On the framework of L_p summations for functions

Sudan Xing

sxing@ualberta.ca

UNIVERSITY OF ALBERTA

Banff International Research Station for Mathematical Innovation and Discovery

数 学 高 等 研 究 院 INSTITUTE FOR ADVANCED STUDY IN MATHEMATICS

(Joint with Michael Roysdon)

Department of Mathematical and Statistical Sciences

University of Alberta

October 19, 2021

Table of contents

Convex bodies background

- $\mathscr{K}^n_{(o)}$: convex bodies (compact convex sets) with the origin o in their interiors.

Convex bodies background

- $\mathscr{K}^n_{(o)}$: convex bodies (compact convex sets) with the origin o in their interiors.
- Minkowski sum: $K, L \in \mathscr{K}^n_{(o)}$, $\alpha, \beta \geq 0$,

$$\alpha K + \beta L = \{\alpha x + \beta y \colon x \in K, y \in L\}.$$

Convex bodies background

- $\mathscr{K}^n_{(o)}$: convex bodies (compact convex sets) with the origin o in their interiors.
- Minkowski sum: $K, L \in \mathscr{K}^n_{(o)}$, $\alpha, \beta \geq 0$,

$$\alpha K + \beta L = \{\alpha x + \beta y \colon x \in K, y \in L\}.$$

- For $K \in \mathscr{K}^n_{(o)}$:

Convex bodies background

- $\mathscr{K}^n_{(o)}$: convex bodies (compact convex sets) with the origin o in their interiors.
- Minkowski sum: $K, L \in \mathscr{K}^n_{(o)}$, $\alpha, \beta \geq 0$,

$$\alpha K + \beta L = \{\alpha x + \beta y : x \in K, y \in L\}.$$

- For $K \in \mathscr{K}^n_{(o)}$:
  - Support function $h_K : S^{n-1} \to \mathbb{R}$,

$$h_K(u) = \max_{x \in K} \langle x, u \rangle \quad \text{for each} \quad u \in S^{n-1}.$$

Convex bodies background

- $\mathscr{K}^n_{(o)}$: convex bodies (compact convex sets) with the origin o in their interiors.
- Minkowski sum: $K, L \in \mathscr{K}^n_{(o)}$, $\alpha, \beta \geq 0$,

$$\alpha K + \beta L = \{\alpha x + \beta y \colon x \in K, y \in L\}.$$

- For $K \in \mathscr{K}^n_{(o)}$:
  - Support function $h_K : S^{n-1} \to \mathbb{R}$,

$$h_K(u) = \max_{x \in K} \langle x, u \rangle \quad \text{for each} \quad u \in S^{n-1}.$$

  - Convex indicator functions $I_K(x) : \mathbb{R}^n \to \mathbb{R}$,

$$I_K(x) = \begin{cases} 0, & \text{if } x \in K, \\ +\infty, & \text{if } x \notin K. \end{cases}$$

Convex bodies background

- $\mathscr{K}^n_{(o)}$: convex bodies (compact convex sets) with the origin o in their interiors.
- Minkowski sum: $K, L \in \mathscr{K}^n_{(o)}$, $\alpha, \beta \geq 0$,

$$\alpha K + \beta L = \{\alpha x + \beta y \colon x \in K, y \in L\}.$$

- For $K \in \mathscr{K}^n_{(o)}$:
  - Support function $h_K : S^{n-1} \to \mathbb{R}$,

$$h_K(u) = \max_{x \in K} \langle x, u \rangle \quad \text{for each} \quad u \in S^{n-1}.$$

  - Convex indicator functions $I_K(x) : \mathbb{R}^n \to \mathbb{R}$,

$$I_K(x) = \begin{cases} 0, & \text{if } x \in K, \\ +\infty, & \text{if } x \notin K. \end{cases}$$

  - Characteristic function $\chi_K(x) : \mathbb{R}^n \to \mathbb{R}$,

$$\chi_K(x) = \begin{cases} 1, & \text{if } x \in K, \\ 0, & \text{if } x \notin K. \end{cases}$$

L_p Minkowski combination

- For $K, L \in \mathscr{K}^n_{(o)}$, $p \geq 1$ and $\alpha, \beta \geq 0$, the L_p Minkowski combination:

$$h_{\alpha \cdot_p K +_p \beta \cdot_p L}(u) = (\alpha h_K(u)^p + \beta h_L(u)^p)^{\frac{1}{p}} .$$

L_p Minkowski combination

✦ For $K, L \in \mathscr{K}^n_{(o)}$, $p \geq 1$ and $\alpha, \beta \geq 0$, the L_p Minkowski combination:

$$h_{\alpha \cdot_p K +_p \beta \cdot_p L}(u) = \left(\alpha h_K(u)^p + \beta h_L(u)^p\right)^{\frac{1}{p}}.$$

★ $p = 1$: the classical Minkowski sum.

L_p Minkowski combination

- For $K, L \in \mathscr{K}^n_{(o)}$, $p \geq 1$ and $\alpha, \beta \geq 0$, the L_p Minkowski combination:

$$h_{\alpha \cdot_p K +_p \beta \cdot_p L}(u) = \left(\alpha h_K(u)^p + \beta h_L(u)^p\right)^{\frac{1}{p}}.$$

  - $p = 1$: the classical Minkowski sum.

- The L_p-Brunn-Minkowski inequality when $p \geq 1$: given $K, L \in \mathscr{K}^n_{(o)}$ and $\alpha, \beta \geq 0$,

$$\mathrm{vol}_n(\alpha \cdot_p K +_p \beta \cdot_p L)^{\frac{p}{n}} \geq \alpha \mathrm{vol}_n(K)^{\frac{p}{n}} + \beta \mathrm{vol}_n(L)^{\frac{p}{n}},$$

  where vol_n denotes the volume (or Lebesgue measure) on $\mathbb{R}^n$.

L_p Minkowski combination

- For $K, L \in \mathscr{K}^n_{(o)}$, $p \geq 1$ and $\alpha, \beta \geq 0$, the L_p Minkowski combination:

$$h_{\alpha \cdot_p K +_p \beta \cdot_p L}(u) = \left(\alpha h_K(u)^p + \beta h_L(u)^p\right)^{\frac{1}{p}} .$$

  - $p = 1$: the classical Minkowski sum.

- The L_p-Brunn-Minkowski inequality when $p \geq 1$: given $K, L \in \mathscr{K}^n_{(o)}$ and $\alpha, \beta \geq 0$,

$$\mathrm{vol}_n(\alpha \cdot_p K +_p \beta \cdot_p L)^{\frac{p}{n}} \geq \alpha \mathrm{vol}_n(K)^{\frac{p}{n}} + \beta \mathrm{vol}_n(L)^{\frac{p}{n}},$$

  where vol_n denotes the volume (or Lebesgue measure) on $\mathbb{R}^n$.

  - $p = 1$: the classical Brunn-Minkowski inequality.

L_p Minkowski combination

- For $K, L \in \mathscr{K}^n_{(o)}$, $p \geq 1$ and $\alpha, \beta \geq 0$, the L_p Minkowski combination:

$$h_{\alpha \cdot_p K +_p \beta \cdot_p L}(u) = \left(\alpha h_K(u)^p + \beta h_L(u)^p\right)^{\frac{1}{p}}.$$

  - $p = 1$: the classical Minkowski sum.

- The L_p-Brunn-Minkowski inequality when $p \geq 1$: given $K, L \in \mathscr{K}^n_{(o)}$ and $\alpha, \beta \geq 0$,

$$\mathrm{vol}_n(\alpha \cdot_p K +_p \beta \cdot_p L)^{\frac{p}{n}} \geq \alpha \mathrm{vol}_n(K)^{\frac{p}{n}} + \beta \mathrm{vol}_n(L)^{\frac{p}{n}},$$

  where vol_n denotes the volume (or Lebesgue measure) on $\mathbb{R}^n$.

  - $p = 1$: the classical Brunn-Minkowski inequality.

- L_p space for convex bodies by Firey: $(\mathscr{K}^n_{(o)}, +_p, \mathrm{vol}_n)$.

L_p Minkowski summation for measurable sets

- $\mathscr{B}$: all measurable sets in $\mathbb{R}^n$.

Non convex case (E. Lutwak, D. Yang and G. Zhang, AAM, 2012)

For $\alpha, \beta \geq 0$, $p \geq 1$, $K, L \in \mathscr{B}$, and $\frac{1}{p} + \frac{1}{q} = 1$,

$$\alpha \cdot_p K +_p \beta \cdot_p L = \left\{ \alpha^{\frac{1}{p}} (1-\lambda)^{\frac{1}{q}} x + \beta^{\frac{1}{p}} \lambda^{\frac{1}{q}} y \colon x \in K, y \in L, 0 \leq \lambda \leq 1 \right\}$$
$$= \bigcup_{0 \leq \lambda \leq 1} \left(\alpha^{\frac{1}{p}} (1-\lambda)^{\frac{1}{q}} K + \beta^{\frac{1}{p}} \lambda^{\frac{1}{q}} L \right).$$

L_p Minkowski summation for measurable sets

- $\mathscr{B}$: all measurable sets in $\mathbb{R}^n$.

Non convex case (E. Lutwak, D. Yang and G. Zhang, AAM, 2012)

For $\alpha, \beta \geq 0$, $p \geq 1$, $K, L \in \mathscr{B}$, and $\frac{1}{p} + \frac{1}{q} = 1$,

$$\begin{aligned} \alpha \cdot_p K +_p \beta \cdot_p L &= \left\{ \alpha^{\frac{1}{p}} (1-\lambda)^{\frac{1}{q}} x + \beta^{\frac{1}{p}} \lambda^{\frac{1}{q}} y \colon x \in K, y \in L, 0 \leq \lambda \leq 1 \right\} \\ &= \bigcup_{0 \leq \lambda \leq 1} \left(\alpha^{\frac{1}{p}} (1-\lambda)^{\frac{1}{q}} K + \beta^{\frac{1}{p}} \lambda^{\frac{1}{q}} L \right). \end{aligned}$$

✦ This definition of the L_p combination agrees with the original one defined by Firey for $K, L \in \mathscr{K}^n_{(o)}$.

L_p Minkowski summation for measurable sets

- $\mathscr{B}$: all measurable sets in $\mathbb{R}^n$.

Non convex case (E. Lutwak, D. Yang and G. Zhang, AAM, 2012)

For $\alpha, \beta \geq 0$, $p \geq 1$, $K, L \in \mathscr{B}$, and $\frac{1}{p} + \frac{1}{q} = 1$,

$$\begin{aligned} \alpha \cdot_p K +_p \beta \cdot_p L &= \left\{ \alpha^{\frac{1}{p}} (1-\lambda)^{\frac{1}{q}} x + \beta^{\frac{1}{p}} \lambda^{\frac{1}{q}} y : x \in K, y \in L, 0 \leq \lambda \leq 1 \right\} \\ &= \bigcup_{0 \leq \lambda \leq 1} \left(\alpha^{\frac{1}{p}} (1-\lambda)^{\frac{1}{q}} K + \beta^{\frac{1}{p}} \lambda^{\frac{1}{q}} L \right). \end{aligned}$$

- ✦ This definition of the L_p combination agrees with the original one defined by Firey for $K, L \in \mathscr{K}^n_{(o)}$.
- ✦ The L_p-Brunn-Minkowski inequality has the same formula for compact sets under this L_p summation for non convex sets.

L_p Minkowski summation for measurable sets

- $\mathscr{B}$: all measurable sets in $\mathbb{R}^n$.

Non convex case (E. Lutwak, D. Yang and G. Zhang, AAM, 2012)

For $\alpha, \beta \geq 0$, $p \geq 1$, $K, L \in \mathscr{B}$, and $\frac{1}{p} + \frac{1}{q} = 1$,

$$\alpha \cdot_p K +_p \beta \cdot_p L = \left\{ \alpha^{\frac{1}{p}} (1-\lambda)^{\frac{1}{q}} x + \beta^{\frac{1}{p}} \lambda^{\frac{1}{q}} y : x \in K, y \in L, 0 \leq \lambda \leq 1 \right\}$$
$$= \bigcup_{0 \leq \lambda \leq 1} \left(\alpha^{\frac{1}{p}} (1-\lambda)^{\frac{1}{q}} K + \beta^{\frac{1}{p}} \lambda^{\frac{1}{q}} L \right).$$

- This definition of the L_p combination agrees with the original one defined by Firey for $K, L \in \mathscr{K}^n_{(o)}$.
- The L_p-Brunn-Minkowski inequality has the same formula for compact sets under this L_p summation for non convex sets.
- Extended L_p space for measurable sets: $(\mathscr{B}, +_p, \mathrm{vol}_n)$.

