A flow approach to the Musielak-Orlicz-Gauss image problem

Caihong Yi

(Joint work with Qi-Rui Li, Weimin Sheng and Deping Ye)

Hangzhou Normal University

Conference on Interaction Between PDEs and Convex Geometry
BIRS-IASM, Oct. 17-22; 2021

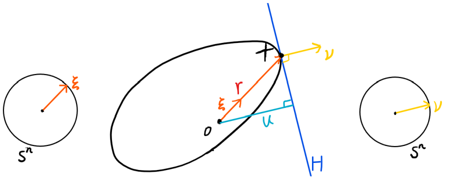
Contents

Contents

Contents

Notations

- Denote by $\mathcal{K}^{n+1}$ the class of convex bodies in $\mathbb{R}^{n+1}$ containing the origin, and $\mathcal{K}_0^{n+1}$ the class of convex bodies in $\mathbb{R}^{n+1}$ containing the origin in their interiors. Let $\Omega \in \mathcal{K}_0^{n+1}$, and $\mathcal{M} = \partial\Omega$ is convex hypersurface in $\mathbb{R}^{n+1}$.
- **Support function** $u : \mathbb{S}^n \to \mathbb{R}$, defined by
$$u(\nu) = \max\{\langle X, \nu \rangle : X \in \Omega\}.$$
- **Radial function** $r : \mathbb{S}^n \to \mathbb{R}$, defined by
$$r(\xi) = \max\{\lambda : \lambda\xi \in \Omega\}.$$
- **The supporting hyperplane** to Ω with unit normal $\nu \in \mathbb{S}^n$
$$H_\Omega(\nu) = \{z \in \mathbb{R}^{n+1} : z \cdot \nu = u_\Omega(\nu)\}$$



Notations

- **The Gauss map** $\boldsymbol{\nu} : \partial\Omega \to \mathbb{S}^n$,

$$\boldsymbol{\nu}(X) = \{\nu \in \mathbb{S}^n : X \cdot \nu = u_\Omega(\nu)\}.$$

  The inverse Gauss map $\boldsymbol{\nu}_\Omega^{-1}$ reparametrizes $\partial\Omega$.

- **The reverse radial Gauss image** of E, $E \subset \mathbb{S}^n$, defined by

$$\boldsymbol{\alpha}_\Omega^*(E) = \{\xi \in \mathbb{S}^n : r_\Omega(\xi)\xi \in \boldsymbol{\nu}_\Omega^{-1}(E)\}.$$

- **The principal curvature radii** of $\mathcal{M}$ at X is given by eigenvalues of $(\nabla^2 u + u g_{\mathbb{S}^n})$, where ∇ the covariant derivative on $\mathbb{S}^n$.

- **Hausdorff metric** $\delta(K, L)$, K, $L \in \mathcal{K}_0^{n+1}$,

$$\delta(K, L) = \max_{x \in \mathbb{S}^n} |u_K(x) - u_L(x)|.$$

- Let $e_1, \cdots, e_n$ be a smooth local orthonormal frame field on $\mathbb{S}^n$, and ∇ be the covariant derivative on $\mathbb{S}^n$,

$$r(\xi) \cdot \xi = u(x) \cdot x + \nabla u(x).$$

Notations

- Minkowski combination $sK + tL$ for two convex bodies $K, L \in \mathcal{K}_0^{n+1}$, $s, t \geq 0$:
$$sK + bL = \{sx + ty : x \in K, y \in L\},$$
or equivalently
$$u(sK + tL, \cdot) = su(K, \cdot) + tu(L, \cdot).$$
- Firey's p-sum $s \cdot K +_p t \cdot L$ for two convex bodies $K, L \in \mathcal{K}_0^{n+1}$, $p > 1$, $s, t \geq 0$, can be defined by its support function
$$u^p(s \cdot K +_p t \cdot L, \cdot) = su^p(K, \cdot) + tu^p(L, \cdot).$$
- $p < 1$, Wulff shape:
$$s \cdot K +_p t \cdot L = \bigcap_{x \in \mathbb{S}^n} \{y \in \mathbb{R}^{n+1} | x \cdot y \leq \left(su_K^p(x) + tu_L^p(x)\right)^{\frac{1}{p}}\}.$$
- When $p = 0$, $s \cdot K +_0 t \cdot L = \bigcap_{x \in \mathbb{S}^n} \{y \in \mathbb{R}^{n+1} | x \cdot y \leq u_K^s u_L^t\}$.

