Toward the correspondence of hook-type $\mathcal{W}$-algebras and $\mathcal{W}$-superalgebras

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based on joint works with T. Creutzig, N. Genra, A. Linshaw and R. Sato

2021/Sep/23

Triality of vertex algebras at the corner

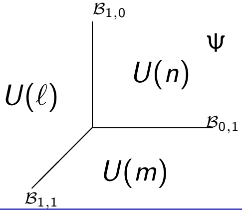
The principal $\mathcal{W}$-algebra $\mathcal{W}^k(\mathfrak{sl}_n)$ enjoys the Feigin-Frenkel duality and GKO type coset construction:

$$\mathcal{W}^{k_1}(\mathfrak{sl}_n) \longrightarrow \mathcal{W}^{k_2}(\mathfrak{sl}_n)$$

$$\mathrm{Com}(V^{k_3}(\mathfrak{sl}_n), V^{k_3-1}(\mathfrak{sl}_n) \otimes L_1(\mathfrak{sl}_n))$$

Rel: $$(k_1+n)(k_2+n)=1, \qquad \frac{1}{k_1+n}+\frac{1}{k_3+n}=1$$

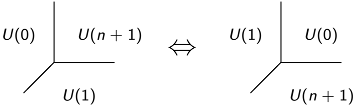
Some vertex algebras appear at boundary of higher dimensional QFTs with boundary conditions, e.g. $V^k(\mathfrak{g})$ appears at the boundary of 3d Chern-Simons theory. Gaiotto–Rapčák explained the above triality as symmetry of boundary conditions for 4d GL-twisted $\mathcal{N}=4$ super Yang–Mills theory and gave a vast generalization in physics context:



$$\mathrm{Com}\left(V(\mathfrak{gl}_{m|\ell}), \mathcal{W}^{\Psi-h^\vee}(\mathfrak{sl}_{n|\ell}, f_{n-m,1^m|1^\ell})\right)$$

$\{(\mathcal{B}_{p,q},\Psi)\}_{(p,q)\in\mathbb{Z}^2/\{\pm1\}}$: boundary conditions $\curvearrowleft PSL_2(\mathbb{Z})$

Feigin–Semikhatov duality



Theorem 1.1 (Creutzig–Linshaw, Creutzig–Genra–N)

For $(k+n+1)(\ell+n)=1$ [a]

$$\textbf{FS}:\ \mathrm{Com}\left(\pi, \mathcal{W}^k(\mathfrak{sl}_{n+1}, f_{n,1})\right) \simeq \mathrm{Com}\left(\pi, \mathcal{W}^\ell(\mathfrak{sl}_{n+1|1}, f_{n+1|1})\right)$$

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[a] We remove the level $(k,\ell)=(-n+\frac{1}{n}, -n+\frac{n}{n+1})$ where the Heisenberg subalgebra degenerates.

The case $n=1$ is actually the Kazama–Suzuki coset construction:

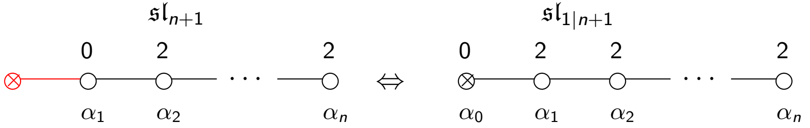
$$\begin{aligned} \textbf{KS}: \quad \mathcal{N}=2\ \mathrm{SCA}_{c.c.=\frac{3k}{k+2}} &\xrightarrow{\simeq} \mathrm{Com}\left(\pi^{\mathrm{diag}}, V^k(\mathfrak{sl}_2)\otimes bc\right) \\ G^+(z) &\mapsto \sqrt{\tfrac{2}{k+2}}\, e(z)\otimes b(z) \\ G^-(z) &\mapsto \sqrt{\tfrac{2}{k+2}}\, f(z)\otimes c(z). \end{aligned}$$