Quermassintegral for convex bodies

- $w_n := \mathrm{vol}_n(B^n)$ where B^n is the n-dimensional unit ball,

Quermassintegral for convex bodies

- $w_n := \text{vol}_n(B^n)$ where B^n is the n-dimensional unit ball,
- $c_{n,j} := \frac{\omega_n}{\omega_{n-j}}$ for $j \in \{0, 1 \cdots, n-1\}$,

Quermassintegral for convex bodies

- $w_n := \text{vol}_n(B^n)$ where B^n is the n-dimensional unit ball,
- $c_{n,j} := \frac{\omega_n}{\omega_{n-j}}$ for $j \in \{0, 1 \cdots, n-1\}$,
- $G_{n,n-j}$: Grassmannian manifold—the $(n-j)$-dimensional subspaces of $\mathbb{R}^n$ equipped with the Haar probability measure $\nu_{n,n-j}$,

Quermassintegral for convex bodies

- $w_n := \mathrm{vol}_n(B^n)$ where B^n is the n-dimensional unit ball,
- $c_{n,j} := \frac{\omega_n}{\omega_{n-j}}$ for $j \in \{0, 1 \cdots, n-1\}$,
- $G_{n,n-j}$: Grassmannian manifold—the $(n-j)$-dimensional subspaces of $\mathbb{R}^n$ equipped with the Haar probability measure $\nu_{n,n-j}$,
- $K|H$: the projections of $K \in \mathscr{K}^n_{(o)}$ on the $(n-j)$-dimensional hyperplane H.

Quermassintegral for convex bodies

- $w_n := \mathrm{vol}_n(B^n)$ where B^n is the n-dimensional unit ball,
- $c_{n,j} := \frac{\omega_n}{\omega_{n-j}}$ for $j \in \{0, 1 \cdots, n-1\}$,
- $G_{n,n-j}$: Grassmannian manifold—the $(n-j)$-dimensional subspaces of $\mathbb{R}^n$ equipped with the Haar probability measure $\nu_{n,n-j}$,
- $K|H$: the projections of $K \in \mathscr{K}^n_{(o)}$ on the $(n-j)$-dimensional hyperplane H.

Kubota's integral formula

For $K \in \mathscr{K}^n_{(o)}$,

$$W_j(K) := c_{n,j} \int_{G_{n,n-j}} \mathrm{vol}_{n-j}(K|H) d\nu_{n,n-j}(H).$$

Quermassintegral for convex bodies

- ✧ $w_n := \mathrm{vol}_n(B^n)$ where B^n is the n-dimensional unit ball,
- ✧ $c_{n,j} := \frac{\omega_n}{\omega_{n-j}}$ for $j \in \{0, 1 \cdots, n-1\}$,
- ✧ $G_{n,n-j}$: Grassmannian manifold—the $(n-j)$-dimensional subspaces of $\mathbb{R}^n$ equipped with the Haar probability measure $\nu_{n,n-j}$,
- ✧ $K|H$: the projections of $K \in \mathscr{K}^n_{(o)}$ on the $(n-j)$-dimensional hyperplane H.

Kubota's integral formula

For $K \in \mathscr{K}^n_{(o)}$,

$$W_j(K) := c_{n,j} \int_{G_{n,n-j}} \mathrm{vol}_{n-j}(K|H) d\nu_{n,n-j}(H).$$

- $j = 0$: $W_0(K) = \mathrm{vol}_n(K)$.

Mixed p-quermassintegrals for convex bodies

Variation formula for quermassintegrals (E. Lutwak, JDG, 1993)

$$\begin{aligned} W_{p,j}(K,L) &= \frac{p}{n-j}\cdot\frac{d}{d\varepsilon}W_j(K+_p\varepsilon\cdot_p L)\Big|_{\varepsilon=0} \\ &= \frac{1}{n-j}\int_{S^{n-1}} h_L(u)^p h_K(u)^{1-p}\,dS_j(K,u). \end{aligned}$$

Mixed p-quermassintegrals for convex bodies

Variation formula for quermassintegrals (E. Lutwak, JDG, 1993)

$$W_{p,j}(K,L) = \frac{p}{n-j}\cdot\frac{d}{d\varepsilon}W_j(K+_p\varepsilon\cdot_p L)\Big|_{\varepsilon=0}$$
$$= \frac{1}{n-j}\int_{S^{n-1}} h_L(u)^p h_K(u)^{1-p}dS_j(K,u).$$

✧ $S_j(K,\cdot)$: the j-th surface area measure for K defined on S^{n-1}.

Mixed p-quermassintegrals for convex bodies

Variation formula for quermassintegrals (E. Lutwak, JDG, 1993)

$$W_{p,j}(K,L) = \frac{p}{n-j} \cdot \frac{d}{d\varepsilon} W_j(K +_p \varepsilon \cdot_p L)\Big|_{\varepsilon=0} = \frac{1}{n-j} \int_{S^{n-1}} h_L(u)^p h_K(u)^{1-p} dS_j(K,u).$$

- $S_j(K,\cdot)$: the j-th surface area measure for K defined on S^{n-1}.
- $S_0(K,\cdot) = S(K,\cdot)$: the surface area measure on S^{n-1}.

Mixed p-quermassintegrals for convex bodies

Variation formula for quermassintegrals (E. Lutwak, JDG, 1993)

$$
\begin{aligned}
W_{p,j}(K,L) &= \frac{p}{n-j}\cdot\frac{d}{d\varepsilon}W_j(K+_p\varepsilon\cdot_p L)\Big|_{\varepsilon=0}\\
&= \frac{1}{n-j}\int_{S^{n-1}} h_L(u)^p h_K(u)^{1-p}dS_j(K,u).
\end{aligned}
$$

- $S_j(K,\cdot)$: the j-th surface area measure for K defined on S^{n-1}.
- $S_0(K,\cdot)=S(K,\cdot)$: the surface area measure on S^{n-1}.
- $j=0$: the classical L_p mixed volume for convex bodies $V_p(K,L)$,

Mixed p-quermassintegrals for convex bodies

Variation formula for quermassintegrals (E. Lutwak, JDG, 1993)

$$\begin{aligned} W_{p,j}(K,L) &= \frac{p}{n-j}\cdot\frac{d}{d\varepsilon}W_j(K+_p\varepsilon\cdot_p L)\Big|_{\varepsilon=0} \\ &= \frac{1}{n-j}\int_{S^{n-1}} h_L(u)^p h_K(u)^{1-p}\,dS_j(K,u). \end{aligned}$$

- $S_j(K,\cdot)$: the j-th surface area measure for K defined on S^{n-1}.
- $S_0(K,\cdot)=S(K,\cdot)$: the surface area measure on S^{n-1}.
- $j=0$: the classical L_p mixed volume for convex bodies $V_p(K,L)$,
- For $K,L\in\mathscr{K}^n_{(o)}$:

$$W_{p,j}(K,L)^{n-j}\geq W_j(K)^{n-j-p}W_j(L)^p,$$

Mixed p-quermassintegrals for convex bodies

Variation formula for quermassintegrals (E. Lutwak, JDG, 1993)

$$
\begin{aligned}
W_{p,j}(K,L) &= \frac{p}{n-j}\cdot\frac{d}{d\varepsilon}W_j(K+_p\varepsilon\cdot_p L)\Big|_{\varepsilon=0}\\
&= \frac{1}{n-j}\int_{S^{n-1}} h_L(u)^p h_K(u)^{1-p}\,dS_j(K,u).
\end{aligned}
$$

- ✧ $S_j(K,\cdot)$: the j-th surface area measure for K defined on S^{n-1}.
- ✧ $S_0(K,\cdot)=S(K,\cdot)$: the surface area measure on S^{n-1}.
- ✧ $j=0$: the classical L_p mixed volume for convex bodies $V_p(K,L)$,
- ✧ For $K,L\in\mathscr{K}^n_{(o)}$:

$$W_{p,j}(K,L)^{n-j}\geq W_j(K)^{n-j-p}W_j(L)^p,$$

★ $j=0$, $V_p(K,L)^n\geq \mathrm{vol}_n(K)^{n-p}\mathrm{vol}_n(L)^p$.

Mixed p-quermassintegrals for convex bodies

Variation formula for quermassintegrals (E. Lutwak, JDG, 1993)

$$\begin{aligned} W_{p,j}(K,L) &= \frac{p}{n-j}\cdot\frac{d}{d\varepsilon}W_j(K+_p\varepsilon\cdot_p L)\Big|_{\varepsilon=0} \\ &= \frac{1}{n-j}\int_{S^{n-1}} h_L(u)^p h_K(u)^{1-p}dS_j(K,u). \end{aligned}$$

- $S_j(K,\cdot)$: the j-th surface area measure for K defined on S^{n-1}.
- $S_0(K,\cdot)=S(K,\cdot)$: the surface area measure on S^{n-1}.
- $j=0$: the classical L_p mixed volume for convex bodies $V_p(K,L)$,
- For $K,L\in\mathscr{K}^n_{(o)}$:

$$W_{p,j}(K,L)^{n-j}\geq W_j(K)^{n-j-p}W_j(L)^p,$$

  - ★ $j=0$, $V_p(K,L)^n\geq \mathrm{vol}_n(K)^{n-p}\mathrm{vol}_n(L)^p$.

- L_p space with quermassintegral for convex bodies: $(\mathscr{K}^n_{(o)},+_p,W_j)$.

Background for functions

- Given $s \in [-\infty, \infty]$ and $a, b \geq 0$, the s-mean of a and b with respect to coefficients $\alpha, \beta \geq 0$ is

$$M_s^{(\alpha,\beta)}(a,b) := \begin{cases} (\alpha a^s + \beta b^s)^{\frac{1}{s}}, & \text{if } s \neq 0, \pm\infty, \\ a^\alpha b^\beta, & \text{if } s = 0, \\ \max\{a,b\}, & \text{if } s = +\infty, \\ \min\{a,b\} & \text{if } s = -\infty, \end{cases}$$

whenever $ab > 0$, and $M_s^{(\alpha,\beta)}(a,b) := 0$ otherwise.

Background for functions

- Given $s \in [-\infty, \infty]$ and $a, b \geq 0$, the s-mean of a and b with respect to coefficients $\alpha, \beta \geq 0$ is

$$M_s^{(\alpha,\beta)}(a,b) := \begin{cases} (\alpha a^s + \beta b^s)^{\frac{1}{s}}, & \text{if } s \neq 0, \pm\infty, \\ a^\alpha b^\beta, & \text{if } s = 0, \\ \max\{a,b\}, & \text{if } s = +\infty, \\ \min\{a,b\} & \text{if } s = -\infty, \end{cases}$$

whenever $ab > 0$, and $M_s^{(\alpha,\beta)}(a,b) := 0$ otherwise.

- A function $f : \mathbb{R}^n \to \mathbb{R}_+$ is s-concave if, for all $x, y \in \mathbb{R}^n$ and any $t \in [0,1]$:

$$f((1-t)x + ty) \geq M_s^{((1-t),t)}(f(x), f(y)).$$

Background for functions

- Given $s \in [-\infty, \infty]$ and $a, b \geq 0$, the s-mean of a and b with respect to coefficients $\alpha, \beta \geq 0$ is

$$M_s^{(\alpha,\beta)}(a,b) := \begin{cases} (\alpha a^s + \beta b^s)^{\frac{1}{s}}, & \text{if } s \neq 0, \pm\infty, \\ a^\alpha b^\beta, & \text{if } s = 0, \\ \max\{a,b\}, & \text{if } s = +\infty, \\ \min\{a,b\} & \text{if } s = -\infty, \end{cases}$$

  whenever $ab > 0$, and $M_s^{(\alpha,\beta)}(a,b) := 0$ otherwise.

