Mahler's conjecture and entropy-transport inequalities

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Interactions between PDE and convex geometry

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I. Blaschke-Santalo's inequality and Mahler's conjecture

1) For sets

Let $K \subset \mathbb{R}^n$ be measurable with finite volume

$K^\circ = \{ y \in \mathbb{R}^n ; \ \langle x, y \rangle \leq 1 \ \forall x \in K \}$

Volume product : $P(K) = |K| \min_z |(K-z)^\circ|$

Blaschke-Santaló : $P(K) \leq P(B_2^n)$ with equality iff K is an ellipsoid

Mahler's conjectures : Let K be a convex body

1) if $K = -K$ then $P(K) \geq P([-1,1]^n) = \frac{4^n}{n!}$

2) For any K $\quad P(K) \geq P(\Delta^n)$ ↳ simplex

A few known results :

- $n=2$: Mahler '39
- K unconditionnal : Saint Raymond '81 ; Meyer '86
- K zonoid : Reisner '86 ; Gordon, Meyer, Reisner '88
- K with many hyperplane symmetries : Barthe - F. '13

- $n=3$ and $K=-K$: Iriyeh-Shibata '20
short proof by F., Hubard, Meyer, Roldan-Pensado and Zvavitch '21
- Isomorphic form: Bourgain-Milman '91:

$\exists c$ s.t. $P(K) \geq \frac{c^n}{n!}$ Kuperberg '08: $c \geq \pi$ for $K=-K$

dream: $c=4$

2) For log-concave functions

Let $\varphi : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ $\mathcal{L}\varphi(y) = \sup_x \langle x,y\rangle - \varphi(x)$

For $z \in \mathbb{R}^n$ let $\varphi_z(x) = \varphi(x+z)$

Functional volume product: $P(\varphi) = \int e^{-\varphi} \times \min_z \int e^{-\mathcal{L}\varphi_z}$

Functional Blaschke-Santaló: $P(\varphi) \leq P\left(\frac{|\cdot|^2}{2}\right) = (2\pi)^n$

- Ball '86: if φ even
- Artstein-Klartag-Milman '04: φ general
- F., Meyer '07: generalized forms; Lehec '07, '08

Functional Mahler's conjectures: let φ be convex

(1) If φ is even then $P(\varphi) \geq P(\|\cdot\|_\infty) = 4^n$

(2) For any φ $P(\varphi) \geq P\left(\sum_{i=1}^n x_i \, I_{\mathbb{R}_+^n}(x)\right) = e^n$

- F., Meyer '08: (1) for φ unconditional (so OK for $n=1$)
(2) for $n=1$

. F., Nakhle '21+: (1) for $n=2$

II. Blaschke-Santalo and transport inequalities

1) AKM: $\forall \varphi$ even $\int e^{-\varphi} \int e^{-\mathcal{L}\varphi} \leq (2\pi)^n$

$\Longleftrightarrow$ $\forall f$ even $\int e^{-f} d\gamma \int e^{Q_1 f} d\gamma \leq 1$ (τ)

where $Q_t f(y) = \inf_x f(x) + \frac{1}{2t}|x-y|^2$

Hamilton-Jacobi semi-group sol° of

$\boxed{\text{PDE}}$ $\begin{cases} \frac{d}{dt} Q_t f = \frac{1}{2} |\nabla Q_t f|^2 \\ Q_0 f = f \end{cases}$

* Proof:

2) Known among specialists:

(τ) for even $\Longleftrightarrow$ $\forall \nu$ even $\forall \mu$

$$W_2^2(\mu,\nu) \leq 2\left(H(\mu|\gamma) + H(\nu|\gamma)\right)$$

$$W_2^2(\mu,\nu) = \inf_{\pi \to \mu, \nu} \int |x-y|^2 d\pi = \sup_f \int Q_{\frac{1}{2}} f \, d\mu - \int f \, d\nu$$

$$H(\mu|\gamma) = \int \log\left(\frac{d\mu}{d\gamma}\right) d\gamma$$

$$\log\left(\int e^{-f} d\gamma\right) = \sup_{\nu}\left(\int(-f)\, d\nu - H(\nu|\gamma)\right)$$