The Musielak-Orlicz-Gauss image problem

- The Musielak-Orlicz function:
  $\mathcal{C} = \{G : (0, \infty) \times \mathbb{S}^n \to \mathbb{R}$ such that G and G_z are continuous on $(0, \infty) \times \mathbb{S}^n\}$.
- The general volume of Ω with respect to the given Lebesgue measure λ on $\mathbb{S}^n$

$$V_{G,\lambda}(\Omega) = \int_{\mathbb{S}^n} G(r_\Omega(\xi), \xi) d\lambda(\xi).$$

- The Musielak-Orlicz addition of continuous functions u and g (Musielak-Orlicz extensions of Firey's p-sum)

$$\Psi(x, u_t(x)) = \Psi(x, u(x)) + tg(x).$$

- Variational formula for the general dual volume

$$\frac{d}{dt} V_{G,\lambda}([u_t])|_{t=0} = \int_{\mathbb{S}^n} g(v) d\widetilde{C}_\Theta(K, v)$$

- Let $\Theta = (G, \Psi, \lambda)$ be a given tripe with $G \in \mathcal{C}$, $\Psi \in \mathcal{C}$, and λ a nonzero finite Lebesgue measure on $\mathbb{S}^n$.
The Musielak-Orlicz-Gauss image measure $\widetilde{C}_\Theta(\Omega, \cdot)$ of $\Omega \in \mathcal{K}_0^{n+1}$ for each Borel set $\omega \subset \mathbb{S}^n$ (Huang-Xing-Ye-Zhu, 2021)

$$\widetilde{C}_\Theta(\Omega, \omega) = \int_{\alpha^*_\Omega(\omega)} \frac{r_\Omega(\xi) G_z(r_\Omega(\xi), \xi)}{\psi(u_\Omega(\alpha_\Omega(\xi)), \alpha_\Omega(\xi))} d\lambda(\xi),$$

where $\psi = z\Psi_z$.

The Musielak-Orlicz-Gauss image problem :
Let $G \in \mathcal{C}$, $\Psi \in \mathcal{C}$, and λ be a nonzero finite Lebesgue measure on $\mathbb{S}^n$. Under what conditions on $\Theta = (G, \Psi, \lambda)$ and a nonzero finite Borel measure μ on $\mathbb{S}^n$ do there exist a $\Omega \in \mathcal{K}_0^{n+1}$ and a constant $\tau \in \mathbb{R}$ such that

$$d\mu = \tau d\widetilde{C}_\Theta(\Omega, \cdot). \quad (1)$$

- Let $\widetilde{C}_{G,\lambda}(\Omega, \cdot) = \widetilde{C}_{(G,\log t,\lambda)}(\Omega, \cdot)$, the Musielak-Orlicz-Gauss image problem can be rewritten as

$$\psi(u_\Omega(\cdot), \cdot)\, d\mu = \tau\, d\widetilde{C}_{G,\lambda}(\Omega, \cdot). \quad (2)$$

- When $d\lambda(\xi) = p_\lambda(\xi)d\xi$ and $d\mu(x) = f(x)dx$, (2) reduces to solving the following Monge-Ampère type equation on $\mathbb{S}^n$:

$$u(u^2 + |\nabla u|^2)^{-\frac{n}{2}} G_z(\sqrt{u^2 + |\nabla u|^2}, \xi) p_\lambda \cdot \det(\nabla^2 u + uI) = \gamma f(x)\psi(u, x). \quad (3)$$

The Musielak-Orlicz-Gauss image problem :

$$u(u^2+|\nabla u|^2)^{-\frac{n}{2}} G_z(\sqrt{u^2+|\nabla u|^2}, \xi) p_\lambda(\xi) \det(\nabla^2 u + uI) = f(x)\psi(u,x).$$

Specially, for $G(r,\xi) = r^q$, $\Psi(u,x) = u^p$ and $p_\lambda(\xi) = 1$.

