A deformed Hermitian Yang-Mills flow

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Interaction Between PDEs and Convex Geometry

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1. Introduction.

Let (M, ω) be a compact Kähler manifold of complex dimension n and χ a closed real $(1,1)$-form on M.

Motivated by mirror symmetry, the deformed Hermitian Yang-Mills (dHYM) equation on (M, ω, χ) is

$$\text{Re}(\chi_u + \sqrt{-1}\omega)^n = \cot\theta_0 \ \text{Im}(\chi_u + \sqrt{-1}\omega)^n. \tag{1}$$

Here $\chi_u = \chi + \sqrt{-1}\partial\bar{\partial}u$ for a real smooth function u on M and θ_0 is the argument of the complex number $\int_M(\chi + \sqrt{-1}\omega)^n$.

The dHYM equation is called supercritical if $\theta_0 \in (0, \pi)$ and hypercritical if $\theta_0 \in (0, \frac{\pi}{2})$.

Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be the eigenvalues of χ_u with respect to ω. If necessary we denote λ by $\lambda(\chi_u)$ and λ_i by $\lambda_i(\chi_u)$ for each $1 \leq i \leq n$. Let $\lambda_i = \cot\theta_i$. Then

$$\begin{aligned}(\chi_u + \sqrt{-1}\omega)^n &= \prod_{i=1}^n (\lambda_i + \sqrt{-1})\,\omega^n \\ &= \frac{\exp(\sqrt{-1}\sum_{i=1}^n \theta_i)}{\prod_{i=1}^n \sin\theta_i}\,\omega^n \\ &= \frac{\cos(\sum_{i=1}^n \theta_i)}{\prod_{i=1}^n \sin\theta_i}\omega^n + \sqrt{-1}\frac{\sin(\sum_{i=1}^n \theta_i)}{\prod_{i=1}^n \sin\theta_i}\omega^n\end{aligned}$$

So the dHYM equation becomes

$$\cos\Big(\sum_{i=1}^n \theta_i\Big) = \cot\theta_0 \sin\Big(\sum_{i=1}^n \theta_i\Big),$$

or

$$\theta(\chi_u) = \theta_0, \tag{2}$$

if we define

$$\theta(\chi_u) := \sum_{i=1}^n \theta_i = \sum_{i=1}^n \operatorname{arccot}\lambda_i.$$

In 2014, Jacob-Yau [2017ma] initiated to study the dHYM equation.

They solved the equation for $n = 2$, by translating it into the complex Monge-Ampère equation which was solved by Yau.

[2017ma] A. Jacob, S.-T. Yau. A special Lagrangian type equation for holomorphic line bundles. Math. Ann. **369**(2017), 869-898.

When $n \geq 3$, Collins-Jacob-Yau [2020cjm] solved the dHYM equation for the supercritical case by assuming the following two conditions hold:

(i) THere exists a subsolution $\underline{u}$, which means $\chi_{\underline{u}}$ satisfies the inequality

$$A_0 := \max_M \max_{1 \leq j \leq n} \sum_{i \neq j} \operatorname{arccot} \lambda_i(\chi_{\underline{u}}) < \theta_0; \tag{3}$$

(ii) $\chi_{\underline{u}}$ also satisfies the inequality

$$B_0 := \max_M \theta(\chi_{\underline{u}}) < \pi. \tag{4}$$

[2020cjm] T. Collins, A. Jacob, S.-T. Yau. $(1,1)$ forms with specified Lagrangian phase: a priori estimates and algebraic obstructions. Camb. J. Math. **8** (2020), 407-452.

When $n = 3$, without condition (4) did Pingali [2019arxiv] then solve the equation by translating it into a mixed Monge-Ampère type equation.

On the other hand, C. Lin [2020arxiv] generalized Collins-Jacob-Yau's result to the Hermitian case (M, ω) with $\partial\bar{\partial}\omega = \partial\bar{\partial}\omega^2 = 0$.