Free field realization and Coset construction

$$\mathcal{W}^k(\mathfrak{sl}_{n+1}, f_{n,1}) \hookrightarrow V^\bullet(\mathfrak{sl}_2) \otimes \pi^{k+h^\vee}_{\mathfrak{h}^\perp} \hookrightarrow \beta\gamma \otimes \mathfrak{u}(1)^{n-1}_{k+h^\vee} \hookrightarrow \widehat{V}_{\mathbb{Z}(1+\sqrt{-1})} \otimes \pi^{k+h^\vee}_{\mathfrak{h}}$$

$$\mathcal{W}^\ell(\mathfrak{sl}_{n+1|1}, f_{n+1|1}) \hookrightarrow V^\bullet(\mathfrak{gl}_{1|1}) \otimes \pi^{\ell+h^\vee}_{\mathfrak{h}^\perp} \hookrightarrow bc \otimes \pi^{\ell+h^\vee}_{\mathfrak{h}} \simeq V_{\mathbb{Z}} \otimes \pi^{\ell+h^\vee}_{\mathfrak{h}}.$$

Schematically, we use weighted Dynkin diagrams



Theorem 1.2 (Creutzig–Genra–N)

For $(k+n+1)(\ell+n)=1$,

$$\textbf{KS}: \mathcal{W}^\ell(\mathfrak{sl}_{n|1}, f_{n+1|1}) \xrightarrow{\simeq} \mathrm{Com}\left(\pi^{\mathrm{diag}}, \mathcal{W}^k(\mathfrak{sl}_{n+1}, f_{n,1}) \otimes V_{\mathbb{Z}}\right),$$

$$\textbf{FST}: \mathcal{W}^k(\mathfrak{sl}_{n+1}, f_{n,1}) \xrightarrow{\simeq} \mathrm{Com}\left(\pi^{\mathrm{diag}}, \mathcal{W}^\ell(\mathfrak{sl}_{n+1|1}, f_{n+1|1}) \otimes V_{\sqrt{-1}\,\mathbb{Z}}\right).$$

Coset construction implies...

$$\mathcal{W}^k(\mathfrak{sl}_{n+1}, f_{n,1}) \simeq \bigoplus_{a\in\mathbb{Z}} \mathscr{C}_+^k(a) \otimes \pi_a^{h^+}, \quad \mathcal{W}^\ell(\mathfrak{sl}_{n+1|1}, f_{n+1|1}) \simeq \bigoplus_{a\in\mathbb{Z}} \mathscr{C}_-^\ell(a) \otimes \pi_a^{h^-},$$

$$\rightsquigarrow \mathscr{C}_+^k(a) \simeq \mathscr{C}_-^\ell(a) \text{ as } \mathscr{C}_+^k(0) \simeq \mathscr{C}_-^\ell(0)\text{-modules!}$$

Question 2.1

Can we interchange $\pi_a^{h^+} \leftrightsquigarrow \pi_a^{h^-}$ $(a \in \mathbb{Z})$ more directly?

- Yes! and given by the **relative semi-infintie cohomology**:

$$H_{\mathrm{rel}}^{\frac{\infty}{2}+n}\left(\mathfrak{gl}_1; \pi_a^H \otimes \pi_b^{\sqrt{-1}\,H}\right) \simeq \delta_{n,0}\delta_{a+b,0}\mathbb{C}[|a\rangle \otimes |b\rangle].$$

We set

$$K_{+\to-} := \bigoplus_{a\in\mathbb{Z}} \pi_{-a}^{\sqrt{-1}\,h^+} \otimes \pi_a^{h^-}, \quad K_{-\to+} := \bigoplus_{a\in\mathbb{Z}} \pi_{-a}^{\sqrt{-1}\,h^-} \otimes \pi_a^{h^+}.$$

$$\simeq V_{\mathbb{Z}} \otimes \pi_{\mathbb{Z}} \qquad\qquad \simeq V_{\sqrt{-1}\,\mathbb{Z}} \otimes \pi_{\sqrt{-1}\,\mathbb{Z}}$$

