A revisit to the affine Bernstein theorem

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Part I. Introduction

Affine maximal surface

Let u be a locally uniformly convex function on $\Omega \subset \mathbb{R}^n$.

- The graph of u defines a hypersurface M in $\mathbb{R}^{n+1}$.
- The *affine metric (Blaschke metric)* g, i.e.,

$$g_{ij} = [\det D^2 u]^{-\frac{1}{n+2}} u_{ij}$$

  gives an affine invariant metric on M.
- The *affine area*

$$A(u) = \int_\Omega \sqrt{\det g_{ij}}\, dx = \int_\Omega [\det D^2 u]^{\frac{1}{n+2}} = \int_M K^{\frac{1}{n+2}} dV_M,$$

  where $K = \frac{\det D^2 u}{(1+|Du|^2)^{\frac{n+2}{2}}}$ is the Gauss curvature.
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Conjecture(S.S. Chern, 1977). An Euclidean complete, affine maximal, locally uniformly convex C^2 hypersurface in $\mathbb{R}^2$ must be an elliptic paraboloid.

- It is proved by Trudinger-Wang in 2000 known as affine Bernstein theorem.
- The Euler-Lagrange equation

$$H[u] := U^{ij} w_{ij} = 0 \tag{1}$$

  on $\mathbb{R}^n$, where (U^{ij}) is the cofactor matrix of the Hessian matrix $D^2 u$, and $w = [\det D^2 u]^{-\frac{n+1}{n+2}}$.
- $H[u]$ is the *affine mean curvature*.
- The conjecture says in $\mathbb{R}^2$, any entire solution to (1) must be quadratic.

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Calabi conjecture

Theorem(E. Calabi, 1982). In dimension two, when the affine metric of the graph is complete, then any entire solution to (1) must be quadratic.

- The completeness represent fairly strong restrictions on the asymptotic behavior of the second derivatives.
- Euclidean complete hypersurfaces are not generally affine complete.
- Trudinger-Wang(2002): Affine completeness implies Euclidean completeness. ($\Rightarrow$ (A different proof to affine Bernstein theorem)
- Open in higher dimensions!

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A (singular) counterexample in higher dimensions

Trudinger-Wang(2000): when $n = 10$,

$$u = \sqrt{|x'|^9 + x_{10}^2} \in W^{2,1}_{loc}(\mathbb{R}^{10}) \cup C^\infty(\mathbb{R}^{10} \setminus \{0\}),$$

where $x' = (x_1, \cdots, x_9)$, is affine maximal.

- There is still no smooth counterexamples!
- The critical dimension is unknown!

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The Monge-Ampère typed fourth order equations

u is a uniformly convex function on $\Omega \subset \mathbb{R}^n$. We study the equation

$$U^{ij} w_{ij} = f \tag{2}$$

on $\mathbb{R}^n$, where (U^{ij}) is the cofactor matrix of the Hessian matrix $D^2 u$, and

$$w = \begin{cases} [\det D^2 u]^{-(1-\theta)}, & \theta \geq 0, \quad \theta \neq 1, \\ \log \det D^2 u, & \theta = 1. \end{cases}$$

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The Monge-Ampère typed fourth order equations

It is the Euler-Lagrange equation of the Monge-Ampère typed functional

$$\mathcal{F}_\theta(u) = A_\theta(u) - \int_\Omega fu\,dx,$$

where

$$A_\theta(u) = \begin{cases} \int_\Omega [\det D^2 u]^\theta\,dx, & \theta > 0, \quad \theta \neq 1, \\ \int_\Omega \log\det D^2 u\,dx, & \theta = 0, \\ \int_\Omega \det D^2 u \log\det D^2 u\,dx, & \theta = 1. \end{cases}$$

The case of $\theta = 0$ (Abreu's equation)

Abreu's equation

$$\sum_{i,j} \frac{\partial^2 u^{ij}}{\partial x_i \partial x_j} = f,$$

where (u^{ij}) is the inverse matrix of $D^2 u$.

