Some results on geometric analysis of Dirac operators

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Introduction

Dirac operator was first introduced by **P.A. Dirac 1928**.
While studing spin-1/2 particles in electron-magnetic fields,
Dirac looked for square root $P = \sqrt{\Delta}$ of $\Delta = -\Sigma_i \partial_{x_i}^2$.

Naturally letting $P := \Sigma_i \gamma_i \partial_{x_i}$, here γ_i's are $n \times n$ matrices, then

$$\gamma_i^2 = -I, \quad \gamma_i \gamma_j + \gamma_j \gamma_i = 0, \forall i \neq j.$$

Algebra generated by this kind of γ_i is called Clifford algebra.

Introduction

Let V be an n-dimensional real vector space, equiped with a inner product $\langle\ ,\ \rangle$.

Definition (Clifford algebra)

The Clifford algebra on V is the algebra generated by all the elements of V and a multiplication "$\cdot$" satisfying

$$v \cdot w + w \cdot v = -2\langle v, w\rangle, \quad \forall v, w \in V. \tag{1}$$

Choosing an orthonormal basis $\{e_1, \cdots, e_n\}$ of V, then

$$e_i^2 = -1, \quad e_i \cdot e_j = -e_j \cdot e_i,\ i \neq j, \quad i = 1, \cdots, n. \tag{2}$$

Introduction

Let (E, M^m) be a vector bundle with metric $\langle , \rangle$ over manifold M.

Definition

$(E, M^m, \langle , \rangle, \cdot, \nabla)$ is a Dirac bundle, if the following properties hold for all $X, Y \in \Gamma(TM), \psi, \varphi \in \Gamma(E)$:

1. $X \cdot Y \cdot \psi + Y \cdot X \cdot \psi = -2 \langle X, Y \rangle \psi$,

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∇ is called a Dirac connection.
The Dirac operator is defined by $D\!\!\!/ := e_i \cdot \nabla_{e_i}$, where e_i is a local orthonormal frame of M.

Introduction

Definition (Spin structure on principal SO(n)-bundle)

Let $(Q, \pi, M^n, SO(n))$ be a principal SO(n)-bundle. A spin structure on Q is a pair (P, Λ) such that
(1) P is a principal Spin(n)-bundle over M;

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(1) P is a principal Spin(n)-bundle over M;
(2) $\Lambda : P \longrightarrow Q$ is a two-sheeted covering map satisfying

$$\begin{array}{ccc} P \times Spin(n) & \longrightarrow & P \\ {\scriptstyle \Lambda\times\lambda}\downarrow & & {\scriptstyle \Lambda}\downarrow \\ Q \times SO(n) & \longrightarrow & Q \end{array}$$

Namely, $\Lambda(pg) = \Lambda(p)\lambda(g), \quad \forall p \in P, \ \forall g \in Spin(n),$
where $\lambda : Spin(n) \to SO(n)$ is the 2-fold covering map.

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Examples (cf. **H.B.Lawson, M.L.Michelson, Spin Geometry,** *Princeton University Press*, Princeton, NJ,1989.):
(i) Homotopy spheres $\mathbb{S}^m (m \geq 2)$.
(ii) Simply-connected Lie groups.
(iii) All the Lie groups, oriented manifolds of dimensions ≤ 3.
(iv) $\mathbb{RP}^n$ with $n = 3 \mod 4$;
$\mathbb{CP}^n$ with n odd, etc.

Introduction

If (M, g) is a spin manifold, then there is a spin bundle ΣM, on which there exists a unique "spin connection", given by

$$\nabla_X \psi = X(\psi) + \frac{1}{2} \sum_{i<j} g(\nabla_X e_i, e_j) e_i \cdot e_j \cdot \psi \tag{3}$$

and it is a metric connection:

$$X\langle \psi, \xi \rangle = \langle \nabla_X \psi, \xi \rangle + \langle \psi, \nabla_X \xi \rangle, \quad \forall X \in \Gamma(TM),\ \psi, \xi \in \Gamma(\Sigma M). \tag{4}$$

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Taking E as the spinor bundle ΣM of M, then the Dirac operator $\not{D}$ is just the classical Dirac operator $\not{\partial} \equiv D$ in geometry (also called the Atiyah-Singer operator).