$$\mathcal{W}^\ell(\mathfrak{sl}_{n+1|1}, f_{n+1|1}) \simeq H_{\mathrm{rel}}^{\frac{\infty}{2}+0}\left(\mathfrak{gl}_1; \mathcal{W}^k(\mathfrak{sl}_{n+1}, f_{n,1}) \otimes V_{\mathbb{Z}} \otimes \pi_{\mathbb{Z}}\right),$$

$$\mathcal{W}^k(\mathfrak{sl}_{n+1}, f_{n,1}) \simeq H_{\mathrm{rel}}^{\frac{\infty}{2}+0}\left(\mathfrak{gl}_1; \mathcal{W}^\ell(\mathfrak{sl}_{n+1|1}, f_{n+1|1}) \otimes V_{\sqrt{-1}\,\mathbb{Z}} \otimes \pi_{\sqrt{-1}\,\mathbb{Z}}\right).$$

Correspondence of module categories

- Can be also applied to modules by

$$K^{\lambda}_{+\to-} := V_{\mathbb{Z}} \otimes \pi^{\lambda}_{\mathbb{Z}}, \quad K^{\lambda}_{-\to+} := V_{\sqrt{-1}\mathbb{Z}} \otimes \pi^{\lambda}_{\sqrt{-1}\mathbb{Z}}.$$

$$\mathcal{W}^k(\mathfrak{sl}_{n+1}, f_{n,1})\text{-mod} \supset \mathbf{KL}^k_{A^+}(n,1) = \bigoplus \mathbf{KL}^{k,\lambda}_{A^+}(n,1), \ (\lambda \in \mathbb{C}/\mathbb{Z}),$$

$$\mathcal{W}^\ell(\mathfrak{sl}_{n+1|1}, f_{n+1|1})\text{-mod} \supset \mathbf{KL}^\ell_{A^-}(n,1) = \bigoplus \mathbf{KL}^{\ell,\lambda}_{A^-}(n,1), \ (\lambda \in \mathbb{C}/\mathbb{Z}).$$

Theorem 2.2 (Creutzig–Genra–N–Sato)

$$\mathbf{KL}^{k,\lambda^+}_{A^+}(n,1) \underset{H^0_{\mathrm{rel},\lambda^-}}{\overset{H^0_{\mathrm{rel},\lambda^+}}{\rightleftarrows}} \mathbf{KL}^{\ell,\lambda^-}_{A^-}(n,1) \quad (\simeq):$$

$$\mathcal{I}_{A^+}\begin{pmatrix} & M_3 & \\ M_1 & & M_2 \end{pmatrix} \simeq \mathcal{I}_{A^-}\begin{pmatrix} & H^0_{\mathrm{rel},\lambda_3^+}(M_3) & \\ H^0_{\mathrm{rel},\lambda_1^+}(M_1) & & H^0_{\mathrm{rel},\lambda_2^+}(M_2) \end{pmatrix}$$

Rational case: Comparision of Fusion rings

Creutzig-Linshaw proved a level-rank duality

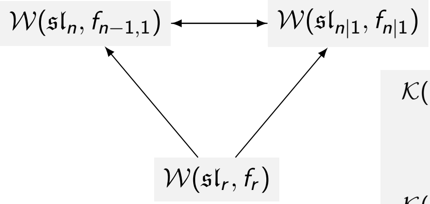
$$\mathrm{Com}\left(\pi, \mathcal{W}_{k(r)}(\mathfrak{sl}_n, f_{n-1,1})\right) \simeq \mathcal{W}_{\alpha(n)}(\mathfrak{sl}_r, f_r), \qquad \begin{array}{l} k(r) = -n + \frac{n+r}{n-1} \\ \alpha(n) = -r + \frac{r+n}{r+1} \end{array}.$$

