1, ξ_2) of length k into the next row as $\sum_\eta (\xi_1 \cdot \eta, \xi_2 \cdot \eta)$, where η is a path of length 1 and $\cdot$ stands for concatenation of paths.

Construction of a subfactor

We use four Bratteli diagrams and their reflections to obtain doubly indexed string algebras A_{kl}. Since the bi-unitary connection gives a basis change of paths of length 2, it also gives a basis change of strings of length 2 so that we have inclusions

$$\begin{array}{ccc} A_{kl} & \subset & A_{k,l+1} \\ \cap & & \cap \\ A_{k+1,l} & \subset & A_{k+1,l+1}. \end{array}$$

This is a commuting square due to the other "half" of bi-unitarity.

Taking the GNS-completions, we have the limit algebras $A_{k,\infty}$ and $A_{\infty,l}$, and they are hyperfinite II_1 factors. We naturally have two subfactors $A_{0,\infty} \subset A_{1,\infty}$ and $A_{\infty,0} \subset A_{\infty,1}$, like $N \subset M$ and $P \subset Q$ before.

An example for the Dynkin diagrams

We give an example of a bi-unitary connection as follows. Fix one of the A-D-E Dynkin diagram and use it for the four Bratteli diagrams. Let n be its Coxeter number and set $\varepsilon = \sqrt{-1}\exp\dfrac{\pi\sqrt{-1}}{2(n+1)}$. We write μ_x for the Perron-Frobenius eigenvector entry for a vertex x. Then a bi-unitary connection is given as follows.

$$\begin{matrix} j & & k \\ & \boxed{W} & \\ l & & m \end{matrix} = \delta_{kl}\varepsilon + \sqrt{\frac{\mu_k\mu_l}{\mu_j\mu_m}}\delta_{jm}\bar{\varepsilon}$$

Figure: A bi-unitary connection on the Dynkin diagram

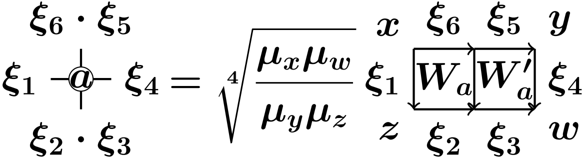
A subfactor and a fusion category

Suppose the subfactor $A_{0,\infty} \subset A_{1,\infty}$ (or equivalently $A_{\infty,0} \subset A_{\infty,1}$) has a finite depth. Consider the bimodules ${}_{A_{0,\infty}}(A_{k,\infty})_{A_{0,\infty}}$ and their irreducible decompositions. We get only finitely many irreducible bimodules in this way and we have a fusion category of bimodules. We have a relative tensor product of bimodules and dual bimodules there.

We also have corresponding tensor products and irreducible decompositions at the level of bi-unitary connections. We then have an equivalent fusion category of bi-unitary connections. This correspondence is given by the open string bimodule construction, due to Asaeda-Haagerup.

A 4-tensor from a bi-unitary connection

Suppose we have a bi-unitary connection W_a. We then define a 4-tensor a as follows.



Here W_a' stands for the horizontal reflection of W_a. We also use the vertical reflection so that we can concatenate 4-tensors as usual. The reflection corresponds to basic construction and the vertical concatenation of 4-tensors corresponds to the product of bi-unitary connections.

A matrix product operator algebra

Suppose we have a 4-tensor corresponding to a commuting square giving a subfactor of finite depth.

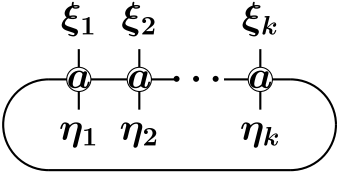
Bultinck-Mariën-Williamson-Şahinoğlu-Haegeman-Verstraete gave an anyon algebra, a finite dimensional C^*-algebra, in this setting and argued that its minimal central projections give anyons describing a two-dimensional topological order. Here an anyon is a new type of quasi-particle more general than a boson and a fermion and it is expected to be useful for constructing a topological quantum computer.

We proved that this anyon algebra is isomorphic to the tube algebra of Ocneanu and anyons correspond to irreducible objects of the Drinfel′d center.

A projector matrix product operator

We define a matrix product operator O_a^k as follows.

$$\sum$$



$$|\,\xi_1\xi_2\cdots\xi_k\rangle\langle\eta_1\eta_2\cdots\eta_k\,|$$

We then set $P^k=\sum_a \frac{d_a}{w}O_a^k$ like Bultinck-Mariën-Williamson-Şahinoğlu-Haegeman-Verstraete. This is a projector matrix product operator (PMPO) and it acts on certain projected entangled pair state (PEPS).

Higher relative commutants of a subfactor

The range of the projector matrix product operator P^k plays an important role in theory of two-dimensional topological order, and we identify it with the higher relative commutant $A'_{\infty,0} \cap A_{\infty,k}$ of the subfactor. This is equal to $A_{0,k}$ if (and only if) the original bi-unitary connection is flat, but we do not assume this flatness here.

We have the inclusion $A'_{\infty,0} \cap A_{\infty,k} \subset A_{0,k}$ due to Ocneanu's compactness argument and he proved that an element in $A'_{\infty,0} \cap A_{\infty,k}$ is characterized as a flat field of strings of length k. A field of strings is an element in a certain string algebra and it is flat if and only if it does not change the form under parallel transport of length 2.

A sketch of a proof

We sketch a proof of the above identification.
It is not difficult to show that if we have a flat field of strings, then it is preserved under the projector matrix product operator P^k because a flat field does not change the form under a parallel transport.
Conversely, take an element in the range of the projector matrix product operator P^k. Then we construct an element $x_m \in A'_{m,0} \cap A_{m,k}$ in a simple manner. Using the Perron-Frobenius theorem, we show that $\{x_m\}_m$ is a Cauchy sequence in the L^2-norm, so it converges to some x in $A'_{\infty,0} \cap A_{\infty,k}$ and gives a flat field of strings.
We next show that all x_m are actually equal to x.
The above two maps are actually mutual inverses.

The Drinfel′d center and Morita equivalence

For getting a fusion category, we used the subfactor $A_{0,\infty} \subset A_{1,\infty}$, but now for the range of the projector matrix product operator, we used the higher relative commutants of the other subfactor $A_{\infty,0} \subset A_{\infty,1}$.
The former is used to get a modular tensor category through the tube algebra and we have description of anyons. The latter produces a series of Hilbert spaces on which Hamiltonians act.
These two subfactors can be quite different, but still the relation between the two is characterized as being opposite Morita equivalent. In particular, they produce complex conjugate topological quantum field theory (TQFT) and have the same Drinfel′d center.