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Atiyah-Singer studied index theory of elliptic operators on compact manifolds. They found on spin manifold there exists Dirac construction and defined the operator: $\partial\!\!\!/\psi := e_i \cdot \nabla_{e_i}\psi$.

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Index theorems and their generalizations and applications lies in the core of modern pure mathematics.

Besides this, Dirac operators are very useful in other topics in mathematics and physics such as the existence of positive scalar curvature (**Gromov-Lawson, Schoen-Yau, W.P.Zhang**), and the positive mass theorem (**E.Witten 1981**) etc.

1 Introduction

2 Dirac equations

3 Extrinsic Eigenvalues

Dirac equations

We consider Dirac type equations on Riemann surfaces M:

$$\partial\!\!\!/\psi = H_{jkl}\langle\psi^j, \psi^k\rangle\psi^l, \tag{5}$$

where Σ is the spin bundle on M, $\Sigma^n := \overbrace{\Sigma \times \cdots \times \Sigma}^{n}$, $n \in \mathbb{Z}_+$, $\psi = (\psi^1, \psi^2, \cdots, \psi^n) \in \Gamma(\Sigma^n)$, and $H_{jkl} = (H^1_{jkl}, H^2_{jkl}, \cdots, H^n_{jkl})$ $\in C^1(M, \mathbb{R}^n)$.

Denote $|\psi| := (\sum_{i=1}^{n} \langle\psi^i, \psi^i\rangle)^{1/2}$.

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Denote $|\psi| := (\sum_{i=1}^{n}\langle\psi^i, \psi^i\rangle)^{1/2}$.

We note that (5) is conformally invariant.

Dirac equations

Motivations:
In **[C.-Jost-Wang, JMP 2007]**, we introduced the following functional:

$$L_c(\phi,\psi) := \frac{1}{2}\int_M [|d\phi|^2 + \langle\psi, \not{D}\psi\rangle - \frac{1}{6}R_{ikjl}\langle\psi^i,\psi^j\rangle\langle\psi^k,\psi^l\rangle]. \qquad (6)$$

We call critical points (ϕ,ψ) of L_c *Dirac-harmonic maps with curvature term*.

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In **[C.-Jost-Wang, JMP 2007]**, we introduced the following functional:

$$L_c(\phi, \psi) := \frac{1}{2}\int_M [|d\phi|^2 + \langle \psi, D\!\!\!/ \psi\rangle - \frac{1}{6} R_{ikjl}\langle \psi^i, \psi^j\rangle\langle \psi^k, \psi^l\rangle]. \quad (6)$$

We call critical points (ϕ, ψ) of L_c *Dirac-harmonic maps with curvature term*.
This functional comes from the supersymmetry σ-model in superstring theory. The only difference is that here the components of ψ are ordinary spinor fields on M, while in physics they take values in a Grassmann algebra.

Dirac equations

The Euler-Lagrange equations of the functional L_c:

$$\not{D}\psi^i = \frac{1}{3} R^i{}_{jkl} \langle \psi^j, \psi^k \rangle \psi^l, \tag{7}$$

$$\tau^i(\phi) - \frac{1}{2} R^i{}_{lmj} \langle \psi^m, \nabla \phi^l \cdot \psi^j \rangle + \frac{1}{12} h^{ip} R_{mkjl;p} \langle \psi^m, \psi^j \rangle \langle \psi^k, \psi^l \rangle = 0, \tag{8}$$

$i = 1, 2, \cdots, n$, where $R^i{}_{jkl}$ is a component of the curvature tensor of N, $\tau(\phi)$ is the tension field of ϕ, and $R_{mkjl;p}$ denotes the covariant derivatives.

Dirac equations

In particular, if ϕ is a constant map, then (7) becomes

$$\not\partial\psi^i = \frac{1}{3}R^i{}_{jkl}\langle\psi^j,\psi^k\rangle\psi^l, \qquad i = 1, 2, \cdots, n, \tag{9}$$

which is a Dirac equation of type (5).

Dirac equations

Another more classical example of type (5) comes from generalized Weierstrass representation of surfaces in three-manifolds (**T.Friedrich, 1998; I.S.Tamanov, 1997**).