Using $\mathcal{K}(\mathcal{W}_{\alpha(n)}(\mathfrak{sl}_r, f_r)) \simeq \mathcal{K}(L_n(\mathfrak{sl}_r))$, this implies

$$\mathcal{W}_{A^+} := \mathcal{W}_{k(r)}(\mathfrak{sl}_n, f_{n-1,1}) \simeq \bigoplus_{i \in \mathbb{Z}_r} \mathbf{L}_{\mathcal{W}}(n\varpi_i) \otimes V_{\frac{ni}{\sqrt{nr}} + \sqrt{nr}\,\mathbb{Z}},$$

$$\mathcal{W}_{A^-} := \mathcal{W}_{\ell(r)}(\mathfrak{sl}_{n|1}, f_{n|1}) \simeq \bigoplus_{i \in \mathbb{Z}_r} \mathbf{L}_{\mathcal{W}}(n\varpi_i) \otimes V_{\frac{(n+r)i}{\sqrt{(n+r)r}} + \sqrt{(n+r)r}\,\mathbb{Z}}.$$

$\Rightarrow$ General theory of simple current extension by lattice can be applied:



$$\mathcal{K}(\mathcal{W}_{A^+}) \simeq \mathcal{K}(L_r(\mathfrak{sl}_n)) \simeq \left(\mathcal{K}(\mathcal{W}_{A^-}) \underset{\mathbb{Z}[\mathbb{Z}_{n+r}]}{\otimes} \mathbb{Z}[\mathbb{Z}_{n(n+r)}]\right)^{\mathbb{Z}_{n+r}}$$

$$\mathcal{K}(\mathcal{W}_{A^-}) \simeq \left(\mathcal{K}(\mathcal{W}_{A^+}) \underset{\mathbb{Z}[\mathbb{Z}_n]}{\otimes} \mathbb{Z}[\mathbb{Z}_{n(n+r)}]\right)^{\mathbb{Z}_n}$$

Toward the general hook-type $\mathcal{W}$-superalgebras

Label	$\mathcal{W}^k_{A+}(n,m)$	$\mathcal{W}^k_{B+}(n,m)$	$\mathcal{W}^k_{C+}(n,m)$	$\mathcal{W}^k_{D+}(n,m)$	$\mathcal{W}^k_{O+}(n,m)$
$\mathfrak{g}$	$\mathfrak{sl}_{n+m}$	$\mathfrak{so}_{2(n+m+1)}$	$\mathfrak{sp}_{2(n+m)}$	$\mathfrak{so}_{2(n+m)+1}$	$\mathfrak{osp}_{1\|2(n+m)}$
$\mathfrak{a}$	$\mathfrak{sl}_n$	$\mathfrak{so}_{2n+1}$	$\mathfrak{sp}_{2n}$	$\mathfrak{so}_{2n+1}$	$\mathfrak{sp}_{2n}$
$\mathfrak{b}$	$\mathfrak{gl}_m$	$\mathfrak{so}_{2m+1}$	$\mathfrak{sp}_{2m}$	$\mathfrak{so}_{2m}$	$\mathfrak{osp}_{1\|2m}$
$k_\mathfrak{b}$	$k+n-1$	$k+2n$	$k+n-\frac{1}{2}$	$k+2n$	$k+n-\frac{1}{2}$

Label	$\mathcal{W}^k_{A-}(n,m)$	$\mathcal{W}^k_{B-}(n,m)$	$\mathcal{W}^k_{C-}(n,m)$	$\mathcal{W}^k_{D-}(n,m)$	$\mathcal{W}^k_{O-}(n,m)$
$\mathfrak{g}$	$\mathfrak{sl}_{n+m\|m}$	$\mathfrak{osp}_{2m+1\|2(n+m)}$	$\mathfrak{osp}_{2(n+m)+1\|2m}$	$\mathfrak{osp}_{2m\|2(n+m)}$	$\mathfrak{osp}_{2(n+m)+2\|2m}$
$\mathfrak{a}$	$\mathfrak{sl}_{n+m}$	$\mathfrak{sp}_{2(n+m)}$	$\mathfrak{so}_{2(n+m)+1}$	$\mathfrak{sp}_{2(n+m)}$	$\mathfrak{so}_{2(n+m)+1}$
$\mathfrak{b}$	$\mathfrak{gl}_m$	$\mathfrak{so}_{2m+1}$	$\mathfrak{sp}_{2m}$	$\mathfrak{so}_{2m}$	$\mathfrak{osp}_{1\|2m}$
$k_\mathfrak{b}$	$-(k+n+m)+1$	$-2(k+n+m)+1$	$-(\frac{1}{2}k+n+m)$	$-2(k+n+m)+1$	$-(\frac{1}{2}k+n+m)$