- When $p = 0$, it becomes the dual Minkowski problem (Huang-Lutwak-Yang-Zhang'16):

$$u(u^2+|\nabla u|^2)^{\frac{q-n-1}{2}} \det(\nabla^2 u + uI) = f(x).$$

  - $0 < q \leq n+1$, existence for even measures, Huang-Lutwak-Yang-Zhang.'16; Böröczky-Henk-Pollehn.'18; Zhao.'18;
  - $q < 0$, existence for general measures, Zhao.'17;
  - $q \in \mathbb{R}$, existence for smooth function f, Li-Sheng-Wang.'18. (Geometric flow method).

  ......

When $q = n+1$, it becomes L_p Minkowski problem (Lutwak'93):

$$u^{1-p}\det(\nabla^2 u + uI) = f(x)$$

- $p = 1$, the classical Minkowski problem settled by Nirenberg'53; Cheng-Yau'76; Pogorelov'78
- $p > 1$ and $p \neq n+1$, existence for even measures. Lutwak'93; Lutwak-Oliker'95
  existence for discrete measures, Hug-Lutwak-Yang-Zhang'05
- $p \geq n+1$, existence and uniqueness of smooth solution, Chou-Wang'06; Guan-Lin'00
- $-n-1 < p < n+1$, weak solution for $f \in L^\infty$ by Chou-Wang'06;.
- $0 < p < 1$, existence for even measures, Haberl-Lutwak-Yang-Zhang'10;
  existence for discrete measures Zhu'15;
  existence for general measures, Chen-Li-Zhu'17.
- $p = 0$ (logarithmic Minkowski problem),
  existence for even measures, Böröczky-Lutwak-Yang-Zhang'13;
  existence for discrete measures, Zhu'14; Böröczky-Hegedus-Zhu'16;
  existence for general measures, Chen-Li-Zhu'19.

......

L_p dual Minkowski problem (Lutwak-Yang-Zhang'18):

$$u^{1-p}(u^2+|\nabla u|^2)^{\frac{q-n-1}{2}}\det(\nabla^2 u+uI)=f(x).$$

- $p \geq q$, existence for discrete measures, Lutwak-Yang-Zhang.'18.
- $pq \geq 0$ and even smooth function f, Chen-Huang-Zhao.'19. (Geometric flow method).
- $\boldsymbol{p>1}$, $q>0$, existence for general measures and $\Omega \in \mathcal{K}^{n+1}$, Böröczky-Fodor.'19.(polytopal solutions to the discrete measures and an approximation argument)
- $\boldsymbol{p>0}$, $q>0$, existence for general measures and $\Omega \in \mathcal{K}^{n+1}$, Chen-Li.'21. (geometric flow method and an approximation argument)
- $p<0, q>0$, existence for even measures, Chen-Chen-Li.'20.

Specially, for $\Psi(u,x)=\Psi(u)$ and $p_\lambda(\xi)=1$, $\widetilde{C}_\Theta(\Omega,\cdot)$ becomes the general dual curvature measure $\widetilde{C}_{G,\psi}(\Omega,\cdot)$ of $\Omega\in\mathcal{K}_0^{n+1}$ for each Borel set $\omega\subset\mathbb{S}^n$ (Gardner-Hug-Weil-Xing-Ye.'19)

$$\widetilde{C}_{G,\psi}(\Omega,\omega)=\int_{\alpha_\Omega^*(\omega)}\frac{r_\Omega(\xi)G_z(r_\Omega(\xi),\xi)}{\psi(u_\Omega(\alpha_\Omega(\xi)))}\,d\xi,$$

where $\psi(t)=t\Psi'(t)$.

