Intertwining Operators/Genus-0 Conformal Blocks Associated to Permutation-Twisted Modules of $V^{\otimes n}$

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Why should we care about permutation orbifold CFT?

- We have a nice twisted-untwisted correspondence for VOA/conformal net modules

(Barron-Dong-Mason, Kac-Longo-Xu, Dong-Xu-Yu, etc.)

- We are able to compute fusion rings/rules among twisted modules using untwisted data.

(*Conformal net*: Kawahigashi-Longo-Mueger, Longo-Xu, Kac-Long-Xu
VOA: Dong-Li-Xu-Yu
Modular functor: Barmeier-Schweigert
Tensor category: EdieMichell-C.Jones-Plavnik, Bischoff-C.Jones, Delaney
etc.)

Explicit modules, but fusion rules are not completely determined

Fusion rings are completely characterized, but objects are "abstract".

The computation of fusion rules suggests a relation:

Genus 0 permutation-twisted chiral CFT

↕

Higher genus untwisted chiral CFT

Goal: Make the above relation precise (and complete) in the VOA setting

- We assume V is a VOA with positive L_0 -grading
- V-modules: $\mathbb{N}$ -gradable (i.e. admissible) modules, where each graded subspace is finite-dimensional.
- To define conformal blocks for untwisted modules, we need data

$$\mathfrak{X} = \{C; x_1, \ldots, x_N; \eta_1, \ldots, \eta_N\}$$

where C is a (not necessarily connected) compact Riemann surface with distinct marked points x_j and holomorphic injective

$$\eta_j : \text{a neighborhood of } x_j \to \mathbb{C}, \qquad \eta_j(x_j) = 0$$

- Associate a V-module W_j to x_j
- A conformal block (CB) associated to $\mathfrak{X}$ and all the modules W_j is a linear functional $\varphi : W_1 \otimes \cdots \otimes W_N \to \mathbb{C}$ "invariant" under the action of V. (Zhu, E.Frenkel-BenZvi)
- If $\mathfrak{X}$ is $\mathbb{P}^1$ with 3 marked points associated with modules W_1, W_2, W_3' then

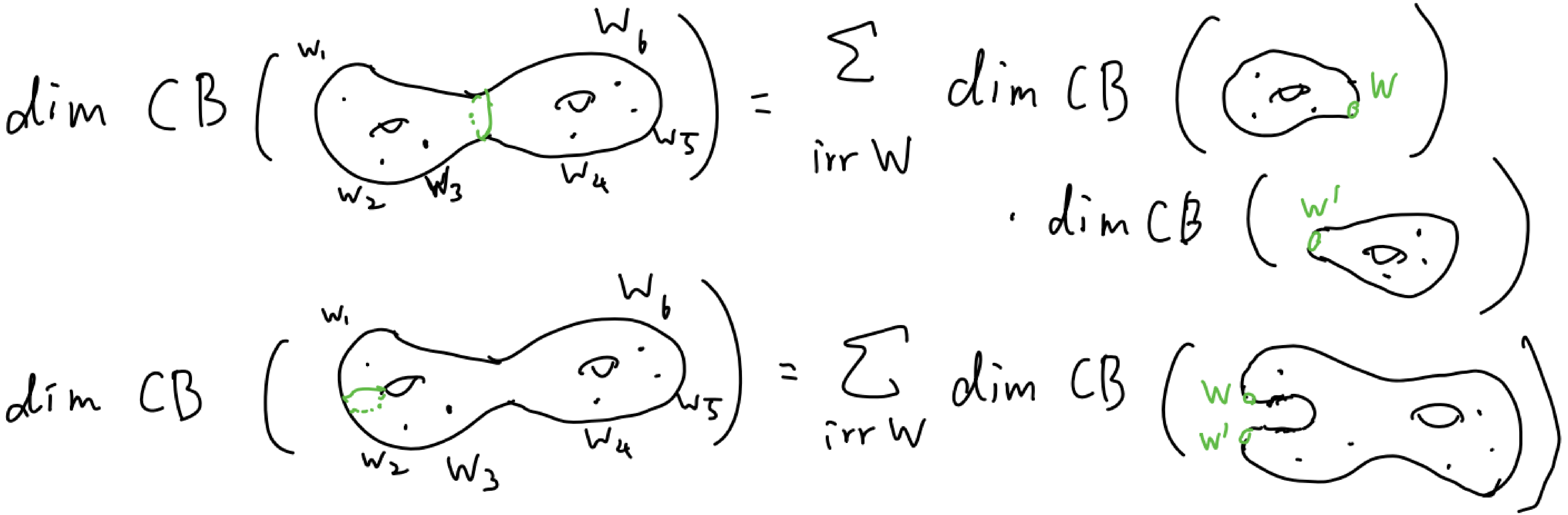
$$\dim\{\text{CB}\} = \text{fusion rule } N_{W_1 W_2}^{W_3}$$

Contragredient to W_3

Theorem (many people, completed by Damiolini-Gibney-Tarasca):
Assume V is CFT-type, C_2-cofinite, rational.