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Theorem (T.Friedrich JGP 1998)

Suppose (M^2, g) is a 2-dimensional orientable Riemannian manifold, $H \in C^\infty(M)$, then the following facts are equivalent:
(1) The universal covering space $\tilde{M}$ of M is isometric immersed into Euclidean space $\mathbb{R}^3$: $(\tilde{M}, g) \to \mathbb{R}^3$ with mean curvature H;

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(1) The universal covering space $\tilde{M}$ *of* M *is isometric immersed into Euclidean space* $\mathbb{R}^3$*:* $(\tilde{M}, g) \to \mathbb{R}^3$ *with mean curvature* H*;*

(2) There is nontrivial solution ψ *for Dirac equation* $\partial\!\!\!/ \psi = H\psi$*, and* $|\psi| \equiv$ *constant.*

Dirac equations

Let M be a compact Riemann surface with fixed spin structure. For any local orthonormal basis $\{e_\alpha\}_{\alpha=1,2}$, one can define the so-called chirality operator $\Gamma := i\, e_1 \cdot e_2 \cdot$ and

$$\Gamma_+ := \frac{1}{2}(Id + \Gamma), \quad \Gamma_- := \frac{1}{2}(Id - \Gamma).$$

Let $U = U(\psi)$, $V = V(\psi)$ be complex functions. We consider the following Dirac equation:

$$\partial\!\!\!/ \psi = [U(\psi)\Gamma_+ + V(\psi)\Gamma_-]\psi. \tag{10}$$

Equation (5) corresponds to the case $U = V = -H|\psi|^2$.

Dirac equations

Surfaces in some 3-Lie groups:

The Dirac equation for surfaces immersed into some three-dimensional Lie groups N take a special form of (10), c.f. **I.S.Tamanov, Russian Mathematical Surveys 2006**:

$$N = SU(2): \qquad U = \bar{V} = -(H - i)|\psi|^2; \tag{11}$$

$$N = Nil: \qquad U = V = -H|\psi|^2 - \frac{i}{2}(|\psi_1|^2 - |\psi_2|^2); \tag{12}$$

$$N = \widetilde{SL_2}: \qquad U = -H|\psi|^2 - i(\frac{3}{2}|\psi_2|^2 - |\psi_1|^2),$$

$$V = -H|\psi|^2 - i(|\psi_2|^2 - \frac{3}{2}|\psi_1|^2). \tag{13}$$

Dirac equations

In **[C.-Jost-Wang, AGAG 2008]**, we considered geometric analysis of the above type of equations and obtained:

Regularity:
Small energy regularity theorem;

Removable singularity theorem;

Blow up analysis:

Energy identity: $\lim_{n\to+\infty} E(\psi_m) = E(\psi) + \sum_{k=1}^{K}\sum_{a=1}^{A_k} E(\xi_k^a).$

Dirac equations

[C.Y.Wang, PAMS 2010] proved that weak solutions of

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on Riemannian surfaces must be smooth, which answered a question raised in **[C.-Jost-Wang, AGAG 2008]**.

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The energy identity was improved by **[M.Zhu, PAMS 2016]**.

Dirac equations

Boundary value problems for Dirac equations:

Dirac equations

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Although the index theorems and Fredholm theorems give us information or criteria for the existence of solutions, in many cases, for an elliptic boundary problem and given boundary data, one needs more direct results about the existence and uniqueness of solutions.

This is our motivation for studying the boundary values problems for Dirac equations.

Dirac equations
Boundary value conditions

Definition (Chiral boundary operator)

Let E be a Dirac bundle, and $G \in \mathrm{End}(E)$ be a chiral operator, i.e.,

$$G^* = G, \quad G^2 = \mathrm{Id}, \quad GX\cdot = -X\cdot G, \quad \nabla G = 0, \quad \forall X \in TM.$$

The chiral boundary operator $\mathcal{B}^{\pm}_{chi}$ is defined by

$$\mathcal{B}^{\pm}_{chi} = \frac{1}{2}\left(\mathrm{Id} \pm \mathbf{n}\cdot G\right).$$

Where $\mathbf{n}$ is the unit normal vector of the boundary ∂M.