The general dual Orlicz Minkowski problem

$$d\mu=\tau d\widetilde{C}_{G,\psi}(\Omega,\cdot),\tag{4}$$

Denote $\widetilde{C}_G(\Omega,\cdot)=\widetilde{C}_{G,1}(\Omega,\cdot)$, the problem can be re-written as

$$\psi(u_\Omega)d\mu=\tau d\widetilde{C}_G(\Omega,\cdot).\tag{5}$$

When $d\mu=f(x)dx$, it is reduced to solving the following equation on $\mathbb{S}^n$:

$$u(u^2+|\nabla u|^2)^{-\frac{n}{2}}G_z(r,\xi)\det(\nabla^2u+uI)=\lambda f(x)\psi(u),\tag{6}$$

- $G_z<0(\Omega\in\mathcal{K}_0^{n+1})$, existence for general measure, Gardner et al.'19;

  existence for smooth function f, Liu-Lu.'20.

  - When $G(r,\xi)=r^q$, $\Psi(u)=u^p$, the result covers the solution to the L_p dual Minkowski problem for $q<0$, $p<0$ (or $p>0$).

- $G_z > 0(\Omega \in \mathcal{K}^{n+1})$, polytopal solutions to the discrete measures and approximation argument, Gardner-Hug-Xing-Ye.'20.

Theorem (Gardner-Hug-Xing-Ye.'20)

Let $G : [0,\infty) \times \mathbb{S}^n \to [0,\infty)$ be continuous, $G_z > 0$ on $(0,\infty) \times \mathbb{S}^n$, and $\psi : (0,\infty) \to (0,\infty)$ be continuous. Suppose that G and ψ satisfy
(I) $zG_z(z,\xi)$ *is continuous on* $[0,\infty) \times \mathbb{S}^n$,
(II) $zG_z(z,\xi) = 0$ *at* $z = 0$ *for* $\xi \in \mathbb{S}^n$,
(III) $\lim_{t\to 0^+} \psi(t)/t = 0$ *and* $\int_1^\infty \frac{\psi(s)}{s} ds = \infty$.
Let μ be finite Borel measure on $\mathbb{S}^n$ that is not concentrated on any closed hemisphere. Then there is a convex body $\Omega \in \mathcal{K}^{n+1}$ such that (5) *holds.*

- When $G(z,\xi) = z^q$, $\Psi(u) = u^p$ and $\psi(t) = t\Psi'(t)$, the result covers the solution to the L_p dual Minkowski problem for $q > 0$, $p > 1$. (The results is obtained by Böröczky-Fodor.'19).

Problem: How to remove the condition $\lim_{t\to 0^+} \psi(t)/t = 0$?

The MOG image problem for the case $G_z < 0$

Let $\mathcal{G}_d$ be the class of continuous functions $G : (0, \infty) \times \mathbb{S}^n \to (0, \infty)$ such that

- $zG_z(z, \xi)$ is continuous on $(0, \infty) \times \mathbb{S}^n$;
- $G_z < 0$ on $(0, \infty) \times \mathbb{S}^n$;
- $\lim_{t\to 0^+} G(z, \xi) = +\infty$ and $\lim_{t\to +\infty} G(z, \xi) = 0$.

Theorem (Huang-Xing-Ye-Zhu'21)

(1) *Let λ and μ be two nonzero finite Borel measures on $\mathbb{S}^n$ that are not concentrated on any closed hemisphere. Suppose that $G \in \mathcal{C}$ and $\Psi \in \mathcal{C}$ such that*
(i)$G \in \mathcal{G}_d$,
(ii) $\Psi_t = \frac{\partial \Psi(t,x)}{\partial t} > 0$ satisfying $\lim_{s\to +\infty} \Psi(s, x) = +\infty$.
Then there exists a $\Omega \in \mathcal{K}_0^{n+1}$ such that (4) holds.