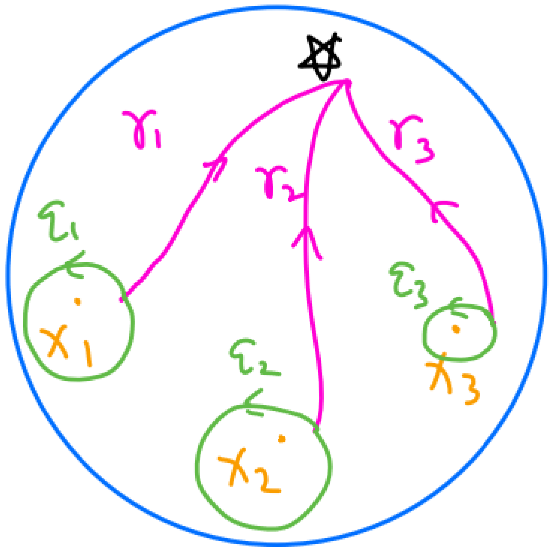
- $\dim\{\text{CB}\}$ is finite and depends only the topology of C, the number of marked points on each connected component, and the modules.
- Factorization property.

Factorization property



Genus-0 twisted conformal blocks

- Let U be a VOA, a finite group $G \le \mathrm{Aut}(U)$. If $g \in G$, a g-twisted module is assumed to satisfy $Y(u, e^{-2i\pi} z) = Y(gu, z)$ where the arg of $e^{-2i\pi} z$ is $-2\pi +$ the arg of z
- We consider $\mathfrak{P} = (\mathbb{P}^1; x_1, x_2, x_3; \eta_1, \eta_2, \eta_3; \gamma_1, \gamma_2, \gamma_3)$



Let $\alpha_j = \gamma_j^{-1} \varepsilon_j \gamma_j$, in the picture, $[\alpha_1], [\alpha_2]$ are free generators of $\Gamma = \pi_1(\mathbb{P}^1 \setminus \{x_1, x_2, x_3\}, \star) \simeq F_2$, and $[\alpha_3]^{-1} = [\alpha_1][\alpha_2]$. Then we associate g_j- twisted module $\mathcal{W}_j$ to x_j (for j=1,2,3) such that $g_3^{-1} = g_1 g_2$.

A CB associated to $\mathfrak{P}$ and $\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3$ is a linear functional $\varphi : \mathcal{W}_1 \otimes \mathcal{W}_2 \otimes \mathcal{W}_3 \to \mathbb{C}$ "invariant under the action of V when moving along $\gamma_1, \gamma_2, \gamma_3$ ".

Now let $E=\{1,2,\ldots,n\}$, let $U=V^{\otimes E}\simeq V^{\otimes n}$, let $G=\mathrm{Perm}(E)$. For each $g\in G$ define $\mathrm{Orb}(g)=\{\text{the set of } g\text{-orbits of } E\}$. Then $\mathcal{W}=\bigotimes_{\omega\in\mathrm{Orb}(g)} W_\omega$ (where each W_ω is a V-module) has a natural structure of g-twisted U-module (by Barron-Dong-Mason).

For j=1,2,3, associate $\mathcal{W}_j=\bigotimes_{\omega\in\mathrm{Orb}(g_j)} W_{j,\omega}$ to the marked point x_j. Each $W_{j,\omega}$ is a V-module. Consider a branched covering $\varphi: C\to\mathbb{P}^1$ which is unbranched outside the finite set $\varphi^{-1}\{x_1,x_2,x_3\}$ and, near each point of this set it looks like $z\mapsto z^k$.