- By computation,

$$\sum_{i,j} \frac{\partial^2 u^{ij}}{\partial x_i \partial x_j} = U^{ij} w_{ij}, \quad w = [\det D^2 u]^{-1}.$$

- Abreu's equation is related to the scalar curvature problem on toric Kähler manifolds.
- The Bernstein theorem means: if $(\mathbb{C}^n, g_u)$ is $(S^1)^n$-invariant and scalar flat, then it is flat.

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Bernstein typed theorems

Theorem(Trudinger-Wang, Jia-Li, Zhou). Let u be an entire convex solution to

$$U^{ij}[(\det D^2 u)^{-(1-\theta)}]_{ij} = 0$$

on $\mathbb{R}^2$. If $0 \le \theta \le \frac{1}{4}$ or $\theta > 1$, u is a quadratic polynomial.

- The case of $\theta = \frac{1}{4}$ solves Chern conjecture.
- $\theta > 1$: Trudinger-Wang(JPDE, 2002).
- $\theta = 1$: $u = e^{x_1} + x_2^2$ is a counterexample.
- $\frac{1}{4} < \theta < 1$: open.
- Open in higher dimensions.

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Part II. Proof of the affine Bernstein theorem(assuming the interior estimstes)

Observation 1

The Monge-Ampère typed equation can be written as a system of two equations for u and w

$$\begin{cases} \det D^2 u = w^{-\frac{1}{1-\theta}} & \text{(Monge-Ampère equation)} \\ U^{ij} D_{ij} w = f & \text{(Linearized Monge-Ampère equation)} \end{cases}$$

Observation 2

- Jorgens-Calabi-Pogorelov: Suppose u is a uniformly convex solution to

$$\det D^2 u = 1 \quad \text{in} \quad \mathbb{R}^n.$$

Then u is a quadratic polynomial.

- Bernstein-Hopf-Mickle: Suppose u is a smooth solution to

$$\sum_{i,j=1}^{2} a_{ij}(x) u_{ij}(x) = 0 \quad \text{in} \quad \mathbb{R}^2, \quad a_{ij} > 0.$$

If $|u(x)| = o(|x|)$ as $|x| \to \infty$, then u is a constant.

- To prove Bernstein theorem, it suffices to show

$$0 < C^{-1} \leq \det D^2 u \leq C, \; x \in \mathbb{R}^2.$$

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Interior estimates

Theorem. Assume $0 \leq \theta \leq \frac{1}{4}$. Let u be a uniformly convex solution to (2) on Ω. Suppose that Ω and u are normalized. Then for any $\Omega' \Subset \Omega$, $0 < \alpha < 1$,

$$\|u\|_{C^{4,\alpha}(\Omega')} \leq C,$$

where C depends on θ, α and $dist(\Omega', \Omega)$.

Assume the a priori estimates hold, we first prove the Bernstein Theorem.

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Let u be a convex function on $\Omega \subset \mathbb{R}^n$. The section centered at $x \in \Omega$ with height $h > 0$

$$S_{h,u}(x) := \{y \in \Omega : u(y) \leq l_x(y) + h\},$$

where $l_x(y) = u(x) + Du(x)(y - x)$ is a support function of u at x.

Lemma(Caffarelli). For any x_0 and $h > 0$, there exists $x \in \mathbb{R}^n$, such that x_0 is the center mass of $S_{h,u}(x)$.

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Preliminaries: Normalization

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and u be a convex function on Ω.

- Ω is normalized if
$$B_{\frac{1}{n}}(x_0) \subset \Omega \subset B_1(x_0),$$
where x_0 is the center of mass of Ω.
- u is normalized on Ω if
$$u|_{\partial\Omega} = 0, \quad \inf_{\Omega} u = -1.$$
- For any Ω, there exists a dilation T with respect to its center of mass, such that $T(\Omega)$ is normalized.

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Proof of the Bernstein Theorem

Assume $u(0) = \inf u = 0$.