The hook-type $\mathcal{W}$-superalgebras have the following simple structure:

$$\mathfrak{sl}_{n+m} = (\mathfrak{sl}_n \oplus \mathfrak{gl}_m) \oplus \left(\mathbb{C}^n \otimes \overline{\mathbb{C}}^m \oplus \overline{\mathbb{C}}^n \otimes \mathbb{C}^m\right)$$

$$\begin{aligned}\mathcal{W}^k_{A^+}(n,m) &\simeq C^k_{A^+}(n,m) \otimes V^{k_\mathfrak{b}}(\mathfrak{gl}_m) \\ &\qquad \oplus C^k_{A^+}(\varpi_1) \otimes \mathbb{V}^{k_\mathfrak{b}}_{\mathfrak{gl}_m}(\mathbb{C}^m) \oplus C^k_{A^+}(\varpi_{m-1}) \otimes \mathbb{V}^{k_\mathfrak{b}}_{\mathfrak{gl}_m}(\overline{\mathbb{C}}^m) \cdots \\ &\simeq \bigoplus_{\lambda \in P_+(\mathfrak{gl}_m)} C^k_{A^+}(\lambda) \otimes \mathbb{V}^{k_\mathfrak{b}}_{\mathfrak{gl}_m}(\lambda)\end{aligned}$$

$$\mathfrak{sl}_{n+m|m} = (\mathfrak{sl}_{n+m} \oplus \mathfrak{gl}_m) \oplus \Pi \left(\mathbb{C}^{n+m} \otimes \overline{\mathbb{C}}^m \oplus \overline{\mathbb{C}}^{n+m} \otimes \mathbb{C}^m\right)$$

$$\begin{aligned}\mathcal{W}^\ell_{A-}(n,m) &\simeq C^\ell_{A-}(n,m) \otimes V^{\ell_\mathfrak{b}}(\mathfrak{gl}_m) \\ &\qquad \oplus C^\ell_{A-}(\varpi_1) \otimes \mathbb{V}^{\ell_\mathfrak{b}}_{\mathfrak{gl}_m}(\mathbb{C}^{0|m}) \oplus C^\ell_{A-}(\varpi_{m-1}) \otimes \mathbb{V}^{\ell_\mathfrak{b}}_{\mathfrak{gl}_m}(\overline{\mathbb{C}}^{0|m}) \cdots \\ &\simeq \bigoplus_{\lambda \in P_+(\mathfrak{gl}_m)} C^\ell_{A-}(\lambda) \otimes \mathbb{V}^{\ell_\mathfrak{b}}_{\mathfrak{gl}_m}(\lambda)\end{aligned}$$

By using $H_{\mathrm{rel}}^{\frac{\infty}{2}+0}$, the gluing object (kernel algebra) is

$$K^{n,m}_{A^+\to A^-} := \bigoplus_{\lambda\in P_+(\mathfrak{gl}_m)} \mathbb{V}^{-k_{\mathfrak{b}}-2h^\vee}_{\mathfrak{gl}_m}(\lambda^\dagger)\otimes \mathbb{V}^{\ell_{\mathfrak{b}}}_{\mathfrak{gl}_m}(\lambda)$$

$$\Rightarrow \mathcal{W}^{\ell}_{A^-}(n,m) \overset{?}{\simeq} H_{\mathrm{rel}}^{\frac{\infty}{2}+0}\left(\mathfrak{gl}_m; \mathcal{W}^{k}_{A^+}(n,m)\otimes K^{n,m}_{A^+\to A^-}\right).$$