3) Lehec noticed that $\forall \varphi$ s.t. $\int x e^{-\varphi} = 0$

one has $\int e^{-\varphi} \int e^{-\mathcal{L}\varphi} \leq (2\pi)^n$

Using these tools Fathi '18 proved that

For ν centered $\forall \mu$ $W_2^2(\mu,\nu) \leq 2\left(H(\mu|\gamma) + H(\nu|\gamma)\right)$

III. Reverse-Santalo and entropy-transport

To η log-concave probability measure with density e^{-V} we associate its moment measure $\nu = \nabla V \# \eta$:

$$\forall f \quad \int f\, d\nu = \int f(\nabla V(x))\, d\eta$$

For ν_1, ν_2 proba we define the maximal correlation optimal transport cost:

$$T(\nu_1,\nu_2) = \sup_{\pi \to \nu_1, \nu_2} \int \langle x,y\rangle\, d\pi = \inf_{f \text{ convex}} \int f\, d\nu_1 + \int \mathcal{L}f\, d\nu_2$$

$H(\eta) = \int \log \frac{d\eta}{dx}\, d\eta$ relative entropy w.r.t Lebesgue

* **Theorem (Gozlan '21):**

(i) $\forall V$ convex on $\mathbb{R}^n$ $\int e^{-V} \int e^{-\mathcal{L}V} \geq c^n$

$\Updownarrow$

(ii) $\forall \eta_1, \eta_2$ log-concave with densities e^{-V_1}, e^{-V_2} and moment measures ν_1 and ν_2:

$$H(\eta_1) + H(\eta_2) \leq -n \log(c e^2) + T(\nu_1, \nu_2)$$

Short pf of (ii) $\Rightarrow$ (i): (F., Gozlan, Zugmeyer '21+)

* Lemma: Let $V: \mathbb{R}^n \to \mathbb{R}$ convex $d\eta = \frac{e^{-V}}{Z}$ proba $\nu = \nabla V \# \eta$ moment measure. Then

① $T(\nu, \eta) = \int x . \nabla V(x)\, d\eta = \int V d\eta + \int \mathcal{L}V d\nu = n$

② $-\log\left(\int e^{-V}\right) = -\int \mathcal{L}V d\nu + H(\eta) + n$

• Pf of (ii) $\Rightarrow$ (i):

• Pf of ①:

- **Pf of ②:**

* **Theorem** (F., Meyer '07): $\varphi : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ convex

1) φ even: $\int_0^{+\infty} e^{-\varphi} \int_0^{+\infty} e^{-\mathcal{L}\varphi} \geq 1$

2) any $\int_{\mathbb{R}} e^{-\varphi} \int_{\mathbb{R}} e^{-\varphi} \geq e$

* **Simple proof** (F, G, Z): φ even

We want $T(\nu_1, \nu_2) \geq 2 + 2\log 2 + H(\eta_1) + H(\eta_2)$

$\eta_1 = e^{-V_1}$ so $H(\eta_1) = -\int V_1 e^{-V_1}$ $\qquad \nu_1 = V_1' \# \eta_1$

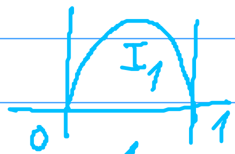
$F_{\nu_1} = F_{\eta_1} \circ V_1'^{-1}$

$$T(\nu_1, \nu_2) = \int_0^1 F_{\nu_1}^{-1}(x)\, F_{\nu_2}^{-1}(x)\, dx$$

$$= \int_0^1 V_1'(F_{\eta_1}^{-1}(x))\, V_2'(F_{\eta_2}^{-1}(x))\, dx$$

$$= \int_0^1 I_1'(x)\, I_2'(x)\, dx \quad \text{where } I_1 = F_{\eta_1}' \circ F_{\eta_1}^{-1}$$

$H(\eta_1) = \int_0^1 \log I_1$



So we want $\int_0^1 I_1' I_2' \geq 2 + 2\log 2 + \int_0^1 \log(I_1 I_2)$