Huang-Zhang-Zhang [2020arxiv] also considered the solution on a compact almost Hermitian manifold for the hypercritical case.

For the parabolic flow method, there are also several results.

Jacob-Yau [2017ma] and Collins-Jacob-Yau [2020cjm] proved the existence and convergence of the line bundle mean curvature flow

$$\begin{cases} u_t = \theta_0 - \theta(\chi_u) \\ u(0) = \underline{u} \end{cases} \tag{5}$$

for the hypercritical case. Here $\underline{u}$ is a subsolution of the dHYM equation such that

$$\theta(\chi_{\underline{u}}) \in (0, \frac{\pi}{2}).$$

Han-Jin [2020arxiv] considered the stability result of the above flow.

Takahashi [2020ijm] proved the existence and convergence of the tangent Lagrangian phase flow

$$\begin{cases} u_t = \tan\Big(\theta_0 - \theta(\chi_u)\Big) \\ u(0) = \underline{u} \end{cases} \tag{6}$$

for the hypercritical case. Here $\underline{u}$ is a subsolution of the dHYM equation such that

$$\theta(\chi_{\underline{u}}) - \theta_0 \in (-\frac{\pi}{2}, \frac{\pi}{2}).$$

There are two problems raised by Collins-Jacob-Yau [2020cjm]. One is whether condition (4) is superfluous.

The other is to find a sufficient and necessary geometric condition on the existence of a solution to the dHYM equation. There are some important progresses made by G. Chen [2021im].

[2021im] G. Chen. The J-equation and the supercritical deformed Hermitian-Yang-Mills equation. Invent. Math. **225** (2021), 529-602.

[2021arxiv] J. Song. Nakai-Moishezon criterions for complex Hessian equations. arxiv: 2012.07956.

Recently, motivated by G. Chen [2021im] and J. Song [2021arxiv], Chu-Lee-Takahashi [2021arxiv] established the following

Theorem. (Chu-Lee-Takahashi) The deformed Hermitian Yang-Mills equation on a compact Kähler manifold (M, ω) with complex dimension n is solvable for the supercritical case if and only if there exists a Kähler metric γ on M such that for any $1 \leq k \leq n$,

$$\int_M \Big(\mathrm{Re}(\chi + \sqrt{-1}\omega)^k - \cot\theta_0 \mathrm{Im}(\chi + \sqrt{-1}\omega)^k \Big) \wedge \gamma^{n-k} \geq 0$$

and for any proper m-dimensional subvariety Y of M and $1 \leq k \leq m$,

$$\int_Y \Big(\mathrm{Re}(\chi + \sqrt{-1}\omega)^k - \cot\theta_0 \mathrm{Im}(\chi + \sqrt{-1}\omega)^k \Big) \wedge \gamma^{m-k} > 0.$$

[2021arxiv] J. Chu, M.-C. Lee, R. Takahashi. A Nakai-Moishezon type criterion for supercritical deformed Hermitian-Yang-Mills equation. arxiv:2105.10725.

Motivated by the concavity of $\cot\theta(\chi_u)$ by G. Chen [2021im], we consider a dHYM flow:

$$\begin{cases} u_t = \cot\theta(\chi_u) - \cot\theta_0, \\ u(x,0) = \underline{u}(x). \end{cases} \tag{7}$$

The main result of this paper is

Theorem 1. (F.–Zhang) Let (M,ω) be a compact Kähler manifold and χ a closed real $(1,1)$ form. Assume that there exists a subsolution $\underline{u}$ of dHYM equation (2) in the sense of (3) which also satisfies (4). Then for the supercritical case, there exists a longtime solution $u(x,t)$ of dHYM flow (7) and it converges to a smooth solution u^∞ to the dHYM equation:

$$\theta(\chi_{u^\infty}) = \theta_0.$$

Hence we reprove the Collins-Jacob-Yau's existence theorem [2020cjm]. Our proof looks like simpler than the one in Collins-Jacob-Yau.