Step 1 For any $h > 0 (h \to +\infty)$,

- there is $x_h \in \mathbb{R}^n$ such that 0 is the centre of mass of $S_{h,u}(x_h)$;
- there is a dilation T_h, such that $\Omega_h := T_h(S_{h,u}(x_h))$ is normalized;
- $$u_h(y) = \frac{u(x) - u(x_h) - Du(x_h)(x - x_h)}{h}, \quad y = T_h(x) \in \Omega_h.$$

Then u_h solves (2) in Ω_h and is normalized, i.e.,

$$\inf_{\Omega_h} u_h = -1, \quad u_h|_{\partial\Omega_h} = 0.$$

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$$\inf_{\Omega_h} u_h = -1, \quad u_h|_{\partial \Omega_h} = 0.$$

Step 2 By the interior estimate, we have

$$\|u_h\|_{C^3(B_{1/2n}(0))} \leq C,$$

where C is independent of h. It implies

$$0 < C^{-1} \leq \det D^2 u_h \leq C \ \text{ in } \ B_{1/2n}(0)$$

and

$$C_1|y|^2 \leq u_h(y) - Du_h(0)y - u_h(0) \leq C_2|y|^2.$$

Note that Ω_h is normalized and

$$\det D_x^2 u = (\det T_h)^2 \cdot h^n \cdot \det D_y^2 u_h.$$

Step 2 By the interior estimate, we have

$$\|u_h\|_{C^3(B_{1/2n}(0))} \le C,$$

where C is independent of h. It implies

$$0 < C^{-1} \le \det D^2 u_h \le C \ \text{ in } \ B_{1/2n}(0)$$

and

$$C_1|y|^2 \le u_h(y) - Du_h(0)y - u_h(0) \le C_2|y|^2.$$

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Step 3 We claim

$$Ch^{-\frac{n}{2}} \le \det T_h \le Ch^{-\frac{n}{2}}.$$

Proof. Change y back to x by $y = T_h(x)$, we have

$$C_1|T_h x|^2 \le \frac{u(x)}{h} \le C_2|T_h x|^2.$$

Let Λ_h, λ_h be the max, min-eigenvalue of T_h. Let $|x| = 1$.

$$C_1\Lambda_h^2 \le \frac{\sup_{|x|=1} u}{h}, \quad C_2\lambda_h^2 \ge \frac{\inf_{|x|=1} u}{h}.$$

i.e.,

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The global determinant estimates follows by letting $h \to \infty$.

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Part III. Interior estimates

Recall: regularity theory of the Linearized Monge-Ampère equation

Theorem(Caffarelli-Gutiérrez, 97'). Assume w is a solution to

$$U^{ij} w_{ij} = f \text{ in } \Omega.$$

If $0 < \Lambda^{-1} \leq \det D^2 u \leq \Lambda$, then

$$\|w\|_{C^{\alpha}(\Omega')} \leq C(\|f\|_{L^{\infty}}, d(\Omega', \partial\Omega), \Lambda), \quad \forall \Omega' \Subset \Omega.$$

- Boundary and higher regularity by Le-Savin, Le-Nguyen, Gutierrez-Nguyen, etc.

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Bootstrapping arguments for interior regularity

$$\begin{cases} \det D^2 u = w^{-\frac{1}{1-\theta}} & \text{(Monge-Ampère equation)} \\ U^{ij} D_{ij} w = f & \text{(Linearized Monge-Ampère equation)} \end{cases}$$

$0 < C_1 \leq \det D^2 u \leq C_2$ (with modulus of convexity estimates)

$\Longrightarrow \|\det D^2 u\|_{C^\alpha} \leq C$ (C^α of the LMA, Cafferelli-Gutierrez)

$\Longrightarrow \|u\|_{C^{2,\alpha}} \leq C$ ($C^{2,\alpha}$ of the MA)

$\Longrightarrow \|\det D^2 u\|_{C^{2,\alpha}} \leq C$ (Schauder estimate)

$\Longrightarrow \|u\|_{C^{4,\alpha}} \leq C \cdots\cdots\cdots$

I will present new proof (jointly with Ling Wang)
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Partial Legendre transform in two dimensions

We consider $n = 2$ and write $u(x) = u(x_1, x_2)$. The partial Legendre transform in the x_1-variable is

$$u^\star(\xi, \eta) = x_1 u_{x_1}(x_1, x_2) - u(x_1, x_2), \tag{3}$$

where

$$(\xi, \eta) = \mathcal{P}(x_1, x_2) := (u_{x_1}, x_2) \in \mathcal{P}(\Omega) := \Omega^\star.$$