Dirac equations

Boundary value conditions

Definition (J-boundary operator)

Let E be a Dirac bundle, and $J \in \mathrm{End}(E)$ be a J-operator, i.e.,

$$J^* = -J, \quad J^2 = -\mathrm{Id}, \quad JX\cdot = X \cdot J, \quad \nabla J = 0, \quad \forall X \in TM.$$

The J-boundary operator $\mathcal{B}_J^\pm$ is defined by

$$\mathcal{B}_J^\pm = \frac{1}{2}\left(\mathrm{Id} \pm \mathbf{n} \cdot J\right).$$

Where $\mathbf{n}$ is the unit normal vector of the boundary ∂M.

Denote by $\mathcal{B}$ be one of $\mathcal{B}_{chi}^\pm$ or $\mathcal{B}_J^\pm$.

Dirac equations

BVP of Dirac equations

Consider the BVP

$$\begin{cases} D\!\!\!/\,\psi = \varphi, & \text{in } \mathring{M}; \\ \mathcal{B}\psi = \mathcal{B}\psi_0, & \text{on } \partial M. \end{cases} \tag{15}$$

Dirac equations
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Consider the BVP

$$\begin{cases} \not{D}\psi = \varphi, & \text{in } \mathring{M}; \\ \mathcal{B}\psi = \mathcal{B}\psi_0, & \text{on } \partial M. \end{cases} \tag{15}$$

Theorem (Bartnik-Chruśiel, Crelle's J. 2005)

The BVP (15) *is solvable in* $H^1(E)$ *if and only if*

$$\int_M \langle \varphi, \eta \rangle + \int_{\partial M} \langle \mathcal{B}\psi_0, \mathbf{n} \cdot \eta \rangle = 0, \quad \forall \eta \in \ker(\not{D}^*, \mathcal{B}^*). \tag{16}$$

Moreover, $\|\psi\|_{H^1(M)} \le C \left(\|\varphi\|_{L^2(M)} + \|\mathcal{B}\psi_0\|_{H^{1/2}(\partial M)} + \|\psi\|_{L^2(M)} \right).$

Suppose $p^* > 1$ if $m = 2$; $p^* > (3m-2)/4$ if $m > 2$.

Theorem (C-Jost-Sun-Zhu, JEMS 2019)

For any $1 < p < p^*$, *the BVP*

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admits a unique solution $\psi \in W^{1,p}(M; E)$, *here* $\varphi \in L^p(M; E)$ *and* $\mathcal{B}\psi_0 \in W^{1-1/p,p}(\partial M; E)$.

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Moreover, ψ *satisfies the following estimate*

$$\|\psi\|_{W^{1,p}(M)} \leq C\left(\|\varphi\|_{L^p(M)} + \|\mathcal{B}\psi_0\|_{W^{1-1/p,p}(\partial M)}\right). \tag{18}$$

Extrinsic Eigenvalues

Submanifold Dirac operators:

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Submanifold Dirac operators:

The submanifold theory for Dirac operators was introduced by **C.Bär 1998**.

Let M^m be a closed spin submanifold embedded in a spin manifold $\bar{M}^{m+n}$.

By Milnor's Lemma there is a unique spin structure on the normal bundle N. Denoted by $\Sigma\bar{M}, \Sigma M$ and ΣN the spinor bundles of $\bar{M}, M$ and N respectively.

Extrinsic Eigenvalues

The spinor bundles $\Sigma\bar{M}|_M = \Sigma M \otimes \Sigma N$ unless m and n are both odd in which case $\Sigma\bar{M}|_M = (\Sigma M \otimes \Sigma N) \oplus (\Sigma M \otimes \Sigma N)$.

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$$\nabla_X^{\Sigma\bar{M}|_M} = \nabla_X^{\Sigma M} \otimes \mathrm{Id} + \mathrm{Id} \otimes \nabla_X^{\Sigma N} + \frac{1}{2}\sum_{\alpha=1}^{n} \bar{\gamma}(A^{\alpha}(X) \cdot \nu_{\alpha}),$$

where A^{α} is the shape operator w.r.t. ν_{α}.