The advantage of our flow is that the imaginary part of the Calabi-Yau functional is constant along the flow.

However, we do not know whether condition (4) is superfluous.

2. Properties.

2.1 The linearized operator. Note

$$\cot\theta(\chi_u) = \frac{\mathrm{Re}(\chi_u + \sqrt{-1}\omega)^n}{\mathrm{Im}(\chi_u + \sqrt{-1}\omega)^n}. \tag{8}$$

Lemma 2. The linearized operator $\mathcal{P}$ of the dHYM flow has the form:

$$\mathcal{P}(v) = v_t - F^{i\bar{j}} v_{i\bar{j}},$$

where

$$F^{i\bar{j}} = \csc^2\theta(\chi_u)\Big(wg^{-1}w + g\Big)^{i\bar{j}},$$

where $g = (g_{i\bar{j}})_{n\times n}$, $w = (w_{i\bar{j}})_{n\times n}$ for $w_{i\bar{j}} = \chi_{i\bar{j}} + u_{i\bar{j}}$, and $D^{i\bar{j}} := (D^{-1})_{i\bar{j}}$ for an invertible Hermitian symmetric matrix D.

2.2 The concavity. Let

$$\theta(\lambda) := \sum_{i=1}^{n} \operatorname{arccot} \lambda_i \quad \text{for } \lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n \tag{9}$$

and

$$\Gamma_\tau := \{\lambda \in \mathbb{R}^n \mid \theta(\lambda) < \tau\} \subset \mathbb{R}^n \quad \text{for } \tau \in (0, \pi).$$

We have the following two useful lemmas.

Lemma 3. (Yuan [2006pams], Wang-Yuan [2014ajm]) If $\theta(\lambda) \le \tau \in (0, \pi)$ for $\lambda = (\lambda_1, \ldots, \lambda_n)$ with $\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_n$, then the following inequalities hold.

(i) $\lambda_{n-1} \ge \cot\frac{\tau}{2} (> 0)$;

(ii) $\lambda_{n-1} \ge |\lambda_n|$; and

(iii) $\lambda_1 + (n-1)\lambda_n \ge 0$.

Moreover, Γ_τ is convex for any $\tau \in (0, \pi)$.

[2006pams] Y. Yuan. Global solutions to special Lagrangian equations. Proc. Amer. Math. Soc. **134**(2006), 1355-1358.

[2014ajm] D. Wang, Y. Yuan. Hessian estimates for special Lagrangian equations with critical and supercritical phases in general dimensions. Amer. J. Math. **136**(2014), 481-499.

Lemma 4. (Chen [2021im]) For any $\tau \in (0, \pi)$, the function $\cot\theta(\lambda)$ on Γ_τ is concave.

proof. When $n = 1$, $\cot\theta(\lambda) = \lambda_1$ is obviously concave. We now assume $n \geq 2$. By the definition of $\theta(\lambda)$, we have

$$\frac{\partial^2 \cot\theta(\lambda)}{\partial\lambda_i \partial\lambda_j} = -2\csc^2\theta(\lambda)\Big(\frac{\lambda_i \delta_{ij}}{(1+\lambda_i^2)^2} - \frac{\cot\theta(\lambda)}{(1+\lambda_i^2)(1+\lambda_j^2)}\Big).$$

Hence the function $\cot\theta(\lambda)$ on Γ_τ is concave if and only if the matrix

$$\Lambda = \Big(\lambda_i \delta_{ij} - \cot\theta(\lambda)\Big)_{n\times n}$$

is posotive definite.