- A function $f\colon \mathbb{R}^n \to \mathbb{R}_+$ is s-concave if, for all $x, y \in \mathbb{R}^n$ and any $t \in [0,1]$:

$$f((1-t)x + ty) \geq M_s^{((1-t),t)}(f(x), f(y)).$$

  - $s = 0$: Log-concave function;

Background for functions

- Given $s \in [-\infty, \infty]$ and $a, b \geq 0$, the s-mean of a and b with respect to coefficients $\alpha, \beta \geq 0$ is

$$M_s^{(\alpha,\beta)}(a,b) := \begin{cases} (\alpha a^s + \beta b^s)^{\frac{1}{s}}, & \text{if } s \neq 0, \pm\infty, \\ a^\alpha b^\beta, & \text{if } s = 0, \\ \max\{a,b\}, & \text{if } s = +\infty, \\ \min\{a,b\} & \text{if } s = -\infty, \end{cases}$$

whenever $ab > 0$, and $M_s^{(\alpha,\beta)}(a,b) := 0$ otherwise.

- A function $f \colon \mathbb{R}^n \to \mathbb{R}_+$ is s-concave if, for all $x, y \in \mathbb{R}^n$ and any $t \in [0,1]$:

$$f((1-t)x + ty) \geq M_s^{((1-t),t)}(f(x), f(y)).$$

  - $s = 0$: Log-concave function;
  - $s = -\infty$: Quasi-concave function.

Borell-Brascamp-Lieb inequality

- Total mass of f: $I(f) = \int_{\mathbb{R}^n} f$.

Borell-Brascamp-Lieb inequality

✧ Total mass of f: $I(f) = \int_{\mathbb{R}^n} f$.

Borell-Brascamp-Lieb inequality (BBL inequality)

Let $t \in [0,1]$ and $s \in [-1/n, \infty]$. Given a triple of measurable functions $h, f, g \colon \mathbb{R}^n \to \mathbb{R}_+$ satisfying the condition

$$h((1-t)x + ty) \geq M_s^{((1-t),t)}(f(x), g(y))$$

for any $x, y \in \mathbb{R}^n$, there is

$$I(h) \geq M_{\frac{s}{1+ns}}^{((1-t),t)}(I(f), I(g)).$$

Borell-Brascamp-Lieb inequality

- Total mass of f: $I(f) = \int_{\mathbb{R}^n} f$.

Borell-Brascamp-Lieb inequality (BBL inequality)

Let $t \in [0,1]$ and $s \in [-1/n, \infty]$. Given a triple of measurable functions $h, f, g : \mathbb{R}^n \to \mathbb{R}_+$ satisfying the condition

$$h((1-t)x + ty) \geq M_s^{((1-t),t)}(f(x), g(y))$$

for any $x, y \in \mathbb{R}^n$, there is

$$I(h) \geq M_{\frac{s}{1+ns}}^{((1-t),t)}(I(f), I(g)).$$

- $s = 0$: Prékopa-Leindler inequality.

Borell-Brascamp-Lieb inequality

✧ Total mass of f: $I(f) = \int_{\mathbb{R}^n} f$.

Borell-Brascamp-Lieb inequality (BBL inequality)

Let $t \in [0,1]$ and $s \in [-1/n, \infty]$. Given a triple of measurable functions $h, f, g : \mathbb{R}^n \to \mathbb{R}_+$ satisfying the condition

$$h((1-t)x + ty) \geq M_s^{((1-t),t)}(f(x), g(y))$$

for any $x, y \in \mathbb{R}^n$, there is

$$I(h) \geq M_{\frac{s}{1+ns}}^{((1-t),t)}(I(f), I(g))\,.$$

- $s = 0$: Prékopa-Leindler inequality.
- Measures having $\left(\frac{1}{s}\right)$-concave density must be $\left(\frac{1}{n+s}\right)$-concave for $s \leq -n$ and $s \geq 0$.

Borell-Brascamp-Lieb inequality

✧ Total mass of f: $I(f) = \int_{\mathbb{R}^n} f$.

Borell-Brascamp-Lieb inequality (BBL inequality)

Let $t \in [0,1]$ and $s \in [-1/n, \infty]$. Given a triple of measurable functions $h, f, g : \mathbb{R}^n \to \mathbb{R}_+$ satisfying the condition

$$h((1-t)x + ty) \geq M_s^{((1-t),t)}(f(x), g(y))$$

for any $x, y \in \mathbb{R}^n$, there is

$$I(h) \geq M_{\frac{s}{1+ns}}^{((1-t),t)}(I(f), I(g)).$$

- $s = 0$: Prékopa-Leindler inequality.
- Measures having $\left(\frac{1}{s}\right)$-concave density must be $\left(\frac{1}{n+s}\right)$-concave for $s \leq -n$ and $s \geq 0$.
- Borell, 1975; Brascamp and Lieb, 1976.

Supremal-convolution

- The minimal function satisfies the BBL inequality:

$$m_{t,s}(z) := \sup_{z=(1-t)x+ty} M_s^{((1-t),t)}(f(x), g(y)).$$

Supremal-convolution

- The minimal function satisfies the BBL inequality:

$$m_{t,s}(z) := \sup_{z=(1-t)x+ty} M_s^{((1-t),t)}(f(x), g(y)).$$

Supremal-convolution (B. Klartag, GAFA, 2007)

Given $s \in [-\infty, \infty]$, $f, g \colon \mathbb{R}^n \to \mathbb{R}_+$, the supremal-convolution of the functions f and g:

$$(f \oplus_s g)(z) := \sup_{z=x+y} \begin{cases} (f(x)^s + g(y)^s)^{\frac{1}{s}}, & \text{if } s \neq 0, \pm\infty, \\ f(x)g(y), & \text{if } s = 0, \\ \max\{f(x), g(y)\}, & \text{if } s = +\infty, \\ \min\{f(x), g(y)\}, & \text{if } s = -\infty. \end{cases}$$

Supremal-convolution

- The minimal function satisfies the BBL inequality:

$$m_{t,s}(z) := \sup_{z=(1-t)x+ty} M_s^{((1-t),t)}(f(x), g(y)).$$

Supremal-convolution (B. Klartag, GAFA, 2007)

Given $s \in [-\infty, \infty]$, $f, g : \mathbb{R}^n \to \mathbb{R}_+$, the supremal-convolution of the functions f and g:

$$(f \oplus_s g)(z) := \sup_{z=x+y} \begin{cases} (f(x)^s + g(y)^s)^{\frac{1}{s}}, & \text{if } s \neq 0, \pm\infty, \\ f(x)g(y), & \text{if } s = 0, \\ \max\{f(x), g(y)\}, & \text{if } s = +\infty, \\ \min\{f(x), g(y)\}, & \text{if } s = -\infty. \end{cases}$$

- $\alpha > 0$,

$$(\alpha \times_s f)(x) := \begin{cases} \alpha^{\frac{1}{s}} f\left(\frac{x}{\alpha}\right), & \text{if } s \neq 0, \pm\infty, \\ f(x)^\alpha, & \text{if } s = 0, \\ f(x), & \text{if } s = \pm\infty. \end{cases}$$

Applications

- The s-concavity is closed under the supremal-convolution operation.

Applications

- The s-concavity is closed under the supremal-convolution operation.
- Functional space equipped with supermal-convolution: $(\mathbb{F}, \oplus_s, I(\cdot))$.
  - $\mathbb{F}$: the set of all integrable functions on $\mathbb{R}^n$.

Applications

- The s-concavity is closed under the supremal-convolution operation.
- Functional space equipped with supermal-convolution: $(\mathbb{F}, \oplus_s, I(\cdot))$.
  - $\mathbb{F}$: the set of all integrable functions on $\mathbb{R}^n$.
- $K, L \subset \mathbb{R}^n$, $t \in [0,1]$:

$$((1-t) \times_s \chi_K) \oplus_s (t \times_s \chi_L) = \chi_{(1-t)K+tL}.$$

Applications

- The s-concavity is closed under the supremal-convolution operation.
- Functional space equipped with supermal-convolution: $(\mathbb{F}, \oplus_s, I(\cdot))$.
  - $\mathbb{F}$: the set of all integrable functions on $\mathbb{R}^n$.
- $K, L \subset \mathbb{R}^n$, $t \in [0,1]$:

$$((1-t) \times_s \chi_K) \oplus_s (t \times_s \chi_L) = \chi_{(1-t)K+tL}.$$

- The BBL inequality: for any $s \geq -1/n$,

$$I\big(((1-t) \times_s f) \oplus_s (t \times_s g)\big) \geq M_{\frac{s}{1+ns}}^{(1-t,t)}(I(f), I(g)).$$

Applications

- The s-concavity is closed under the supremal-convolution operation.
- Functional space equipped with supermal-convolution: $(\mathbb{F}, \oplus_s, I(\cdot))$.
  - $\mathbb{F}$: the set of all integrable functions on $\mathbb{R}^n$.
- $K, L \subset \mathbb{R}^n$, $t \in [0,1]$:

$$((1-t) \times_s \chi_K) \oplus_s (t \times_s \chi_L) = \chi_{(1-t)K+tL}.$$

- The BBL inequality: for any $s \geq -1/n$,

$$I\big(((1-t) \times_s f) \oplus_s (t \times_s g)\big) \geq M_{\frac{s}{1+ns}}^{(1-t,t)}(I(f), I(g)).$$

  - $f = \chi_K$, $K \subset \mathbb{R}^n$: $I(f) = \mathrm{vol}_n(K)$.

Applications

- The s-concavity is closed under the supremal-convolution operation.
- Functional space equipped with supermal-convolution: $(\mathbb{F}, \oplus_s, I(\cdot))$.
  - $\mathbb{F}$: the set of all integrable functions on $\mathbb{R}^n$.
- $K, L \subset \mathbb{R}^n$, $t \in [0,1]$:

$$((1-t) \times_s \chi_K) \oplus_s (t \times_s \chi_L) = \chi_{(1-t)K+tL}.$$

- The BBL inequality: for any $s \geq -1/n$,

$$I\big(((1-t) \times_s f) \oplus_s (t \times_s g)\big) \geq M^{(1-t,t)}_{\frac{s}{1+ns}}(I(f), I(g)).$$

  - $f = \chi_K$, $K \subset \mathbb{R}^n$: $I(f) = \text{vol}_n(K)$.
  - $\frac{1}{n} > \frac{s}{1+ns}$ for $s \geq -1/n$:

$$V((1-t)K + tL) \geq M^{(1-t,t)}_{\frac{s}{1+ns}}(V(K), V(L)).$$

Applications

- The s-concavity is closed under the supremal-convolution operation.
- Functional space equipped with supermal-convolution: $(\mathbb{F}, \oplus_s, I(\cdot))$.
  - $\mathbb{F}$: the set of all integrable functions on $\mathbb{R}^n$.
- $K, L \subset \mathbb{R}^n$, $t \in [0,1]$:

$$((1-t)\times_s \chi_K)\oplus_s(t\times_s \chi_L) = \chi_{(1-t)K+tL}.$$

- The BBL inequality: for any $s \geq -1/n$,

$$I\big(((1-t)\times_s f)\oplus_s(t\times_s g)\big) \geq M_{\frac{s}{1+ns}}^{(1-t,t)}(I(f), I(g)).$$

  - $f = \chi_K$, $K \subset \mathbb{R}^n$: $I(f) = \mathrm{vol}_n(K)$.
  - $\frac{1}{n} > \frac{s}{1+ns}$ for $s \geq -1/n$:

$$V((1-t)K + tL) \geq M_{\frac{s}{1+ns}}^{(1-t,t)}(V(K), V(L)).$$

  - $s = 0$:

$$V((1-t)K + tL) \geq V(K)^{1-t} V(L)^t.$$

Infimal-convolution

- $\mathrm{Cvx}(\mathbb{R}^n) := \{u : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, proper (non-empty domain), convex, l.s.c$\}$, where l.s.c. denotes lower semi-continuous.

Infimal-convolution

- $\mathrm{Cvx}(\mathbb{R}^n) := \{u : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, proper (non-empty domain), convex, l.s.c$\}$, where l.s.c. denotes lower semi-continuous.

Infimal-convolution for $u, v \in \mathrm{Cvx}(\mathbb{R}^n)$

$$(u \Box v)(x) = \inf_{y \in \mathbb{R}^n} \{u(x - y) + v(y)\}.$$

Infimal-convolution

- $\mathrm{Cvx}(\mathbb{R}^n) := \{u : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, proper (non-empty domain), convex, l.s.c$\}$, where l.s.c. denotes lower semi-continuous.

Infimal-convolution for $u, v \in \mathrm{Cvx}(\mathbb{R}^n)$

$$(u \square v)(x) = \inf_{y \in \mathbb{R}^n} \{u(x-y) + v(y)\}.$$

- $(\alpha \times u)(x) = \alpha u(x/\alpha)$.