Extrinsic Eigenvalues

The spinorial curvature operator satisfies

$$
\begin{aligned}
& R^{\Sigma \bar{M}|_M}(X, Y) \\
& = R^{\Sigma M}(X, Y) \otimes \mathrm{Id} + \mathrm{Id} \otimes R^{\Sigma N}(X, Y) + \frac{1}{4} \sum_{\alpha=1}^{n} \gamma([A^\alpha(X), A^\alpha(Y)]) \otimes \mathrm{Id} \\
& + \frac{1}{4} \sum_{\alpha,\beta=1}^{n} \left(\left\langle A^\alpha(X), A^\beta(Y) \right\rangle - \left\langle A^\alpha(Y), A^\beta(X) \right\rangle \right) \mathrm{Id} \otimes \gamma^\perp(\nu_\alpha \cdot \nu_\beta) \\
& + \frac{1}{2} \sum_{\alpha=1}^{n} \bar{\gamma}\left(((\nabla_X A)^\alpha(Y) - (\nabla_Y A)^\alpha(X)) \cdot \nu_\alpha\right).
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The Weitzenböck formula:

$$\left(D^{\Sigma N}\right)^2 = \left(\nabla^{\Sigma M \otimes \Sigma N}\right)^* \nabla^{\Sigma M \otimes \Sigma N} + \mathcal{R}^{\Sigma N},$$

where

$$\mathcal{R}^{\Sigma N} = \frac{1}{2}\bar{\gamma}(e_i \cdot e_j) R^{\Sigma M \otimes \Sigma N}(e_i, e_j).$$

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Friedrich 1980 first derived the lower bound of the first eigenvalues of the Dirac operator D (in terms of the scalar curvature S_M and dimension m of the underling manifold M):

$$\lambda^2(D) \geq \frac{m}{4(m-1)} \inf S_M. \tag{19}$$

Extrinsic Eigenvalues

Since then, various kinds of estimates in terms of intrinsic geometric quantities have been proved.

A well known result of **Hijazi 1986** states that

$$\lambda^2(D) \geq \frac{m}{4(m-1)}\lambda_1(L_M) \tag{20}$$

for $m \geq 3$, where $L_M = -\frac{4(m-1)}{m-2}\Delta + S_M$ is the Yamabe operator of M.

Extrinsic Eigenvalues

If $m = 2$, **C.Bär 1992** proved that

$$\lambda^2(D) \geq \frac{4\pi(1 - g_M)}{\text{Area}(M)}, \tag{21}$$

where g_M is the genus of M.
The equality in (19), (20) or (21) gives an Einstein metric.

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On the other hand, **O.Hijazi, S.Montiel and X.Zhang, 2001** established eigenvalue estimates for Dirac operator on embedded hypersurfaces and submanifolds in terms of the mean curvature, the Yamabe number, and the energy-momentum tensor etc. under some extra assumptions.

Extrinsic Eigenvalues

We proved the following lower bound estimates for $D^{\Sigma N}$:

Theorem (C.-Sun, Math.Z. 2021)

Let M^m be a closed spin submanifold isometrically embedded in a spin manifold $\bar{M}^{m+n}$. Suppose $n = 1$ or $\bar{M}$ is locally conformally flat.

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Theorem (C.-Sun, Math.Z. 2021)

Let M^m be a closed spin submanifold isometrically embedded in a spin manifold $\bar{M}^{m+n}$. Suppose $n = 1$ or $\bar{M}$ is locally conformally flat.
Then any eigenvalue λ of the Dirac operator $D^{\Sigma N}$ of the twisted bundle $\Sigma M \otimes \Sigma N$ satisfies

$$\lambda^2 \geq \begin{cases} \dfrac{4\pi(1-g_M)}{\mathrm{Area}(M)} - \dfrac{(n-1)\int_M \left|\mathring{A}\right|^2}{2\,\mathrm{Area}(M)}, & m = 2, \\ \dfrac{m}{4(m-1)}\lambda_1(L), & m > 2. \end{cases}$$

Extrinsic Eigenvalues

Theorem (Conti.)

Here $\mathring{A}$ is the traceless part of the shape operator A, $\lambda_1(L)$ (if $m > 2$) is the first eigenvalue of the operator L defined by

$$L = -\frac{4(m-1)}{m-2}\Delta + S_M - (n-1)\left|\mathring{A}\right|^2 .$$

Moreover, if $\lambda \neq 0$, then the equality implies that the Ricci curvature of M satisfies

$$Ric = \frac{4(m-1)\lambda^2}{m^2} g + (n-1)\sum_{\alpha=1}^{n}\left(\mathring{A}^\alpha\right)^2 .$$

Extrinsic Eigenvalues

Remark.