Without loss of generality, we assume $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. Since $\theta(\lambda) \in (0, \pi)$, by Lemma 3(1), we have $\lambda_{n-1} > 0$. $\square$

2.3 Parabolic subsolution. Motivated by B. Guan's definition [2014dmj] of a subsolution of fully nonlinear equations, Székelyhidi [2019jdg] gave a weaker version of a subsolution and Collins-Jacob-Yau [2020cjm] used it to the dHYM equation which is equivalent to (3).

[2014dmj] B. Guan. Second-order estimates and regularity for fully nonlinear elliptic equations on Riemannian manifolds. Duke Math. J. 163(2014), 1491-1524.

[2018jdg] G. Székelyhidi. Fully non-linear elliptic equations on compact Hermitian manifolds. J. Differential Geom. 109(2018), 337-378.

On the other hand, Phong-Tô [2017arxiv] modified Székelyhidi's definition to the parabolic case. We use their definition to the dHYM flow.

Definition 5. A smooth function $\underline{u}(x,t)$ on $M \times [0,T)$ is called a subsolution of the dHYM flow if there exists a constant $\delta > 0$ such that for any $(x,t) \in M \times [0,T)$, the subset of $\mathbb{R}^{n+1}$

$$S_\delta(x,t) := \Big\{(\mu,\tau) \in \mathbb{R}^n \times \mathbb{R} \mid \mu_i > -\delta \text{ for each } i, \tau > -\delta, \text{and} \\ \cot\theta\Big(\lambda(\chi_{\underline{u}(x,t)}) + \mu\Big) - \underline{u}_t(x,t) + \tau = \cot\theta_0\Big\}$$

is uniformly bounded.

[2017arxiv] D. H. Phong, D. Tô. Fully non-linear parabolic equations on compact Hermitian manifolds. arXiv: 1711.10697.

We have the following observation.

Lemma 6. If $\underline{u}$ is a subsolution of the dHYM equation with $B_0 < \pi$, then the function $\underline{u}(x,t) = \underline{u}(x)$ on $M \times [0, \infty)$ is also a subsolution of the dHYM flow.

2.4 The Calabi-Yau Functional. Recall the definition of the Calabi-Yau functional by Collins-Yau [2021apde]: for any $v \in C^2(M, \mathbb{R})$,

$$\mathrm{CY}_{\mathbb{C}}(v) := \frac{1}{n+1} \sum_{i=0}^{n} \int_M v(\chi_v + \sqrt{-1}\omega)^i \wedge (\chi + \sqrt{-1}\omega)^{n-i}.$$

Let $v(s) \in C^{2,1}(M \times [0, T], \mathbb{R})$ be a variation of the function v, i.e., $v(0) = v$. The integration by parts gives

$$\frac{d}{ds}\mathrm{CY}_{\mathbb{C}}(v(s)) = \int_M \frac{\partial v(s)}{\partial s}\Big(\chi_{v(s)} + \sqrt{-1}\omega\Big)^n. \tag{10}$$

[2021apde] T. Collins, S.-T. Yau. Moment Maps, Nonlinear PDE and Stability in Mirror Symmetry, I: Geodesics. Ann. PDE 7, 11(2021).

Lemma 7. Let $u(x,t)$ be a solution of the dHYM flow. Then

$$\text{Im}\Big(\text{CY}_{\mathbb{C}}(u(\cdot,t))\Big) = \text{Im}\Big(\text{CY}_{\mathbb{C}}(\underline{u})\Big).$$

Proof. Denote by $u(t) := u(x,t)$ for simplicity.