Examples of branched coverings:

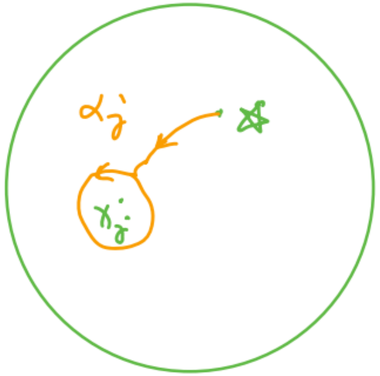
(a) $\varphi:\mathbb{P}^1\xrightarrow{z^2}\mathbb{P}^1$ with branched points $0,\infty$

(b) elliptic curve $w^2=z(z-a)(z-b)\xrightarrow{z}\mathbb{P}^1$ with branched points $0,a,b,\infty$

Describe $\varphi : C \to \mathbb{P}^1$

- We have 1-1 correspondence $\langle g_1, g_2, g_3 \rangle$-orbit Ω of E $\updownarrow$ connected component C_Ω of C

- Define an action $\Gamma \curvearrowright E$ sending $[\alpha_j] \mapsto g_j$. The restriction $\Gamma \curvearrowright \Omega$ is transitive, which is equivalent to a coset action $\Gamma \curvearrowright \Gamma / \Gamma_\Omega$ for a cofinite subgroup $\Gamma_\Omega \le \Gamma$.



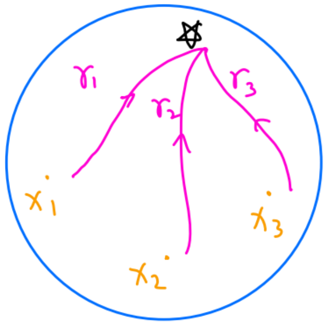
Then $\Gamma_\Omega \le \Gamma$ corresponds to $\varphi : C_\Omega \setminus \varphi^{-1}\{x_1, x_2, x_3\} \to \mathbb{P}^1 \setminus \{x_1, x_2, x_3\}$ via the "(cofinite) subgroup $\leftrightarrow$ (finite) covering space" correspondence.

- Moreover, we have 1-1 correspondence $\mathrm{Orb}(g_j)$ $\updownarrow$ $\varphi^{-1}(x_j)$; ω $\updownarrow$ $\tilde{x}_{j,\omega}$ $\leftarrow$ associate $W_{j,\omega}$

- Near $\tilde{x}_{j,\omega}$, φ is equivalent to $z \mapsto z^{|\omega|}$ where $|\omega|$ is the size of ω.

Theorem (G.) A linear functional $\varphi : \mathcal{W}_1 \otimes \mathcal{W}_2 \otimes \mathcal{W}_3 = \bigotimes_{j=1,2,3} \bigotimes_{\omega \in \mathrm{Orb}(g_j)} W_{j,\omega} \to \mathbb{C}$

is a CB associated to



and $\mathcal{W}_1, \mathcal{W}_2, \mathcal{W}_3$ iff it is a CB associated to the branched covering C, the set of marked points

$$\varphi^{-1}\{x_1, x_2, x_3\} = \{\tilde{x}_{j,\omega} : j = 1, 2, 3, \omega \in \mathrm{Orb}(g_j)\}$$

with suitable local coordinates, and the associated V-modules $W_{j,\omega}$.

C_2-cofinite or rational is not assumed in this theorem

Note: If C_Ω corresponds to the $\langle g_1, g_2, g_3 \rangle$ - orbit Ω with size $|\Omega|$, then by Riemann-Hurwitz formula,

$$\mathrm{genus}(C_\Omega) = 1 - |\Omega| + \frac{1}{2} \sum_{j=1,2,3} \sum_{\substack{\omega \in \mathrm{Orb}(g_j) \\ \omega \subset \Omega}} (|\omega| - 1)$$

Applications and outlook

- The currently existing VOA/Conforma Net correspondences (Carpi-Kawahigashi-Longo-Weiner, Henriques-Tener, Raymond-Tanimoto-Tener ...) are genus-0 by nature. Now we know that doing such genus-0 correspondence for permutation-twisted CFT amounts to establishing higher genus correspondence for untwisted CFT.
- We have a new explanation of why multi-interval Jones-Wassermann subfactors/muti-interval Connes fusion are related to higher genus CFT. (Asked e.g. by Wassermann in Proceedings ICM 1994.)
- Problem: In VOA, understand genus-1 (or higher genus) data and phenomena (e.g. modular invariance, mapping class group rep.) from the point of view of genus-0 permutation orbifolds (e.g. their G-crossed braided fusion categories). And vice versa!
- Problem: For a completely rational conformal net $\mathcal{A}$ with g_1-, g_2-, g_1g_2-permutation twisted modules $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$(or possibly with more modules), construct an explicit isomorphism between $\mathrm{Hom}_{\mathcal{A}}(\mathcal{H}_1 \boxtimes \mathcal{H}_2, \mathcal{H}_3)$ and a space of conformal blocks for untwisted modules (in the sense of Bartels-Douglas-Henriques).

Thank you!