If M is a hypersurface, i.e., $n = 1$, then $D^{\Sigma N} = D$ is just the classical Dirac operator on M.

Extrinsic Eigenvalues

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In this case, our Theorem is reduced to the above mentioned Hijazi's result for $m \geq 3$ and Bär's result for $m = 2$.

Extrinsic Eigenvalues

Sketch of proof

First,
Denoted $\bar{P}$ by the Schouten tensor:

$$\bar{P}_{AB} := \frac{1}{n+m-2}\left(\bar{R}ic_{AB} - \frac{\bar{S}}{2(n+m-1)}\bar{g}_{AB}\right), \quad 1 \leq A, B \leq n+m,$$

the Weyl tensor $\bar{W}$ is given by

$$\bar{W}_{ABCD} := \bar{R}_{ABCD} - \left(\bar{P}_{AC}\bar{g}_{BD} + \bar{P}_{BD}\bar{g}_{AC} - \bar{P}_{AD}\bar{g}_{BC} - \bar{P}_{BC}\bar{g}_{AD}\right).$$

Extrinsic Eigenvalues

Lemma

For the curvature term $\mathcal{R}^{\Sigma N} = \frac{1}{2}\bar{\gamma}(e_i \cdot e_j) R^{\Sigma M \otimes \Sigma N}(e_i, e_j)$ *in the Weitzenböck formula, we have*

$$\mathcal{R}^{\Sigma N} = \frac{S_M - (n-1)\left|\mathring{A}\right|^2}{4} - \frac{1}{8}\bar{W}_{ij\alpha\beta}\bar{\gamma}(e_i \cdot e_j \cdot \nu_\alpha \cdot \nu_\beta)$$
$$- \frac{n}{4}\sum_{i=1}^{m}\sum_{\beta=1}^{n}\left(\bar{\gamma}\left(\mathring{A}^\beta(e_i) \cdot \nu_\beta\right) - \frac{1}{n}\sum_{\alpha=1}^{n}\bar{\gamma}\left(\mathring{A}^\alpha(e_i) \cdot \nu_\alpha\right)\right)^2 .$$

Extrinsic Eigenvalues

Second, For every $f \in C^\infty(M)$, we have the weighted spinorial Reilly formula established in **[C-Jost-Sun-Zhu, JEMS 2019]**:

$$\begin{aligned} &\frac{m-1}{m}\int_M \exp(f)\left|D^{\Sigma N}\psi\right|^2 \\ &= \int_M \exp(f)\left(\frac{m-1}{2}\Delta f - \frac{(m-1)(m-2)}{4}\left|\nabla f\right|^2 + \mathcal{R}_\psi^{\Sigma N}\right)|\psi|^2 \quad (22) \\ &+ \int_M \exp((1-m)f)\left|P^{\Sigma N}\left(\exp\left(\frac{m}{2}f\right)\psi\right)\right|^2, \end{aligned}$$

where $\mathcal{R}_\psi^{\Sigma N}|\psi|^2 = \left(\mathcal{R}^{\Sigma N}\psi, \psi\right)$, and $P^{\Sigma N}$ is the twistor operator defined by $P_X^{\Sigma N}\psi := \nabla_X^{\Sigma M \otimes \Sigma N}\psi + \frac{1}{m}\underline{\gamma}(X)D^{\Sigma N}\psi$, and $\underline{\gamma} = \gamma \otimes \mathrm{Id}$.

Extrinsic Eigenvalues

Third, Suppose ψ is an eigenspinor of $D^{\Sigma N}$ associated with λ, i.e.,

$$D^{\Sigma N}\psi = \lambda\psi.$$

Then the weighted spinorial Reilly formula implies

$$\frac{m-1}{m}\lambda^2 \int_M e^f |\psi|^2$$
$$\geq \int_M e^f \left(\frac{m-1}{2}\Delta f - \frac{(m-1)(m-2)}{4} |\nabla f|^2 + \frac{S_M - (n-1)\left|\mathring{A}^2\right|}{4} \right) |\psi|^2 . \tag{23}$$

Extrinsic Eigenvalues

Fourth, we choose $f \in C^{\infty}(M)$ as the unique solution to the following PDE (for $m = 2$):

$$\Delta f + \kappa_M - \frac{n-1}{2}\mathring{A}^2 = \frac{4\pi(1-g_M)}{\text{Area}(M)} - \frac{(n-1)\int_M \left|\mathring{A}\right|^2}{2\,\text{Area}(M)}, \quad \int_M f = 0,$$

on M.