$$\begin{aligned}
&\frac{d}{dt}\text{Im}\Big(\text{CY}_{\mathbb{C}}(u(t))\Big) = \text{Im}\frac{d}{dt}\text{CY}_{\mathbb{C}}(u(t)) \\
=&\int_M \frac{\partial u(t)}{\partial t}\text{Im}(\chi_{u(t)} + \sqrt{-1}\omega)^n \\
=&\int_M \Big(\frac{\text{Re}(\chi_{u(t)} + \sqrt{-1}\omega)^n}{\text{Im}(\chi_{u(t)} + \sqrt{-1}\omega)^n} - \cot\theta_0\Big)\text{Im}(\chi_{u(t)} + \sqrt{-1}\omega)^n \\
=&\int_M \text{Re}(\chi_{u(t)} + \sqrt{-1}\omega)^n - \cot\theta_0 \int_M \text{Im}(\chi_{u(t)} + \sqrt{-1}\omega)^n \\
=&\int_M \text{Re}(\chi + \sqrt{-1}\omega)^n - \cot\theta_0 \int_M \text{Im}(\chi + \sqrt{-1}\omega)^n \\
=&0,
\end{aligned}$$

where each equality is successively by (10), (7) and (8), Stokes' theorem, and the definition of θ_0. Hence the conclusion holds as $u(0) = \underline{u}$. $\square$

3. Estimates.

We assume that u is the solution of dHYM flow (7) in $M \times [0, T)$, where T is the maximal existence time. By showing the uniform a priori estimates, we can prove $T = \infty$.

3.1 The u_t-estimate.

Lemma 8. For any $(x, t) \in M \times [0, T)$,

$$\min_M u_t|_{t=0} \leq u_t(x, t) \leq \max_M u_t|_{t=0}; \tag{11}$$

in particular,

$$0 < \min_M \theta(\chi_{\underline{u}(x)}) \leq \theta(\chi_{u(x,t)}) \leq B_0 < \pi. \tag{12}$$

Proof. The u_t satisfies the equation:

$$(u_t)_t = F^{i\bar{j}}(u_t)_{i\bar{j}}.$$

By the maximum principle, u_t attains its maximum and minimum on the initial time, i.e., inequality (11) holds, i.e.,

$$\min_M \cot\theta(\chi_{\underline{u}}) \le u_t(x,t) + \cot\theta_0 \le \max_M \cot\theta(\chi_{\underline{u}}),$$

or

$$\min_M \cot\theta(\chi_{\underline{u}}) \le \cot\theta(\chi_{u(x,t)}) \le \max_M \cot\theta(\chi_{\underline{u}}).$$

Thus we obtain

$$0 < \min_M \theta(\chi_{\underline{u}}) \le \theta(\chi_{u(x,t)}) \le \max_M \theta(\chi_{\underline{u}}) = B_0.$$

□

We have an useful corollary of the above lemma.

Corollary 9. Let $\lambda_n(x,t)$ be the minimum eigenvalue of χ_u with respect to the metric ω at (x,t). Then

$$\max_{M\times[0,T)}|\lambda_n| \le A_1 \quad \text{for} \quad A_1 := |\cot B_0| + \left|\cot\left(\frac{\min_M \theta(\chi_{\underline{u}})}{n}\right)\right|.$$

Proof. By Lemma 8, we have

$$0 < \frac{\min_M \theta(\chi_{\underline{u}})}{n} \le \frac{\theta(\chi_u)}{n} \le \operatorname{arccot}\lambda_n \le B_0 < \pi.$$

Hence we have

$$\cot B_0 \le \lambda_n \le \cot\left(\frac{\min_M \theta(\chi_{\underline{u}})}{n}\right).$$

□

3.2 The C^0-estimate. We first prove a Harnack type inequality along the dHYM flow.

Lemma 10. Let u be the solution of the dHYM flow on $M \times [0, T)$. Then for any $T_0 < T$ we have the following Harnack type inequality:

$$\sup_{M\times[0,T_0]} u(x,t) \leq C\Big(-\inf_{M\times[0,T_0]}\big(u(x,t) - \underline{u}(x)\big) + 1\Big).$$