Infimal-convolution

✦ $\mathrm{Cvx}(\mathbb{R}^n) := \{u : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, proper (non-empty domain), convex, l.s.c$\}$, where l.s.c. denotes lower semi-continuous.

Infimal-convolution for $u, v \in \mathrm{Cvx}(\mathbb{R}^n)$

$$(u \Box v)(x) = \inf_{y \in \mathbb{R}^n} \{u(x-y) + v(y)\}.$$

- $(\alpha \times u)(x) = \alpha u(x/\alpha)$.
- $u, v \in \mathrm{Cvx}(\mathbb{R}^n) \Rightarrow u \Box v \in \mathrm{Cvx}(\mathbb{R}^n)$.

Infimal-convolution

✦ $\mathrm{Cvx}(\mathbb{R}^n) := \{u : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, proper (non-empty domain), convex, l.s.c$\}$, where l.s.c. denotes lower semi-continuous.

Infimal-convolution for $u, v \in \mathrm{Cvx}(\mathbb{R}^n)$

$$(u \square v)(x) = \inf_{y \in \mathbb{R}^n} \{u(x-y) + v(y)\}.$$

- $(\alpha \times u)(x) = \alpha u(x/\alpha)$.
- $u, v \in \mathrm{Cvx}(\mathbb{R}^n) \Rightarrow u \square v \in \mathrm{Cvx}(\mathbb{R}^n)$.
- $$\mathrm{epi}(u \square v) = \mathrm{epi}(u) + \mathrm{epi}(v),$$

  where "$+$" denotes the Minkowski sum in $\mathbb{R}^n$ and

  $$\mathrm{epi}(u) = \{(x, y) \in \mathbb{R}^n \times \mathbb{R} \colon y \geq u(x)\}.$$

Legendre transformation

- Legendre transformation: $u^*: \mathrm{Cvx}(\mathbb{R}^n) \to \mathrm{Cvx}(\mathbb{R}^n)$

$$u^*(x) = \sup_{y \in \mathbb{R}^n} [\langle x, y \rangle - u(y)].$$

Legendre transformation

✦ Legendre transformation: $u^* : \mathrm{Cvx}(\mathbb{R}^n) \to \mathrm{Cvx}(\mathbb{R}^n)$

$$u^*(x) = \sup_{y \in \mathbb{R}^n} [\langle x, y \rangle - u(y)].$$

★ $u^{**} = u$.

★ $u \leq v \Rightarrow u^* \geq v^*$.

- Legendre transformation: $u^*: \mathrm{Cvx}(\mathbb{R}^n) \to \mathrm{Cvx}(\mathbb{R}^n)$

$$u^*(x) = \sup_{y \in \mathbb{R}^n} [\langle x, y \rangle - u(y)].$$

  - $u^{**} = u$.
  - $u \leq v \Rightarrow u^* \geq v^*$.

- $(I_K)^* = h_K$ for $K \in \mathscr{K}^n_{(o)}$.

Legendre transformation

- **Legendre transformation**: $u^*: \mathrm{Cvx}(\mathbb{R}^n) \to \mathrm{Cvx}(\mathbb{R}^n)$

$$u^*(x) = \sup_{y \in \mathbb{R}^n} [\langle x, y \rangle - u(y)].$$

  - $u^{**} = u$.
  - $u \leq v \Rightarrow u^* \geq v^*$.

- $(I_K)^* = h_K$ for $K \in \mathscr{K}^n_{(o)}$.

- $\alpha, \beta \geq 0$:

$$((\alpha \times u) \Box (\beta \times v))^* = \alpha u^* + \beta v^*.$$

Families of functions

- Super-coercive function:

$$C_s(\mathbb{R}^n) := \left\{ u \in \mathrm{Cvx}(\mathbb{R}^n) : \; u(o) = 0, \lim_{\|x\| \to \infty} \frac{u(x)}{\|x\|} = +\infty \right\} \subset \mathrm{Cvx}(\mathbb{R}^n).$$

Families of functions

- Super-coercive function:

$$C_s(\mathbb{R}^n) := \left\{ u \in \mathrm{Cvx}(\mathbb{R}^n) : \; u(o) = 0, \lim_{\|x\| \to \infty} \frac{u(x)}{\|x\|} = +\infty \right\} \subset \mathrm{Cvx}(\mathbb{R}^n).$$

- s-concave functions:

$$\mathscr{F}_s(\mathbb{R}^n) = \left\{ f : \mathbb{R}^n \to \mathbb{R}_+, f \text{ is } s\text{-concave}, \text{u.s.c}, f \in L^1(\mathbb{R}^n), f(o) = \|f\|_\infty > 0 \right\}.$$

  - u.s.c.: upper semi-continuous.

Families of functions

- Super-coercive function:

$$C_s(\mathbb{R}^n) := \left\{ u \in \mathrm{Cvx}(\mathbb{R}^n):\ u(o) = 0, \lim_{\|x\| \to \infty} \frac{u(x)}{\|x\|} = +\infty \right\} \subset \mathrm{Cvx}(\mathbb{R}^n).$$

- s-concave functions:

$$\mathscr{F}_s(\mathbb{R}^n) = \left\{ f : \mathbb{R}^n \to \mathbb{R}_+, f \text{ is } s\text{-concave, u.s.c}, f \in L^1(\mathbb{R}^n), f(o) = \|f\|_\infty > 0 \right\}.$$

  - u.s.c.: upper semi-continuous.
  - $\mathscr{F}_0(\mathbb{R}^n)$: the class of all such log-concave functions.

Families of functions

- Super-coercive function:

$$C_s(\mathbb{R}^n) := \left\{ u \in \mathrm{Cvx}(\mathbb{R}^n) : \; u(o) = 0, \lim_{\|x\| \to \infty} \frac{u(x)}{\|x\|} = +\infty \right\} \subset \mathrm{Cvx}(\mathbb{R}^n).$$

- s-concave functions:

$$\mathscr{F}_s(\mathbb{R}^n) = \left\{ f : \mathbb{R}^n \to \mathbb{R}_+, f \text{ is } s\text{-concave}, \text{u.s.c}, f \in L^1(\mathbb{R}^n), f(o) = \|f\|_\infty > 0 \right\}.$$

  - u.s.c.: upper semi-continuous.
  - $\mathscr{F}_0(\mathbb{R}^n)$: the class of all such log-concave functions.
  - $\mathscr{F}_{-\infty}(\mathbb{R}^n)$: the class of all such quasi-concave functions.

Asplund summation

- Base function: $\mathscr{F}_s(\mathbb{R}^n) \to C_s(\mathbb{R}^n)$. For $f \in \mathscr{F}_s(\mathbb{R}^n)$, the base function $u_f \in C_s(\mathbb{R}^n)$ satisfies
$$f(x) = (1 - s u_f(x))_+^{\frac{1}{s}},$$
where $a_+ = \max\{a, 0\}$.

Asplund summation

- Base function: $\mathscr{F}_s(\mathbb{R}^n) \to C_s(\mathbb{R}^n)$. For $f \in \mathscr{F}_s(\mathbb{R}^n)$, the base function $u_f \in C_s(\mathbb{R}^n)$ satisfies
$$f(x) = (1 - s u_f(x))_+^{\frac{1}{s}},$$
where $a_+ = \max\{a, 0\}$.
  - L. Rotem, ADV, 2013.

Asplund summation

- Base function: $\mathscr{F}_s(\mathbb{R}^n) \to C_s(\mathbb{R}^n)$. For $f \in \mathscr{F}_s(\mathbb{R}^n)$, the base function $u_f \in C_s(\mathbb{R}^n)$ satisfies

$$f(x) = (1 - su_f(x))_+^{\frac{1}{s}},$$

where $a_+ = \max\{a, 0\}$.

  - ★ L. Rotem, ADV, 2013.
  - $s = 0$: $f(x) = e^{-u_f(x)}$.

Asplund summation

- Base function: $\mathscr{F}_s(\mathbb{R}^n) \to C_s(\mathbb{R}^n)$. For $f \in \mathscr{F}_s(\mathbb{R}^n)$, the base function $u_f \in C_s(\mathbb{R}^n)$ satisfies
$$f(x) = (1 - su_f(x))_+^{\frac{1}{s}},$$
where $a_+ = \max\{a, 0\}$.
    - L. Rotem, ADV, 2013.
    - $s = 0$: $f(x) = e^{-u_f(x)}$.
- $f = \chi_K$, $K \in \mathscr{K}^n_{(o)}$: $u_f = I_K$.

Asplund summation

- Base function: $\mathscr{F}_s(\mathbb{R}^n) \to C_s(\mathbb{R}^n)$. For $f \in \mathscr{F}_s(\mathbb{R}^n)$, the base function $u_f \in C_s(\mathbb{R}^n)$ satisfies
$$f(x) = (1 - su_f(x))_+^{\frac{1}{s}},$$
where $a_+ = \max\{a, 0\}$.
  - L. Rotem, ADV, 2013.
  - $s = 0$: $f(x) = e^{-u_f(x)}$.
- $f = \chi_K$, $K \in \mathscr{K}^n_{(o)}$: $u_f = I_K$.
- Asplund summation: $f, g \in \mathscr{F}_s(\mathbb{R}^n)$ and $\alpha > 0$:

Asplund summation

- Base function: $\mathscr{F}_s(\mathbb{R}^n) \to C_s(\mathbb{R}^n)$. For $f \in \mathscr{F}_s(\mathbb{R}^n)$, the base function $u_f \in C_s(\mathbb{R}^n)$ satisfies
$$f(x) = (1 - su_f(x))_+^{\frac{1}{s}},$$
where $a_+ = \max\{a, 0\}$.
  - L. Rotem, ADV, 2013.
  - $s = 0$: $f(x) = e^{-u_f(x)}$.
- $f = \chi_K$, $K \in \mathscr{K}^n_{(o)}$: $u_f = I_K$.
- Asplund summation: $f, g \in \mathscr{F}_s(\mathbb{R}^n)$ and $\alpha > 0$:
  - $f \star_s g$:
$$u_{f \star_s g}(x) = (u_f \Box u_g)(x),$$

Asplund summation

- Base function: $\mathscr{F}_s(\mathbb{R}^n) \to C_s(\mathbb{R}^n)$. For $f \in \mathscr{F}_s(\mathbb{R}^n)$, the base function $u_f \in C_s(\mathbb{R}^n)$ satisfies

$$f(x) = (1 - s u_f(x))_+^{\frac{1}{s}},$$

  where $a_+ = \max\{a, 0\}$.

  - L. Rotem, ADV, 2013.
  - $s = 0$: $f(x) = e^{-u_f(x)}$.

- $f = \chi_K$, $K \in \mathscr{K}^n_{(o)}$: $u_f = I_K$.

- Asplund summation: $f, g \in \mathscr{F}_s(\mathbb{R}^n)$ and $\alpha > 0$:
  - $f \star_s g$:

$$u_{f \star_s g}(x) = (u_f \Box u_g)(x),$$

  - $\alpha \cdot_s f$:

$$u_{\alpha \cdot_s f}(x) = \alpha u_f\left(\frac{x}{\alpha}\right).$$

Asplund summation

- Base function: $\mathscr{F}_s(\mathbb{R}^n) \to C_s(\mathbb{R}^n)$. For $f \in \mathscr{F}_s(\mathbb{R}^n)$, the base function $u_f \in C_s(\mathbb{R}^n)$ satisfies

$$f(x) = (1 - s u_f(x))_+^{\frac{1}{s}},$$

  where $a_+ = \max\{a, 0\}$.

  - ★ L. Rotem, ADV, 2013.
  - $s = 0$: $f(x) = e^{-u_f(x)}$.

- $f = \chi_K$, $K \in \mathscr{K}^n_{(o)}$: $u_f = I_K$.

- Asplund summation: $f, g \in \mathscr{F}_s(\mathbb{R}^n)$ and $\alpha > 0$:

  - $f \star_s g$:

$$u_{f \star_s g}(x) = (u_f \Box u_g)(x),$$

  - $\alpha \cdot_s f$:

$$u_{\alpha \cdot_s f}(x) = \alpha u_f\left(\frac{x}{\alpha}\right).$$

  - ★ Functional space equipped with Asplund summation: $(\mathbb{F}, \star_s, I(\cdot))$.