- The partial Legendre transform is widely used in Monge-Ampère equations.
  $(\det D^2 u = f(x_1, x_2) \Longrightarrow f(u^\star_\xi, \eta) u^\star_{\xi\xi} + u^*_{\eta\eta} = 0)$
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Partial Legendre transform in two dimensions

By computation, we have

$$\frac{\partial(\xi,\eta)}{\partial(x_1,x_2)} = \begin{pmatrix} u_{x_1x_1} & u_{x_1x_2} \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad \frac{\partial(x_1,x_2)}{\partial(\xi,\eta)} = \begin{pmatrix} \frac{1}{u_{x_1x_1}} & -\frac{u_{x_1x_2}}{u_{x_1x_1}} \\ 0 & 1 \end{pmatrix}.$$

Hence,

$$u^\star_\xi = x_1, \; u^\star_\eta = -u_{x_2},$$
$$u^\star_{\xi\xi} = \frac{1}{u_{x_1x_1}}, \; u^\star_{\eta\eta} = -\frac{\det D^2 u}{u_{x_1x_1}}, \; u^\star_{\xi\eta} = -\frac{u_{x_1x_2}}{u_{x_1x_1}}.$$

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The equation under partial Legendre transform

Proposition. Let u be a uniformly convex solution to (2) in Ω. Then in $\Omega^\star = \mathcal{P}(\Omega)$, its partial Legendre transform $u^\star$ satisfies

$$w^\star w^\star_{\xi\xi} + w^\star_{\eta\eta} + (\theta - 1) {w^\star_\xi}^2 + \frac{\theta - 2}{w^\star} {w^\star_\eta}^2 = 0, \tag{4}$$

Here $w^\star = -\frac{u^\star_{\eta\eta}}{u^\star_{\xi\xi}} (= \det D^2 u)$.

Proof

In order to derive the equation under partial Legendre transform, we consider the associated functionals of (2)

$$A_\theta(u) = \begin{cases} \int_\Omega [\det D^2 u]^\theta \, dx, & \theta > 0, \quad \theta \neq 1, \\ \int_\Omega \log \det D^2 u \, dx, & \theta = 0, \\ \int_\Omega \det D^2 u \log \det D^2 u \, dx, & \theta = 1. \end{cases}$$

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By

$$\det D^2 u = -\frac{u^\star_{\eta\eta}}{u^\star_{\xi\xi}}, \quad dxdy = u^\star_{\xi\xi}\, d\xi d\eta,$$

we have

$$\begin{aligned}
A_\theta(u) &= \int_{\Omega^\star} \left(-\frac{u^\star_{\eta\eta}}{u^\star_{\xi\xi}}\right)^\theta u^\star_{\xi\xi}\, d\xi d\eta \\
&= \int_{\Omega^\star} (-u^\star_{\eta\eta})^\theta {u^\star_{\xi\xi}}^{1-\theta}\, d\xi d\eta := A^\star_\theta(u^\star),\ \theta \in (0,1); \\
A_0(u) &= \int_{\Omega^\star} \log\left(-\frac{u^\star_{\eta\eta}}{u^\star_{\xi\xi}}\right) u^\star_{\xi\xi}\, d\xi d\eta := A^\star_0(u^\star); \\
A_1(u) &= \int_{\Omega^\star} \left(-\frac{u^\star_{\eta\eta}}{u^\star_{\xi\xi}}\right) \log\left(-\frac{u^\star_{\eta\eta}}{u^\star_{\xi\xi}}\right) u^\star_{\xi\xi}\, d\xi d\eta := A^\star_1(u^\star).
\end{aligned}$$

It suffices to derive the Euler-Lagrange equation of $A^\star_\theta$.