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on M.

Therefore, according to the above inequality (23), we have

$$\lambda^2 \geq \frac{4\pi(1-g_M)}{\text{Area}(M)} - \frac{(n-1)\int_M \left|\mathring{A}\right|^2}{2\,\text{Area}(M)}.$$

Extrinsic Eigenvalues

For the limit case, since $P^{\Sigma N}\left(\exp\left(\frac{m}{2}f\right)\psi\right) = 0$, we deduce that

$$\nabla_X^{\Sigma M \otimes \Sigma N}\psi + \frac{\lambda}{m}\underline{\gamma}(X)\psi + \frac{m}{2}X(f)\psi + \frac{1}{2}\underline{\gamma}\left(X \cdot \nabla f\right)\psi = 0. \qquad (24)$$

A direct computation gives

$$\begin{aligned} &\frac{m-1}{m}\left(D^{\Sigma N}\right)^2\psi = \left(P^{\Sigma N}\right)^* P^{\Sigma N}\psi + \mathcal{R}^{\Sigma N}\psi \\ =&\left[\frac{m-1}{2}\Delta f - \frac{(m-1)(m-2)}{4}\left|\nabla f\right|^2 + \frac{S_M - (n-1)\left|\mathring{A}\right|^2}{4}\right]\psi \\ &-\frac{m-1}{m}\lambda\underline{\gamma}(\nabla f)\psi. \end{aligned}$$

Extrinsic Eigenvalues

Notice that in the limit case,

$$\frac{m-1}{2}\Delta f - \frac{(m-1)(m-2)}{4}\left|\nabla f\right|^2 + \frac{S_M - (n-1)\left|\mathring{A}\right|^2}{4} = \frac{m-1}{m}\lambda^2.$$

We conclude that

$$\frac{m-1}{m}\lambda\underline{\gamma}(\nabla f)\psi = 0.$$

Since $\lambda \neq 0$ and $\psi \neq 0$ everywhere, we know that f is a constant and $f = 0$ according to the normalizing condition. Hence,

$$\nabla_X^{\Sigma M \otimes \Sigma N}\psi + \frac{\lambda}{m}\underline{\gamma}(X)\psi = 0,$$

Extrinsic Eigenvalues

which implies that

$$\sum_{i=1}^{m} \bar{\gamma}(e_i) R^{\Sigma M \otimes \Sigma N}(e_i, e_j)\psi = \frac{2(m-1)\lambda^2}{m^2} \bar{\gamma}(e_j)\psi.$$

Applying Gauss equations and Ricci equations, a direct computation gives

$$\sum_{i=1}^{m} \bar{\gamma}(e_i) R^{\Sigma M \otimes \Sigma N}(e_i, e_j)\psi = \frac{1}{2}\bar{\gamma}\left(Ric(e_j)\right)\psi + \frac{1-n}{2}\sum_{\alpha=1}^{n} \bar{\gamma}\left(\left(\mathring{A}^{\alpha}\right)^2 (e_j)\right)\psi.$$

Extrinsic Eigenvalues

Thus

$$\frac{1}{2}\bar{\gamma}\left(Ric(e_j)\right)\psi + \frac{1-n}{2}\sum_{\alpha=1}^{n}\bar{\gamma}\left(\left(\mathring{A}^{\alpha}\right)^2(e_j)\right)\psi = \frac{2(m-1)\lambda^2}{m^2}\bar{\gamma}(e_j)\psi. \tag{25}$$

Since ψ vanishes nowhere on M, then (25) implies that

$$Ric = \frac{4(m-1)\lambda^2}{m^2}g + (n-1)\sum_{\alpha=1}^{n}\left(\mathring{A}^{\alpha}\right)^2.$$

Thank You!