Properties for Asplund summation

- For any $t \in [0,1]$ and $f, g \in \mathscr{F}_s(\mathbb{R}^n)$, the supremal-convolution coincides with the Asplund summation with coefficients $((1-t), t)$:

$$[((1-t) \cdot_s f) \star_s (t \cdot_s g)] = [((1-t) \times_s f) \oplus_s (t \times_s g)];$$

Properties for Asplund summation

- For any $t \in [0,1]$ and $f, g \in \mathscr{F}_s(\mathbb{R}^n)$, the supremal-convolution coincides with the Asplund summation with coefficients $((1-t), t)$:

$$[((1-t)\cdot_s f) \star_s (t \cdot_s g)] = [((1-t)\times_s f) \oplus_s (t \times_s g)];$$

- $$[((1-t)\times_s f) \oplus_s (t \times_s g)] = \left[1 - s((1-t)u_f^* + t u_g^*)^*\right]_+^{\frac{1}{s}}.$$

Properties for Asplund summation

- For any $t \in [0,1]$ and $f, g \in \mathscr{F}_s(\mathbb{R}^n)$, the supremal-convolution coincides with the Asplund summation with coefficients $((1-t), t)$:

$$[((1-t)\cdot_s f)\star_s (t \cdot_s g)] = [((1-t)\times_s f)\oplus_s (t \times_s g)];$$

- $$[((1-t)\times_s f)\oplus_s (t \times_s g)] = \left[1 - s((1-t)u_f^* + tu_g^*)^*\right]_+^{\frac{1}{s}}.$$

- For $u \in C_s(\mathbb{R}^n)$, consider the integral operator $J_s \colon C_s(\mathbb{R}^n) \to \mathbb{R}_+$

$$J_s(u) = \int_{\mathbb{R}^n} [1 - su(x)]_+^{\frac{1}{s}}\, dx.$$

Properties for Asplund summation

- For any $t \in [0,1]$ and $f, g \in \mathscr{F}_s(\mathbb{R}^n)$, the supremal-convolution coincides with the Asplund summation with coefficients $((1-t), t)$:

$$[((1-t)\cdot_s f)\star_s(t\cdot_s g)] = [((1-t)\times_s f)\oplus_s(t\times_s g)];$$

- $$[((1-t)\times_s f)\oplus_s(t\times_s g)] = \left[1 - s((1-t)u_f^* + tu_g^*)^*\right]_+^{\frac{1}{s}}.$$

- For $u \in C_s(\mathbb{R}^n)$, consider the integral operator $J_s \colon C_s(\mathbb{R}^n) \to \mathbb{R}_+$

$$J_s(u) = \int_{\mathbb{R}^n} [1 - su(x)]_+^{\frac{1}{s}}\, dx.$$

- The BBL inequality:

$$J_s(((1-t)\times u)\Box(t\times v)) \geq M_{\frac{s}{1+ns}}^{((1-t),t)}(J_s(u), J_s(v)).$$

$L_{p,s}$ supremal-convolution

$L_{p,s}$ supremal-convolution for $p \geq 1$[1]

For $f : \mathbb{R}^n \to \mathbb{R}_+$ and $g : \mathbb{R}^n \to \mathbb{R}_+$ for $s \geq 0$, $p \geq 1$, and $1/p + 1/q = 1$,

$$[f \oplus_{p,s} g](z) := \sup_{0 \leq \lambda \leq 1} \left(\sup_{z=(1-\lambda)^{\frac{1}{q}} x + \lambda^{\frac{1}{q}} y} M_s^{\left((1-\lambda)^{\frac{1}{q}}, \lambda^{\frac{1}{q}}\right)} (f(x), g(y)) \right)$$
$$= \sup_{0 \leq \lambda \leq 1} \left([(1-\lambda)^{\frac{1}{q}} \times_s f] \oplus_s [\lambda^{\frac{1}{q}} \times_s g](z) \right).$$

[1] M. Roysdon and S. Xing, *On L_p-Brunn-Minkowski type and L_p-isoperimetric type inequalities for measures,* Trans. Amer. Math. Soc., 374 (2021), 5003–5036.

$L_{p,s}$ supremal-convolution

$L_{p,s}$ supremal-convolution for $p \geq 1$[1]

For $f : \mathbb{R}^n \to \mathbb{R}_+$ and $g : \mathbb{R}^n \to \mathbb{R}_+$ for $s \geq 0$, $p \geq 1$, and $1/p + 1/q = 1$,

$$[f \oplus_{p,s} g](z) := \sup_{0 \leq \lambda \leq 1} \left(\sup_{z=(1-\lambda)^{\frac{1}{q}} x + \lambda^{\frac{1}{q}} y} M_s^{\left((1-\lambda)^{\frac{1}{q}}, \lambda^{\frac{1}{q}}\right)}(f(x), g(y)) \right)$$
$$= \sup_{0 \leq \lambda \leq 1} \left([(1-\lambda)^{\frac{1}{q}} \times_s f] \oplus_s [\lambda^{\frac{1}{q}} \times_s g](z) \right).$$

- $\alpha > 0$: $(\alpha \times_{p,s} f)(x) = \alpha^{s/p} f\left(\frac{x}{\alpha^{1/p}}\right)$.

[1] M. Roysdon and S. Xing, *On L_p-Brunn-Minkowski type and L_p-isoperimetric type inequalities for measures,* Trans. Amer. Math. Soc., 374 (2021), 5003–5036.

$L_{p,s}$ supremal-convolution

$L_{p,s}$ supremal-convolution for $p \geq 1$[1]

For $f : \mathbb{R}^n \to \mathbb{R}_+$ and $g : \mathbb{R}^n \to \mathbb{R}_+$ for $s \geq 0$, $p \geq 1$, and $1/p + 1/q = 1$,

$$[f \oplus_{p,s} g](z) := \sup_{0 \leq \lambda \leq 1} \left(\sup_{z=(1-\lambda)^{\frac{1}{q}} x + \lambda^{\frac{1}{q}} y} M_s^{\left((1-\lambda)^{\frac{1}{q}}, \lambda^{\frac{1}{q}}\right)} (f(x), g(y)) \right)$$
$$= \sup_{0 \leq \lambda \leq 1} \left([(1-\lambda)^{\frac{1}{q}} \times_s f] \oplus_s [\lambda^{\frac{1}{q}} \times_s g](z) \right).$$

- $\alpha > 0$: $(\alpha \times_{p,s} f)(x) = \alpha^{s/p} f\left(\frac{x}{\alpha^{1/p}}\right)$.
- $f = \chi_K$, $g = \chi_L$, $K, L \in \mathscr{B}$:

$$((1-t) \times_{p,s} f) \oplus_{p,s} (t \times_{p,s} g) = \chi_{(1-t)\cdot_p K + t \cdot_p L}.$$

[1] M. Roysdon and S. Xing, *On L_p-Brunn-Minkowski type and L_p-isoperimetric type inequalities for measures,* Trans. Amer. Math. Soc., 374 (2021), 5003–5036.

$L_{p,s}$ supremal-convolution

$L_{p,s}$ supremal-convolution for $p \geq 1$[1]

For $f : \mathbb{R}^n \to \mathbb{R}_+$ and $g : \mathbb{R}^n \to \mathbb{R}_+$ for $s \geq 0$, $p \geq 1$, and $1/p + 1/q = 1$,

$$[f \oplus_{p,s} g](z) := \sup_{0 \leq \lambda \leq 1} \left(\sup_{z=(1-\lambda)^{\frac{1}{q}} x + \lambda^{\frac{1}{q}} y} M_s^{\left((1-\lambda)^{\frac{1}{q}}, \lambda^{\frac{1}{q}}\right)} (f(x), g(y)) \right)$$
$$= \sup_{0 \leq \lambda \leq 1} \left([(1-\lambda)^{\frac{1}{q}} \times_s f] \oplus_s [\lambda^{\frac{1}{q}} \times_s g](z) \right).$$

- $\alpha > 0$: $(\alpha \times_{p,s} f)(x) = \alpha^{s/p} f\left(\frac{x}{\alpha^{1/p}}\right)$.
- $f = \chi_K$, $g = \chi_L$, $K, L \in \mathscr{B}$:

$$((1-t) \times_{p,s} f) \oplus_{p,s} (t \times_{p,s} g) = \chi_{(1-t)\cdot_p K + t \cdot_p L}.$$

- Functional space equipped with $L_{p,s}$ supremal-convolution: $(\mathbb{F}, \oplus_{p,s}, I(\cdot))$.

[1] M. Roysdon and S. Xing, *On L_p-Brunn-Minkowski type and L_p-isoperimetric type inequalities for measures,* Trans. Amer. Math. Soc., 374 (2021), 5003–5036.

Properties of $L_{p,s}$ supremal-convolution

Let $f, g, h\colon \mathbb{R}^n \to \mathbb{R}_+$ be arbitrary, not identically zero, functions defined on $\mathbb{R}^n$, and let $s \in [-\infty, \infty]$, $p \geq 1$, and $\alpha, \beta, \gamma > 0$.

Properties of $L_{p,s}$ supremal-convolution

Let $f, g, h\colon \mathbb{R}^n \to \mathbb{R}_+$ be arbitrary, not identically zero, functions defined on $\mathbb{R}^n$, and let $s \in [-\infty, \infty]$, $p \geq 1$, and $\alpha, \beta, \gamma > 0$.

★ Homogeneity:

$$(\alpha \times_{p,s} f) \oplus_{p,s} (\beta \times_{p,s} g) = (\alpha+\beta) \times_{p,s} \left[\left(\frac{\alpha}{\alpha+\beta} \times_{p,s} f \right) \oplus_{p,s} \left(\frac{\beta}{\alpha+\beta} \times_{p,s} g \right) \right]$$

for $s \neq \pm\infty$.

Properties of $L_{p,s}$ supremal-convolution

Let $f, g, h\colon \mathbb{R}^n \to \mathbb{R}_+$ be arbitrary, not identically zero, functions defined on $\mathbb{R}^n$, and let $s \in [-\infty, \infty]$, $p \geq 1$, and $\alpha, \beta, \gamma > 0$.

- ★ Homogeneity:

  $$(\alpha \times_{p,s} f) \oplus_{p,s} (\beta \times_{p,s} g) = (\alpha + \beta) \times_{p,s} \left[\left(\frac{\alpha}{\alpha + \beta} \times_{p,s} f \right) \oplus_{p,s} \left(\frac{\beta}{\alpha + \beta} \times_{p,s} g \right) \right]$$

  for $s \neq \pm\infty$.

- ★ Measurability: $(\alpha \times_{p,s} f) \oplus_{p,s} (\beta \times_{p,s} g)$ is measurable whenever both f and g are Borel measurable.

Properties of $L_{p,s}$ supremal-convolution

Let $f, g, h\colon \mathbb{R}^n \to \mathbb{R}_+$ be arbitrary, not identically zero, functions defined on $\mathbb{R}^n$, and let $s \in [-\infty, \infty]$, $p \geq 1$, and $\alpha, \beta, \gamma > 0$.

- ★ Homogeneity:

$$(\alpha \times_{p,s} f) \oplus_{p,s} (\beta \times_{p,s} g) = (\alpha+\beta) \times_{p,s} \left[\left(\frac{\alpha}{\alpha+\beta} \times_{p,s} f \right) \oplus_{p,s} \left(\frac{\beta}{\alpha+\beta} \times_{p,s} g \right) \right]$$

for $s \neq \pm\infty$.

- ★ Measurability: $(\alpha \times_{p,s} f) \oplus_{p,s} (\beta \times_{p,s} g)$ is measurable whenever both f and g are Borel measurable.
- ★ Commutativity: $(\alpha \times_{p,s} f) \oplus_{p,s} (\beta \times_{p,s} g) = (\beta \times_{p,s} g) \oplus_{p,s} (\alpha \times_{p,s} f)$.

Properties of $L_{p,s}$ supremal-convolution

Let $f, g, h \colon \mathbb{R}^n \to \mathbb{R}_+$ be arbitrary, not identically zero, functions defined on $\mathbb{R}^n$, and let $s \in [-\infty, \infty]$, $p \geq 1$, and $\alpha, \beta, \gamma > 0$.

- ★ Homogeneity:

  $$(\alpha \times_{p,s} f) \oplus_{p,s} (\beta \times_{p,s} g) = (\alpha+\beta) \times_{p,s} \left[\left(\frac{\alpha}{\alpha+\beta} \times_{p,s} f \right) \oplus_{p,s} \left(\frac{\beta}{\alpha+\beta} \times_{p,s} g \right) \right]$$

  for $s \neq \pm\infty$.

- ★ Measurability: $(\alpha \times_{p,s} f) \oplus_{p,s} (\beta \times_{p,s} g)$ is measurable whenever both f and g are Borel measurable.