Proof. For any $t \in [0, T_0]$, we have $\theta(\chi_{u(t)}) \leq B_0 < \pi$ by Lemma 8. Then by the convexity of $\Gamma_{\omega,B_0} := \{\alpha \in \Lambda^{1,1}(M,\mathbb{R}) \mid \theta(\alpha) < B_0\}$ in Lemma 3, we have

$$\theta(\chi_{su+(1-s)\underline{u}}) \leq B_0.$$

Denote $\eta_0 := B_0/6 + 5\pi/6$ for convenience. Then $B_0 < \eta_0 < \pi$. Hence,

$$\begin{aligned}
&\frac{\operatorname{Im}(\chi_{su(t)+(1-s)\underline{u}} + \sqrt{-1}\omega)^n}{\omega^n} \\
&= \prod_{k=1}^{n} (1 + \lambda_k^2(\chi_{su(t)+(1-s)\underline{u}}))^{\frac{1}{2}} \sin(\theta(\chi_{su(t)+(1-s)\underline{u}}) \\
&\geq \begin{cases} \sin \eta_0, \ \text{if } \theta(\chi_{su(t)+(1-s)\underline{u}}) \geq \dfrac{\pi}{6} \\ \sqrt{1+\lambda_1^2} \sin \operatorname{arccot} \lambda_1 = 1, \ \text{if } \theta(\chi_{su(t)+(1-s)\underline{u}}) < \dfrac{\pi}{6} \end{cases} \\
&\geq \sin \eta_0 \triangleq c_0.
\end{aligned} \tag{13}$$

By Lemma 7, the imaginary part of the Calabi-Yau functional is constant along the flow. Hence,

$$
\begin{aligned}
0 =&\mathrm{Im}\Big(\mathsf{CY}_{\mathbb{C}}(u(t))\Big) - \mathrm{Im}\Big(\mathsf{CY}_{\mathbb{C}}(\underline{u})\Big) \\
=&\int_0^1 \frac{d}{ds}\mathrm{Im}\Big(\mathsf{CY}_{\mathbb{C}}(su(t) + (1-s)\underline{u})\Big)ds \\
=&\int_0^1 \int_M (u(t) - \underline{u})\mathrm{Im}\Big(\chi_{su(t)+(1-s)\underline{u}} + \sqrt{-1}\omega\Big)^n ds \\
=&\int_M (u(t) - \underline{u})\Big(\int_0^1 \mathrm{Im}\Big(\chi_{su(t)+(1-s)\underline{u}} + \sqrt{-1}\omega\Big)^n ds\Big). \qquad (14)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\int_M (u-\underline{u})\omega^n \\
=&\int_M (u-\underline{u})\omega^n - \frac{1}{c_0}\int_M (u-\underline{u})\Big(\int_0^1 \mathrm{Im}\Big(\chi_{su(t)+(1-s)\underline{u}} + \sqrt{-1}\omega\Big)^n ds\Big) \\
=&\frac{1}{c_0}\int_M -(u-\underline{u})\underbrace{\Big(-c_0\omega^n + \int_0^1 \mathrm{Im}\Big(\chi_{su(t)+(1-s)\underline{u}} + \sqrt{-1}\omega\Big)^n ds\Big)}_{\text{This term is positive by inequality (13)}} \\
\le&\frac{-\inf_{M\times[0,T_0]}(u-\underline{u})}{c_0}\int_M\Big(-c_0\omega^n + \int_0^1 \mathrm{Im}\Big(\chi_{su(t)+(1-s)\underline{u}} + \sqrt{-1}\omega\Big)^n ds\Big) \\
=&\frac{-\inf_{M\times[0,T_0]}(u-\underline{u})}{c_0}\Big(-c_0\int_M\omega^n + \int_0^1 \mathrm{Im}\int_M\Big(\chi_{su(t)+(1-s)\underline{u}} + \sqrt{-1}\omega\Big)^n ds\Big) \\
=&\frac{-\inf_{M\times[0,T_0]}(u-\underline{u})}{c_0}\Big(-c_0\int_M\omega^n + \mathrm{Im}\int_M\Big(\chi + \sqrt{-1}\omega\Big)^n\Big) \\
\le&c_0^{-1}\mathrm{Im}\int_M\Big(\chi + \sqrt{-1}\omega\Big)^n\Big(-\inf_{M\times[0,T_0]}(u-\underline{u})\Big) \\
=&C\Big(-\inf_{M\times[0,T_0]}(u-\underline{u})\Big),
\end{aligned}
$$

where $C = c_0^{-1}\mathrm{Im}\int_M(\chi + \sqrt{-1}\omega)^n$.