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The key estimate

For simplicity, we change notations and write new equation as

$$uu_{xx} + u_{yy} = (1-\theta)u_x^2 + \frac{2-\theta}{u}u_y^2. \tag{5}$$

We have the following interior gradient estimate

Theorem. Assume u is a solution to (5) on $B_R(0)$ and satisfies $0 < \lambda \le u \le \Lambda$. Then there exists $\alpha, C > 0$ depending on λ, Λ, R and θ, such that

$$\int_{B_R(0)} |\nabla u|^3 (R^2 - x^2 - y^2)^\alpha \, dV \le C. \tag{6}$$

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Proof

Let $z = v\phi(u)\eta(x, y)$, where

$$\begin{aligned} v &= \sqrt{u_x^2 + u_y^2 + 1}, \\ \eta &= (R^2 - x^2 - y^2)^\alpha, \ \alpha > 3, \\ \phi(u) &= Au^{\theta-2} - \frac{u}{2\theta^2 - 9\theta + 9}, \ A \geq \frac{\Lambda^{3-\theta}}{2\theta^2 - 9\theta + 9} + 1. \end{aligned}$$

Compute $uz_{xx} + z_{yy}$, integration by parts, $\cdots\cdots$.

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The interior estimate of the fourth order equation

Theorem. Assume $n = 2$ and $\theta \in [0, 1]$. Let $\Omega \subset \mathbb{R}^2$ be a convex domain and u be a uniformly convex solution to equation (2) on Ω satisfying

$$0 < \lambda < \det D^2 u \leq \Lambda.$$

Then for any $\Omega' \Subset \Omega$, there exists a constant $C > 0$ depending on $\sup_\Omega |u|$, λ, Λ and $dist(\Omega', \partial\Omega)$, such that

$$\|u\|_{C^{4,\alpha}(\Omega')} \leq C.$$

The modulus of convexity

For a convex function on $\mathbb{R}^n$, the *modulus of convexity* of u

$$m_u(t) = \inf\{u(x) - \ell_z(x) : |x - z| > t\},$$

where $t > 0$ and ℓ_z is the supporting function of u at z.

- For a strictly convex function, $m_u > 0$.
- Heinz: in two dimensions, when $\det D^2 u \geq \lambda > 0$, there exists $C > 0$ depending on λ such that $m_u \geq C > 0$.

For the partial Legendre transform $(\xi, \eta) = \mathcal{P}(x, y) = (u_x, y)$,

Lemma(J. K. Liu). There exists a constant $\delta > 0$ depending on m_u, such that $B_\delta(0) \subset \mathcal{P}(B_R)$.

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Proof of interior estimates

For $p \in \Omega$, let $R = \frac{dist(p,\partial\Omega)}{2}$. W. L. O. G, we assume $\mathcal{P}(p) = 0$.

- $\sup_{B_R(p)} |Du| \leq C$ for $C > 0$ depending on R and $\sup_\Omega |u|$.
- By Liu's Lemma, there exists $\delta > 0$, s.t. $B_\delta(0) \subset \mathcal{P}(B_R(p))$.
- By the interior gradient estimate(Key lemma) of the new equation

  $$w^\star w^\star_{\xi\xi} + w^\star_{\eta\eta} + (\theta - 1) w^{\star 2}_\xi + \frac{\theta - 2}{w^\star} w^{\star 2}_\eta = 0,$$

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  $$\|w^\star\|_{W^{1,3}(B_{\frac{3\delta}{4}}(0))} \leq C.$$

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- $\sup_{B_R(p)} |Du| \leq C$ for $C > 0$ depending on R and $\sup_\Omega |u|$.
- By Liu's Lemma, there exists $\delta > 0$, s.t. $B_\delta(0) \subset \mathcal{P}(B_R(p))$.
- By the interior gradient estimate(Key lemma) of the new equation

$$w^\star w^\star_{\xi\xi} + w^\star_{\eta\eta} + (\theta - 1) w^{\star 2}_\xi + \frac{\theta - 2}{w^\star} w^{\star 2}_\eta = 0,$$

we have

$$\|w^\star\|_{W^{1,3}(B_{\frac{3\delta}{4}}(0))} \leq C.$$

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Proof of interior estimates

- Note that $n = 2$. By Sobolev theorem, we have the C^α estimate of $w^\star$.
- By the interior $W^{2,p}$-estimate of the new equation, we have $\|w^\star\|_{W^{2,\frac{3}{2}}(B_{\frac{\delta}{2}}(0))} \leq C$, which implies $W^{1,6}$ estimate of $\nabla w^\star$.
- Repeating this arguments, we have all the higher order estimates.
- Transforming back by partial Legendre transform, we obtain the interior estimate of u.

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Thank you for your attention!