- ★ Commutativity: $(\alpha \times_{p,s} f) \oplus_{p,s} (\beta \times_{p,s} g) = (\beta \times_{p,s} g) \oplus_{p,s} (\alpha \times_{p,s} f)$.

- ✧ Associativity:

  $$[(\alpha \times_{p,s} f) \oplus_{p,s} (\beta \times_{p,s} g)] \oplus_{p,s} (\gamma \times_{p,s} h) \neq (\alpha \times_{p,s} f) \oplus_{p,s} [(\beta \times_{p,s} g) \oplus_{p,s} (\gamma \times_{p,s} h)].$$

Properties of $L_{p,s}$ supremal-convolution

Let $f, g, h\colon \mathbb{R}^n \to \mathbb{R}_+$ be arbitrary, not identically zero, functions defined on $\mathbb{R}^n$, and let $s \in [-\infty, \infty]$, $p \geq 1$, and $\alpha, \beta, \gamma > 0$.

- ★ Homogeneity:

$$(\alpha \times_{p,s} f) \oplus_{p,s} (\beta \times_{p,s} g) = (\alpha+\beta) \times_{p,s} \left[\left(\frac{\alpha}{\alpha+\beta} \times_{p,s} f \right) \oplus_{p,s} \left(\frac{\beta}{\alpha+\beta} \times_{p,s} g \right) \right]$$

  for $s \neq \pm\infty$.

- ★ Measurability: $(\alpha \times_{p,s} f) \oplus_{p,s} (\beta \times_{p,s} g)$ is measurable whenever both f and g are Borel measurable.
- ★ Commutativity: $(\alpha \times_{p,s} f) \oplus_{p,s} (\beta \times_{p,s} g) = (\beta \times_{p,s} g) \oplus_{p,s} (\alpha \times_{p,s} f)$.
- ✧ Associativity:

$$[(\alpha \times_{p,s} f) \oplus_{p,s} (\beta \times_{p,s} g)] \oplus_{p,s} (\gamma \times_{p,s} h) \neq (\alpha \times_{p,s} f) \oplus_{p,s} [(\beta \times_{p,s} g) \oplus_{p,s} (\gamma \times_{p,s} h)].$$

Remark: For $p = 1$, the first three rules recover the result of S. G. Bobkov, A. Colesanti, I. Fragalà, MM, 2014. The last Associativity rule only works for $p = 1$.

$L_{p,s}$ inf-sup convolution

- $p : 0 < p < 1,\ 1/p + 1/q = 1,$

$L_{p,s}$ inf-sup convolution

- $p : 0 < p < 1,\ 1/p + 1/q = 1,$
- $s \in [-\infty, \infty],$

$L_{p,s}$ inf-sup convolution

- $p : 0 < p < 1$, $1/p + 1/q = 1$,
- $s \in [-\infty, \infty]$,
- $f, g : \mathbb{R}^n \to \mathbb{R}_+$ Borel measurable functions with $o \in int(supp(f))$, $o \in int(supp(g))$,

$L_{p,s}$ inf-sup convolution

- $p : 0 < p < 1$, $1/p + 1/q = 1$,
- $s \in [-\infty, \infty]$,
- $f, g : \mathbb{R}^n \to \mathbb{R}_+$ Borel measurable functions with $o \in int(supp(f))$, $o \in int(supp(g))$,
- $\alpha, \beta > 0$.

$L_{p,s}$ inf-sup convolution

- $p : 0 < p < 1,\ 1/p + 1/q = 1,$
- $s \in [-\infty, \infty],$
- $f, g : \mathbb{R}^n \to \mathbb{R}_+$ Borel measurable functions with $o \in int(supp(f))$, $o \in int(supp(g))$,
- $\alpha, \beta > 0.$

$L_{p,s}$ inf-sup-convolution

$$[\alpha \times_{p,s} f \oplus_{p,s} \beta \times_{p,s} g](z)$$
$$:= \inf_{0 \le \lambda \le 1} \left[\sup_{z = \alpha^{\frac{1}{p}} (1-\lambda)^{\frac{1}{q}} x + \beta^{\frac{1}{p}} \lambda^{\frac{1}{q}} y} M_s^{\left((1-\lambda)^{\frac{1}{q}}, \lambda^{\frac{1}{q}}\right)} \left(\alpha^{\frac{1}{sp}} f(x), \beta^{\frac{1}{sp}} g(y)\right) \right].$$

L_p-Borell-Brascamp-Lieb inequality

- $p : p \geq 1,\ 1/p + 1/q = 1,$

L_p-Borell-Brascamp-Lieb inequality

- $p : p \geq 1$, $1/p + 1/q = 1$,
- $s : s \geq 0$,

L_p-Borell-Brascamp-Lieb inequality

- $p : p \geq 1$, $1/p + 1/q = 1$,
- $s : s \geq 0$,
- $f, g, h \colon \mathbb{R}^n \to \mathbb{R}_+$ bounded integrable functions,

L_p-Borell-Brascamp-Lieb inequality

- $p : p \geq 1$, $1/p + 1/q = 1$,
- $s : s \geq 0$,
- $f, g, h \colon \mathbb{R}^n \to \mathbb{R}_+$ bounded integrable functions,
- For $\lambda \in [0,1]$ and $t \in [0,1]$, denote L_p coefficients

$$C_{p,\lambda,t} := (1-t)^{\frac{1}{p}}(1-\lambda)^{\frac{1}{q}}, \quad D_{p,\lambda,t} := t^{\frac{1}{p}}\lambda^{\frac{1}{q}}.$$

L_p-Borell-Brascamp-Lieb inequality

- $p : p \geq 1$, $1/p + 1/q = 1$,
- $s : s \geq 0$,
- $f, g, h \colon \mathbb{R}^n \to \mathbb{R}_+$ bounded integrable functions,
- For $\lambda \in [0,1]$ and $t \in [0,1]$, denote L_p coefficients

$$C_{p,\lambda,t} := (1-t)^{\frac{1}{p}}(1-\lambda)^{\frac{1}{q}}, \quad D_{p,\lambda,t} := t^{\frac{1}{p}}\lambda^{\frac{1}{q}}.$$

M. Roysdon and S. Xing, Trans. Amer. Math. Soc., 2021

If

$$h\left(C_{p,\lambda,t}x + D_{p,\lambda,t}y\right) \geq \left[C_{p,\lambda,t}f(x)^s + D_{p,\lambda,t}g(y)^s\right]^{\frac{1}{s}}$$

for every $x \in \operatorname{supp}(f)$, $y \in \operatorname{supp}(g)$ and every $\lambda \in [0,1]$, then

$$I(h) \geq M_{\frac{ps}{1+ns}}^{((1-t),t)}\left(I(f), I(g)\right).$$

L_p-Borell-Brascamp-Lieb inequality

- $p : p \geq 1$, $1/p + 1/q = 1$,
- $s : s \geq 0$,
- $f, g, h \colon \mathbb{R}^n \to \mathbb{R}_+$ bounded integrable functions,
- For $\lambda \in [0,1]$ and $t \in [0,1]$, denote L_p coefficients

$$C_{p,\lambda,t} := (1-t)^{\frac{1}{p}}(1-\lambda)^{\frac{1}{q}}, \quad D_{p,\lambda,t} := t^{\frac{1}{p}}\lambda^{\frac{1}{q}}.$$

M. Roysdon and S. Xing, Trans. Amer. Math. Soc., 2021

If

$$h\left(C_{p,\lambda,t}x + D_{p,\lambda,t}y\right) \geq \left[C_{p,\lambda,t}f(x)^s + D_{p,\lambda,t}g(y)^s\right]^{\frac{1}{s}}$$

for every $x \in \operatorname{supp}(f)$, $y \in \operatorname{supp}(g)$ and every $\lambda \in [0,1]$, then

$$I(h) \geq M_{\frac{ps}{1+ns}}^{((1-t),t)}\left(I(f), I(g)\right).$$

- Method: Revolution bodies & L_p Brunn-Minkowski inequality.

L_p-Borell-Brascamp-Lieb inequality

L_p-Borell-Brascamp-Lieb inequality for $s \in (-\infty, \infty)$

[1] Let $p \geq 1$, $-\infty < s < \infty$, and $t \in (0,1)$. Let $f, g, h\colon \mathbb{R}^n \to \mathbb{R}_+$ be a triple of bounded integrable functions and

$$h\left(C_{p,\lambda,t}x + D_{p,\lambda,t}y\right) \geq \left[C_{p,\lambda,t}f(x)^s + D_{p,\lambda,t}g(y)^s\right]^{\frac{1}{s}}$$

for every $x \in \mathrm{supp}(f)$, $y \in \mathrm{supp}(g)$ and every $\lambda \in [0,1]$. Then for $\gamma = \frac{s}{1+ns}$,

$$I(h) \geq \begin{cases} M_{p\gamma}^{((1-t),t)}\left(I(f), I(g)\right), & \text{if } s \geq -\frac{1}{n}, \\ \min\left\{\left[C_{p,\lambda,t}\right]^{\frac{1}{\gamma}} I(f), \left[D_{p,\lambda,t}\right]^{\frac{1}{\gamma}} I(g)\right), & \text{if } s < -\frac{1}{n}. \end{cases}$$

1

[1] M. Roysdon and S. Xing, *On the framework of L_p summations for functions*, 2021.

L_p-Borell-Brascamp-Lieb inequality

L_p-Borell-Brascamp-Lieb inequality for $s \in (-\infty, \infty)$

[1] Let $p \geq 1$, $-\infty < s < \infty$, and $t \in (0,1)$. Let $f, g, h \colon \mathbb{R}^n \to \mathbb{R}_+$ be a triple of bounded integrable functions and

$$h\left(C_{p,\lambda,t}x + D_{p,\lambda,t}y\right) \geq \left[C_{p,\lambda,t}f(x)^s + D_{p,\lambda,t}g(y)^s\right]^{\frac{1}{s}}$$

for every $x \in \mathrm{supp}(f)$, $y \in \mathrm{supp}(g)$ and every $\lambda \in [0,1]$. Then for $\gamma = \frac{s}{1+ns}$,

$$I(h) \geq \begin{cases} M_{p\gamma}^{((1-t),t)}\left(I(f), I(g)\right), & \text{if } s \geq -\frac{1}{n}, \\ \min\left\{[C_{p,\lambda,t}]^{\frac{1}{\gamma}}\, I(f), [D_{p,\lambda,t}]^{\frac{1}{\gamma}}\, I(g)\right), & \text{if } s < -\frac{1}{n}. \end{cases}$$

- $p = 1, s \geq -\frac{1}{n}$: the classical BBL inequality.

1

[1] M. Roysdon and S. Xing, *On the framework of L_p summations for functions*, 2021.

L_p-Borell-Brascamp-Lieb inequality

L_p-Borell-Brascamp-Lieb inequality for $s \in (-\infty, \infty)$

[1] Let $p \geq 1$, $-\infty < s < \infty$, and $t \in (0,1)$. Let $f, g, h\colon \mathbb{R}^n \to \mathbb{R}_+$ be a triple of bounded integrable functions and

$$h\left(C_{p,\lambda,t}x + D_{p,\lambda,t}y\right) \geq \left[C_{p,\lambda,t}f(x)^s + D_{p,\lambda,t}g(y)^s\right]^{\frac{1}{s}}$$

for every $x \in \operatorname{supp}(f)$, $y \in \operatorname{supp}(g)$ and every $\lambda \in [0,1]$. Then for $\gamma = \frac{s}{1+ns}$,

$$I(h) \geq \begin{cases} M_{p\gamma}^{((1-t),t)}\left(I(f), I(g)\right), & \text{if } s \geq -\frac{1}{n}, \\ \min\left\{[C_{p,\lambda,t}]^{\frac{1}{\gamma}}\, I(f), [D_{p,\lambda,t}]^{\frac{1}{\gamma}}\, I(g)\right), & \text{if } s < -\frac{1}{n}. \end{cases}$$

- $p = 1, s \geq -\frac{1}{n}$: the classical BBL inequality.
- $p = 1, s < -\frac{1}{n}$: the case solved by S. Dancs and B. Uhrin, JMAA, 1980.

1

[1] M. Roysdon and S. Xing, *On the framework of L_p summations for functions*, 2021.