It turn out that $K^{n,m}_{A^+\to A^-}$ and $K^{n,m}_{A^-\to A^+}$ have a common shape:

$$A^c[\mathfrak{gl}_m,k] := \bigoplus_{\lambda\in P_+(\mathfrak{sl}_m)} \mathbb{V}^{k_1}_{\mathfrak{sl}_m}(\lambda)\otimes \mathbb{V}^{k_2}_{\mathfrak{sl}_m}(\lambda^\dagger)\otimes V_{\frac{s(\lambda)}{\sqrt{cm}}+\sqrt{cm}\mathbb{Z}}\otimes \pi_{\sqrt{cm}\mathbb{Z}}$$

with $\frac{1}{k_1+m}+\frac{1}{k_2+m}=c$ $(c\in\mathbb{Z})$. Looking at the lowest wieght subspaces at each component, we find $\mathbb{C}[GL_m]\simeq\bigoplus_\lambda L_\lambda\otimes L_{\lambda^\dagger}$ and indeed the case $c=0$ is the chiral differential operators

$$\mathscr{D}^{\mathrm{ch}}_{GL_m,k_1} := \mathrm{Ind}^{\widehat{\mathfrak{gl}_m},k_1}_{\widehat{\mathfrak{gl}_m}[\![t]\!]\oplus\mathbb{C}K}\mathbb{C}[J_\infty GL_m].$$

For $c\neq 0$, it is related to $\mathbb{C}_q[GL_m]$ (Moriwaki). One exapmle is

$$L_1(\mathfrak{d}(2,1;-\alpha))\simeq \bigoplus \mathbb{V}^{-1+\frac{1}{\alpha}}_{\mathfrak{sl}_2}(n)\otimes\mathbb{V}^{-1+\alpha}_{\mathfrak{sl}_2}(n)\otimes V_{\frac{n}{\sqrt{2}}+\sqrt{2}\mathbb{Z}}$$

$$\simeq \bigoplus \mathbb{V}^{-1+\frac{1}{\alpha}}_{\mathfrak{osp}_{1|2}}(n)\otimes \mathbb{V}^{\frac{1}{2}\alpha}_{\mathfrak{so}_3}(2n).$$

Kernel algebras and Howe duality

Theorem 2.3 (Creutzig-Linshaw-N-Sato, work in progress)

Let $(X,Y)=(A,A),(B,O),(C,C),(D,D),(O,B)$. *For* $k,\ell\in\mathbb{C}\backslash\mathbb{Q}$ *under duality relation, we have*

$$\mathcal{W}^{\ell}_{Y^-}(n,m)\simeq H^{\frac{\infty}{2}+0}_{\mathrm{rel}}\left(\mathfrak{b},\mathcal{W}^{k}_{X^+}(n,m)\otimes A^{1}[\mathfrak{b},\alpha_+]\right),$$
$$\mathcal{W}^{k}_{Y^+}(n,m)\simeq H^{\frac{\infty}{2}+0}_{\mathrm{rel}}\left(\mathfrak{b},\mathcal{W}^{\ell}_{X^-}(n,m)\otimes A^{-1}[\mathfrak{b},\alpha_-]\right).$$

Therefore, $C^{k}_{X^+}(n,m)\simeq C^{\ell}_{Y^-}(n,m)$ *holds for all cases.* [a]

[a] Creutzig–Linshaw proved it except for $(B,O),(C,C),(O,B)$ where $\mathbb{Z}_2$-orbiforlds was taken.

For the (B,O) case, kernel algebras should be related to

$$\mathbb{C}[\mathrm{Hom}(\mathbb{C}^{2m+1},\mathbb{C}^{2m|1})]/^{\exists}\mathcal{I}\simeq\bigoplus L^{\mathfrak{so}_{2m+1}}_{s_\lambda}\otimes L^{\mathfrak{osp}_{1|2m}}_{\lambda}.$$

In general, associative algebras A which are multiplicity-free representations as $(\mathfrak{g}_1,\mathfrak{g}_2)$-bimodules serve as "gluing objects" between tensor categories for quantum groups/ vertex superalgeras. Candidates are found in (quantum) Howe duality (or its generalization).

Thank you for listening!