Therefore we have

$$\int_M u(x,t)\omega^n \le C\Big(- \inf_{M\times[0,T_0]}\Big(u(x,t) - \underline{u}(x)\Big) + 1\Big). \qquad (15)$$

On the other hand, let $G(x,z)$ be Green's function of the metric ω on M. Then for any $(x,t)\in M\times[0,T_0]$,

$$u(x,t) = \Big(\int_M \omega^n\Big)^{-1}\int_M u(z,t)\omega^n - \int_{z\in M}\Delta_\omega u(z,t)G(x,z)\omega^n.$$

Since $\Delta_\omega u > -\mathrm{tr}_\omega\chi > -C_0$ and $G(x,y)$ is bounded from below, there exits a uniform constant C such that

$$u(x,t) \le \Big(\int_M \omega^n\Big)^{-1}\int_M u(z,t)\omega^n + C. \qquad (16)$$

Combing (15) with (16), we obtain the desired estimate. $\square$

Now we prove the C^0 estimate similar as Phong-Tô [2017arxiv].

Proposition 11. Along the dHYM flow, there exists a uniform constant M_0 independent of T such that

$$|u|_{C^0(M\times[0,T))} \le M_0.$$

Proof. Combining (14) with (13) implies for any $t \in [0,T)$,

$$\sup_{x\in M}(u(x,t) - \underline{u}(x)) \ge 0.$$

Combing the above inequality with the concavity of the equation, we can apply Lemma 1 in Phong-Tô [2017arxiv]: there exists a uniform constant C_1 such that

$$\inf_{M\times[0,T_0]}(u(x,t) - \underline{u}(x)) \ge -C_1 \qquad \text{for any } T_0 < T.$$

Then combing this estimate with the Harnack type inequality in Lemma 10, we have

$$\sup_{M\times[0,T_0]} u \le C.$$

□

3.3 The gradient estimate. We can prove the gradient estimate following the argument in the elliptic case by Collins-Yau [2018arxiv].

Proposition 12. Let u be the solution of dHYM flow (7). There exists a uniform constant M_1 such that

$$\max_{M\times[0,T)} |\nabla u|_\omega \leq M_1.$$

3.4 Second order estimates. In the elliptic case, Collins-Jacob-Yau [2020cjm] used an auxiliary function containing the gradient term which modifies the one in Hou-Ma-Wu [2010mrl]. Our auxiliary function does not contain the gradient term.

Proposition 14. There exists a uniform constant M_2 such that

$$\sup_{M\times[0,T)} |\partial\bar{\partial}u|_\omega \leq M_2.$$

4. Proof of the main theorem.

So we can prove the uniform a priori estimates up to the second order. By the concavity of $\theta(\chi_u(x,t))$, we have the uniform $C^{2,\alpha}$ estimates and then the higher estimates hold. Thus we have the longtime existence.

The proof of the convergence is the similar as the one in Phong-Tô [2017arxiv]. We can prove $u(x,t)$ converges exponentially to a function u^∞. By the uniform C^k estimates of $u(x,t)$ for all $k \in \mathbb{N}$, $u(x,t)$ converges to u^∞ in C^∞ and u^∞ satisfies

$$\theta(\chi_{u^\infty}) := \sum_{i=1}^{n} \mathrm{arccot}\lambda_i(\chi_{u^\infty}) = \theta_0.$$

Thank You!