L_p-Borell-Brascamp-Lieb inequality

L_p-Borell-Brascamp-Lieb inequality for $s \in (-\infty, \infty)$

[1] Let $p \geq 1$, $-\infty < s < \infty$, and $t \in (0,1)$. Let $f, g, h\colon \mathbb{R}^n \to \mathbb{R}_+$ be a triple of bounded integrable functions and

$$h\left(C_{p,\lambda,t}x + D_{p,\lambda,t}y\right) \geq \left[C_{p,\lambda,t}f(x)^s + D_{p,\lambda,t}g(y)^s\right]^{\frac{1}{s}}$$

for every $x \in \operatorname{supp}(f)$, $y \in \operatorname{supp}(g)$ and every $\lambda \in [0,1]$. Then for $\gamma = \frac{s}{1+ns}$,

$$I(h) \geq \begin{cases} M_{p\gamma}^{((1-t),t)}\left(I(f), I(g)\right), & \text{if } s \geq -\frac{1}{n}, \\ \min\left\{\left[C_{p,\lambda,t}\right]^{\frac{1}{\gamma}} I(f), \left[D_{p,\lambda,t}\right]^{\frac{1}{\gamma}} I(g)\right), & \text{if } s < -\frac{1}{n}. \end{cases}$$

- $p = 1, s \geq -\frac{1}{n}$: the classical BBL inequality.
- $p = 1, s < -\frac{1}{n}$: the case solved by S. Dancs and B. Uhrin, JMAA, 1980.
- Method: Optimal transportation & Classic BBL inequality.

1

[1] M. Roysdon and S. Xing, *On the framework of L_p summations for functions*, 2021.

$L_{p,s}$ concavity definitions

- $p : p \geq 1,\ 1/p + 1/q = 1,$

$L_{p,s}$ concavity definitions

- $p : p \geq 1,\ 1/p + 1/q = 1,$
- $s \in [-\infty, +\infty],$

$L_{p,s}$ concavity definitions

- $p : p \geq 1,\ 1/p + 1/q = 1,$
- $s \in [-\infty, +\infty],$

$L_{p,s}$-concave measures

A non-negative measure μ on $\mathbb{R}^n$ is $L_{p,s}$-concave if, for any pair of Borel measurable sets $A, B \subset \mathbb{R}^n$, one has

$$\mu\left(C_{p,\lambda,t}A + D_{p,\lambda,t}B\right) \geq M_s^{(C_{p,\lambda,t},D_{p,\lambda,t})}(\mu(A), \mu(B))$$

for every $\lambda \in [0,1]$ and $t \in [0,1]$.

$L_{p,s}$ concavity definitions

- $p : p \geq 1,\ 1/p + 1/q = 1,$
- $s \in [-\infty, +\infty],$

$L_{p,s}$-concave measures

A non-negative measure μ on $\mathbb{R}^n$ is $L_{p,s}$-concave if, for any pair of Borel measurable sets $A, B \subset \mathbb{R}^n$, one has

$$\mu\left(C_{p,\lambda,t}A + D_{p,\lambda,t}B\right) \geq M_s^{(C_{p,\lambda,t},D_{p,\lambda,t})}(\mu(A), \mu(B))$$

for every $\lambda \in [0, 1]$ and $t \in [0, 1]$.

- $s = -\infty$: $L_{p,s}$-quasi-concave measure,

$L_{p,s}$ concavity definitions

- $p : p \geq 1,\ 1/p + 1/q = 1,$
- $s \in [-\infty, +\infty],$

$L_{p,s}$-concave measures

A non-negative measure μ on $\mathbb{R}^n$ is $L_{p,s}$-concave if, for any pair of Borel measurable sets $A, B \subset \mathbb{R}^n$, one has

$$\mu\left(C_{p,\lambda,t}A + D_{p,\lambda,t}B\right) \geq M_s^{(C_{p,\lambda,t},D_{p,\lambda,t})}(\mu(A), \mu(B))$$

for every $\lambda \in [0, 1]$ and $t \in [0, 1]$.

- $s = -\infty$: $L_{p,s}$-quasi-concave measure,
- $s = 0$: $L_{p,s}$-log-concave measure,

$L_{p,s}$ concavity definitions

- $p : p \geq 1,\ 1/p + 1/q = 1,$
- $s \in [-\infty, +\infty],$

$L_{p,s}$-concave measures

A non-negative measure μ on $\mathbb{R}^n$ is $L_{p,s}$-concave if, for any pair of Borel measurable sets $A, B \subset \mathbb{R}^n$, one has

$$\mu\left(C_{p,\lambda,t}A + D_{p,\lambda,t}B\right) \geq M_s^{(C_{p,\lambda,t},D_{p,\lambda,t})}(\mu(A), \mu(B))$$

for every $\lambda \in [0,1]$ and $t \in [0,1]$.

- $s = -\infty$: $L_{p,s}$-quasi-concave measure,
- $s = 0$: $L_{p,s}$-log-concave measure,
- $p = 1$: the classical s-concave measure,

$L_{p,s}$ concavity definitions

- $p : p \geq 1,\ 1/p + 1/q = 1,$
- $s \in [-\infty, +\infty],$

$L_{p,s}$-concave measures

A non-negative measure μ on $\mathbb{R}^n$ is $L_{p,s}$-concave if, for any pair of Borel measurable sets $A, B \subset \mathbb{R}^n$, one has

$$\mu\left(C_{p,\lambda,t}A + D_{p,\lambda,t}B\right) \geq M_s^{(C_{p,\lambda,t},D_{p,\lambda,t})}(\mu(A), \mu(B))$$

for every $\lambda \in [0,1]$ and $t \in [0,1]$.

- $s = -\infty$: $L_{p,s}$-quasi-concave measure,
- $s = 0$: $L_{p,s}$-log-concave measure,
- $p = 1$: the classical s-concave measure,
- If the measure $\mu : \mathbb{R}^n \to \mathbb{R}$ is replaced by the function $f : \mathbb{R}^n \to \mathbb{R}$, then we obtain $L_{p,s}$ concavities definitions for functions.

$L_{p,s}$ concavities

- $p : p \geq 1$, $1/p + 1/q = 1$,

$L_{p,s}$ concavities

- $p : p \geq 1$, $1/p + 1/q = 1$,
- $s, \beta :\in [-\infty, +\infty]$ be such that $s + \beta \geq 0$,

$L_{p,s}$ concavities

- $p : p \geq 1,\ 1/p + 1/q = 1,$
- $s, \beta :\in [-\infty, +\infty]$ be such that $s + \beta \geq 0,$
- f: $L_{p,s}$-concave,
  g: $L_{p,\beta}$-concave.

$L_{p,s}$ concavities

- $p : p \geq 1$, $1/p + 1/q = 1$,
- $s, \beta :\in [-\infty, +\infty]$ be such that $s + \beta \geq 0$,
- f: $L_{p,s}$-concave,
  g: $L_{p,\beta}$-concave.

Then the convolution of f and g,

$$(f * g)(z) = \int_{\mathbb{R}^n} f(x) g(z - x) dx$$

$L_{p,s}$ concavities

- $p : p \geq 1,\ 1/p + 1/q = 1,$
- $s, \beta :\in [-\infty, +\infty]$ be such that $s + \beta \geq 0,$
- f: $L_{p,s}$-concave,
  g: $L_{p,\beta}$-concave.

Then the convolution of f and g,

$$(f * g)(z) = \int_{\mathbb{R}^n} f(x)g(z-x)dx$$

1. is $L_{p,(s^{-1}+\beta^{-1}+n)^{-1}}$-concave whenever $\frac{s\beta}{s+\beta} \in \left[-\frac{1}{n}, +\infty\right),$

$L_{p,s}$ concavities

- $p : p \geq 1$, $1/p + 1/q = 1$,
- $s, \beta :\in [-\infty, +\infty]$ be such that $s + \beta \geq 0$,
- f: $L_{p,s}$-concave,
  g: $L_{p,\beta}$-concave.

Then the convolution of f and g,

$$(f * g)(z) = \int_{\mathbb{R}^n} f(x)g(z-x)dx$$

1. is $L_{p,(s^{-1}+\beta^{-1}+n)^{-1}}$-concave whenever $\frac{s\beta}{s+\beta} \in \left[-\frac{1}{n}, +\infty\right)$,
2. is $L_{p,(s^{-1}+\beta^{-1}+n)}$-quasi-concave whenever $\frac{s\beta}{s+\beta} \in \left(-\infty, -\frac{1}{n}\right)$.

$L_{p,s}$ concavities

- $p : p \geq 1,\ 1/p + 1/q = 1$,
- $s, \beta :\in [-\infty, +\infty]$ be such that $s + \beta \geq 0$,
- f: $L_{p,s}$-concave,
  g: $L_{p,\beta}$-concave.

Then the convolution of f and g,

$$(f * g)(z) = \int_{\mathbb{R}^n} f(x) g(z - x) dx$$

1. is $L_{p,(s^{-1}+\beta^{-1}+n)^{-1}}$-concave whenever $\frac{s\beta}{s+\beta} \in \left[-\frac{1}{n}, +\infty\right)$,
2. is $L_{p,(s^{-1}+\beta^{-1}+n)}$-quasi-concave whenever $\frac{s\beta}{s+\beta} \in \left(-\infty, -\frac{1}{n}\right)$.

- $p = 1$: Uhrin, JMAA, 1980.

The $L_{p,s}$ Asplund summation for $p \geq 1$

- Given $\alpha, \beta \geq 0$ and $u, v \in C_s(\mathbb{R}^n)$, the L_p additions of u, v (base functions):

$$[(\alpha \boxtimes_p u) \boxplus_p (\beta \boxtimes_p v)](x) := \{(\alpha(u^*(x))^p + \beta(v^*(x))^p)^{1/p}\}^*.$$

The $L_{p,s}$ Asplund summation for $p \geq 1$

✧ Given $\alpha, \beta \geq 0$ and $u, v \in C_s(\mathbb{R}^n)$, the L_p additions of u, v (base functions):

$$[(\alpha \boxtimes_p u) \boxplus_p (\beta \boxtimes_p v)](x) := \{(\alpha(u^*(x))^p + \beta(v^*(x))^p)^{1/p}\}^*.$$

The $L_{p,s}$ *Asplund summation* for s-concave functions (M. Roysdon and S. Xing, 2021)

For $p \geq 1$, $s \in (-\infty, \infty)$, given $f, g \in \mathscr{F}_s(\mathbb{R}^n)$, the $L_{p,s}$ Asplund summation with weights $\alpha, \beta \geq 0$ is

$$(\alpha \cdot_{p,s} f) \star_{p,s} (\beta \cdot_{p,s} g) := \left(1 - s\left[(\alpha \boxtimes_p u_f) \boxplus_p (\beta \boxtimes_p u_g)\right]\right)_+^{\frac{1}{s}}.$$

The $L_{p,s}$ Asplund summation for $p \geq 1$

- Given $\alpha, \beta \geq 0$ and $u, v \in C_s(\mathbb{R}^n)$, the L_p additions of u, v (base functions):

$$[(\alpha \boxtimes_p u) \boxplus_p (\beta \boxtimes_p v)](x) := \{(\alpha(u^*(x))^p + \beta(v^*(x))^p)^{1/p}\}^*.$$

The $L_{p,s}$ *Asplund summation* for s-concave functions (M. Roysdon and S. Xing, 2021)

For $p \geq 1$, $s \in (-\infty, \infty)$, given $f, g \in \mathscr{F}_s(\mathbb{R}^n)$, the $L_{p,s}$ Asplund summation with weights $\alpha, \beta \geq 0$ is

$$(\alpha \cdot_{p,s} f) \star_{p,s} (\beta \cdot_{p,s} g) := \Big(1 - s\big[(\alpha \boxtimes_p u_f) \boxplus_p (\beta \boxtimes_p u_g)\big]\Big)_+^{\frac{1}{s}}.$$

- $p = 1, s = 0$: Asplund summations for log-concave functions by A. Colesanti and Fragalà (2013), Cordero-Erausquin and Klartag (2015), etc.

The $L_{p,s}$ Asplund summation for $p \geq 1$

✧ Given $\alpha, \beta \geq 0$ and $u, v \in C_s(\mathbb{R}^n)$, the L_p additions of u, v (base functions):

$$[(\alpha \boxtimes_p u) \boxplus_p (\beta \boxtimes_p v)](x) := \{(\alpha(u^*(x))^p + \beta(v^*(x))^p)^{1/p}\}^*.$$

The $L_{p,s}$ *Asplund summation* for s-concave functions (M. Roysdon and S. Xing, 2021)

For $p \geq 1$, $s \in (-\infty, \infty)$, given $f, g \in \mathscr{F}_s(\mathbb{R}^n)$, the $L_{p,s}$ Asplund summation with weights $\alpha, \beta \geq 0$ is

$$(\alpha \cdot_{p,s} f) \star_{p,s} (\beta \cdot_{p,s} g) := \Big(1 - s\big[(\alpha \boxtimes_p u_f) \boxplus_p (\beta \boxtimes_p u_g)\big]\Big)_+^{\frac{1}{s}}.$$

✧ $p = 1, s = 0$: Asplund summations for log-concave functions by A. Colesanti and Fragalà (2013), Cordero-Erausquin and Klartag (2015), etc.

✧ $p \geq 1, s = 0$: L_p Asplund summations for log-concave functions by N. Fang, S. Xing and D. Ye, 2020.

The $L_{p,s}$ Asplund summation for $p \geq 1$

✧ Given $\alpha, \beta \geq 0$ and $u, v \in C_s(\mathbb{R}^n)$, the L_p additions of u, v (base functions):

$$[(\alpha \boxtimes_p u) \boxplus_p (\beta \boxtimes_p v)](x) := \{(\alpha(u^*(x))^p + \beta(v^*(x))^p)^{1/p}\}^*.$$

The $L_{p,s}$ *Asplund summation* for s-concave functions (M. Roysdon and S. Xing, 2021)

For $p \geq 1$, $s \in (-\infty, \infty)$, given $f, g \in \mathscr{F}_s(\mathbb{R}^n)$, the $L_{p,s}$ Asplund summation with weights $\alpha, \beta \geq 0$ is

$$(\alpha \cdot_{p,s} f) \star_{p,s} (\beta \cdot_{p,s} g) := \Big(1 - s\big[(\alpha \boxtimes_p u_f) \boxplus_p (\beta \boxtimes_p u_g)\big]\Big)_+^{\frac{1}{s}}.$$

✧ $p = 1, s = 0$: Asplund summations for log-concave functions by A. Colesanti and Fragalà (2013), Cordero-Erausquin and Klartag (2015), etc.

✧ $p \geq 1, s = 0$: L_p Asplund summations for log-concave functions by N. Fang, S. Xing and D. Ye, 2020.

✧ Functional space equipped with $L_{p,s}$ Asplund summation: $(\mathbb{F}, \star_{p,s}, I(\cdot))$.

The $L_{p,s}$ Asplund summation for $0 < p < 1$

- Support function for $f = (1 - su_f)_+^{1/s} \in \mathscr{F}_s(\mathbb{R}^n)$:

$$h_f = (u_f)^*.$$

The $L_{p,s}$ Asplund summation for $0 < p < 1$

- Support function for $f = (1 - su_f)_+^{1/s} \in \mathscr{F}_s(\mathbb{R}^n)$:

$$h_f = (u_f)^*.$$

- s-Alexandrov Function:

$$A[u]_s = (1 - su^*)_+^{1/s}$$

for $s \in (-\infty, \infty)$.

The $L_{p,s}$ Asplund summation for $0 < p < 1$

- Support function for $f = (1 - su_f)_+^{1/s} \in \mathscr{F}_s(\mathbb{R}^n)$:

$$h_f = (u_f)^*.$$

- s-Alexandrov Function:

$$A[u]_s = (1 - su^*)_+^{1/s}$$

for $s \in (-\infty, \infty)$.

The $L_{p,s}$ Asplund summation for $0 < p < 1$

For $0 < p < 1$, $s \in (-\infty, \infty)$, given $f, g \in \mathscr{F}_s(\mathbb{R}^n)$ with $h_f, h_g \geq 0$, the $L_{p,s}$ Asplund summation $\alpha \cdot_{p,s} f \star_{p,s} \beta \cdot_{p,s} g$ with weights $\alpha, \beta \geq 0$ is

$$\alpha \cdot_{p,s} f \star_{p,s} \beta \cdot_{p,s} g := A\left[\left(\alpha h_f^p + \beta h_g^p\right)^{1/p}\right]_s.$$

The $L_{p,s}$ Asplund summation for $0 < p < 1$

- Support function for $f = (1 - su_f)_+^{1/s} \in \mathscr{F}_s(\mathbb{R}^n)$:

$$h_f = (u_f)^*.$$

- s-Alexandrov Function:

$$A[u]_s = (1 - su^*)_+^{1/s}$$

for $s \in (-\infty, \infty)$.

The $L_{p,s}$ Asplund summation for $0 < p < 1$

For $0 < p < 1$, $s \in (-\infty, \infty)$, given $f, g \in \mathscr{F}_s(\mathbb{R}^n)$ with $h_f, h_g \geq 0$, the $L_{p,s}$ Asplund summation $\alpha \cdot_{p,s} f \star_{p,s} \beta \cdot_{p,s} g$ with weights $\alpha, \beta \geq 0$ is

$$\alpha \cdot_{p,s} f \star_{p,s} \beta \cdot_{p,s} g := A\left[\left(\alpha h_f^p + \beta h_g^p\right)^{1/p}\right]_s.$$

- $s = 0$: Asplund summations for log-concave functions by L. Rotem, JAA.

Quermassintegral for functions

- Projection function $\big(P_H f\big)(z) := \sup_{y \in H^\perp} f(z+y), \qquad f \in \mathscr{F}_s(\mathbb{R}^n).$

Quermassintegral for functions

- Projection function $\big(P_H f\big)(z) := \sup_{y \in H^\perp} f(z+y), \qquad f \in \mathscr{F}_s(\mathbb{R}^n).$

Quermassintegral of function

For functions in $\mathscr{F}_s(\mathbb{R}^n)$ and $j \in \{0, \cdots, n-1\}$, the j-th quermassintegral of function f,

$$W_j(f) := c_{n,j} \int\limits_{G_{n,n-j}} \int\limits_{H} P_H f(x) dx \, d\nu_{n,n-j}(H).$$

Quermassintegral for functions

- Projection function $\big(P_H f\big)(z) := \sup_{y \in H^\perp} f(z+y), \qquad f \in \mathscr{F}_s(\mathbb{R}^n).$

Quermassintegral of function

For functions in $\mathscr{F}_s(\mathbb{R}^n)$ and $j \in \{0, \cdots, n-1\}$, the j-th quermassintegral of function f,

$$W_j(f) := c_{n,j} \int_{G_{n,n-j}} \int_H P_H f(x) dx \, d\nu_{n,n-j}(H).$$

- $W_j(f) = \int_0^\infty W_j(\{x \in \mathbb{R}^n : f(x) \geq t\}) dt.$

Quermassintegral for functions

✦ Projection function $\big(P_H f\big)(z) := \sup_{y \in H^\perp} f(z+y), \qquad f \in \mathscr{F}_s(\mathbb{R}^n)$.

Quermassintegral of function

For functions in $\mathscr{F}_s(\mathbb{R}^n)$ and $j \in \{0, \cdots, n-1\}$, the j-th quermassintegral of function f,

$$W_j(f) := c_{n,j} \int_{G_{n,n-j}} \int_H P_H f(x) dx\, d\nu_{n,n-j}(H).$$

✧ $W_j(f) = \int_0^\infty W_j(\{x \in \mathbb{R}^n : f(x) \geq t\}) dt.$

✧ $f = \chi_K$: $W_j(f) = W_j(K)$, for $K \in \mathscr{K}^n_{(o)}$.

Quermassintegral for functions

- Projection function $\big(P_H f\big)(z) := \sup_{y\in H^\perp} f(z+y), \qquad f \in \mathscr{F}_s(\mathbb{R}^n).$

Quermassintegral of function

For functions in $\mathscr{F}_s(\mathbb{R}^n)$ and $j \in \{0, \cdots, n-1\}$, the j-th quermassintegral of function f,

$$W_j(f) := c_{n,j} \int_{G_{n,n-j}} \int_H P_H f(x)dx\, d\nu_{n,n-j}(H).$$

- $W_j(f) = \int_0^\infty W_j(\{x \in \mathbb{R}^n : f(x) \geq t\})dt.$
- $f = \chi_K$: $W_j(f) = W_j(K)$, for $K \in \mathscr{K}^n_{(o)}$.
- $\alpha \in [-1, \frac{1}{n-j}]$, $\gamma \in [-\alpha, \infty)$, $f, g \in \mathscr{F}_\alpha(\mathbb{R}^n)$, $p \geq 1$:
  $W_j((1-t) \times_{p,\alpha} f \oplus_{p,\alpha} t \times_{p,\alpha} g) \geq [(1-t)W_j(f)^\beta + tW_j(g)^\beta]^{1/\beta}$, $\beta = \frac{p\alpha\gamma}{\alpha+\gamma}$.

$L_{p,s}$ mixed quermassintegral

Variation formula of quermassintegral

The $L_{p,s}$ mixed quermassintegral for s-concave functions $f, g \in \mathscr{F}_s(\mathbb{R}^n)$ is

$$\begin{aligned} W^s_{p,j}(f,g) & \\ := \quad & \frac{1}{n-j} \lim_{\varepsilon \to 0} \frac{W_j(f \star_{p,s} \varepsilon \cdot_{p,s} g) - W_j(f)}{\varepsilon} \\ = \quad & \frac{1}{n-j} \int_{\mathbb{R}^n} \frac{[1 - s(u_f)_H(x)]_+^{\frac{1}{s}-1} (u_g^*)_H(\nabla(u_f)_H(x))^p}{\|x\|^j} (u_f^*)_H(\nabla(u_f)_H(x))^{1-p} dx. \end{aligned}$$

$L_{p,s}$ mixed quermassintegral

Variation formula of quermassintegral

The $L_{p,s}$ mixed quermassintegral for s-concave functions $f, g \in \mathscr{F}_s(\mathbb{R}^n)$ is

$$
\begin{aligned}
&W^s_{p,j}(f,g)\\
:=\quad & \frac{1}{n-j}\lim_{\varepsilon\to 0}\frac{W_j(f\star_{p,s}\varepsilon\cdot_{p,s}g)-W_j(f)}{\varepsilon}\\
=\quad & \frac{1}{n-j}\int_{\mathbb{R}^n}\frac{[1-s(u_f)_H(x)]_+^{\frac{1}{s}-1}(u_g^*)_H(\nabla(u_f)_H(x))^p}{\|x\|^j}(u_f^*)_H(\nabla(u_f)_H(x))^{1-p}dx.
\end{aligned}
$$

- $s=0$: $W^0_{p,j}(f,g)=\frac{1}{n-j}\int_{\mathbb{R}^n}\frac{e^{-(u_f)_H(x)}(u_g^*)_H(\nabla(u_f)_H(x))^p(u_f^*)_H(\nabla(u_f)_H(x))^{1-p}}{\|x\|^j}dx.$

$L_{p,s}$ mixed quermassintegral

Variation formula of quermassintegral

The $L_{p,s}$ mixed quermassintegral for s-concave functions $f, g \in \mathscr{F}_s(\mathbb{R}^n)$ is

$$\begin{aligned} & W^s_{p,j}(f,g) \\ := \quad & \frac{1}{n-j} \lim_{\varepsilon \to 0} \frac{W_j(f \star_{p,s} \varepsilon \cdot_{p,s} g) - W_j(f)}{\varepsilon} \\ = \quad & \frac{1}{n-j} \int_{\mathbb{R}^n} \frac{[1 - s(u_f)_H(x)]_+^{\frac{1}{s}-1} (u_g^*)_H(\nabla(u_f)_H(x))^p}{\|x\|^j} (u_f^*)_H(\nabla(u_f)_H(x))^{1-p} dx. \end{aligned}$$

- $s = 0$: $W^0_{p,j}(f,g) = \frac{1}{n-j} \int_{\mathbb{R}^n} \frac{e^{-(u_f)_H(x)} (u_g^*)_H(\nabla(u_f)_H(x))^p (u_f^*)_H(\nabla(u_f)_H(x))^{1-p}}{\|x\|^j} dx.$
- $j = 0$, $s = 0$:
  (i) L. Rotem's work for $0 < p < 1$;
  (ii) N. Fang, S. Xing and D. Ye's work for $p \geq 1$;
  (iii) $f(x) = \chi_K$, $g = \chi_L$ for $K, L \in \mathscr{K}^n_{(o)}$:

$$W^1_{p,0}(f,g) = V_p(K,L) = \frac{1}{n} \int_{S^{n-1}} h_L^p(u) h_K^{1-p} dS(K,u).$$

Thank you very much!!!