(2) *Let λ and μ be two nonzero finite even Borel measures on $\mathbb{S}^n$ that are not concentrated on any closed hemisphere. Suppose that $G \in \mathcal{C}$ and $\Psi \in \mathcal{C}$ such that*
(i)$G(z, \xi) = G(z, -\xi)$ and $\Psi(t, x) = \Psi(-t, x)$,
(ii)$G \in \mathcal{G}_d$ and $\Psi \in \mathcal{G}_d$.
Then there exists a $\Omega \in \mathcal{K}_e^{n+1}$ such that (4) holds.

The MOG image problem for the case $G_z > 0$

Let $\mathcal{G}_I^0$ be the class of continuous functions $G : [0,\infty) \times \mathbb{S}^n \to [0,\infty)$ such that

- $zG_z(z,\xi)$ is continuous on $[0,\infty) \times \mathbb{S}^n$;
- $G_z > 0$ on $(0,\infty) \times \mathbb{S}^n$;
- $G(0,\xi) = 0$ and $zG_z(z,\xi) = 0$ at $z = 0$ for $\xi \in \mathbb{S}^n$.

Theorem (Li-Sheng-Ye-Yi'21)

Let $G \in \mathcal{G}_I^0$, $\Psi \in \mathcal{G}_I^0$ and λ be a nonzero finite Borel measure on $\mathbb{S}^n$. Assume the following conditions on G, λ and Ψ.
(i) $d\lambda(\xi) = p_\lambda(\xi)d\xi$ *where* $p_\lambda : \mathbb{S}^n \to (0,\infty)$ *is continuous.*
(ii) *For all $x \in \mathbb{S}^n$, the following holds:*

$$\lim_{s\to+\infty} \Psi(s,x) = +\infty. \tag{7}$$

Let μ be a nonzero finite Borel measure on $\mathbb{S}^n$ that is not concentrated on any closed hemisphere. Then there is a convex body $\Omega \in \mathcal{K}^{n+1}$ such that (2) holds, with the constant $\tau = \frac{1}{\widetilde{C}_{G,\lambda}(\Omega,\mathbb{S}^n)} \int_{\mathbb{S}^n} \psi(u_\Omega(x),x)\, d\mu(x)$.

The suitably designed curvature flow

Our proof is based on the study of a suitably designed parabolic flow and the use of approximation argument.

Let $G \in \mathcal{G}_I^0$, $\Psi \in \mathcal{G}_I^0$ and $\psi = z\Psi_z$.

$$\begin{cases} \textbf{Case 1:} \quad \liminf_{s \to 0^+} \frac{sG_z(s,x)}{\psi(s,x)} = \infty, \quad \text{for all } x \in \mathbb{S}^n. \\ \textbf{Case 2:} \quad \liminf_{s \to 0^+} \frac{sG_z(s,x)}{\psi(s,x)} < \infty, \quad \text{for some } \ x \in \mathbb{S}^n. \end{cases} \tag{8}$$

In Case 1, the convex body Ω satisfies Musielak-Orlicz-Gauss image problem $\Longrightarrow \Omega \in \mathcal{K}_0^{n+1}$.

- Specially, when $G(s,x) = s^q$ and $\psi(s,x) = s^p$, it extends the L_p dual Minkowski problem for the case $p > q > 0$.

In Case 2, the convex body Ω satisfies Musielak-Orlicz-Gauss image problem $\Longrightarrow \Omega \in \mathcal{K}^{n+1}$.

- Specially, when $G(s,x) = s^q$ and $\psi(s,x) = s^p$, it extends the L_p dual Minkowski problem for the case $q \geq p > 0$

The suitably designed curvature flow

Let G, Ψ, f and p_λ be smooth positive functions, suppose that $X(\cdot, t)$ be a smooth solution to the flow (9) , and $\mathcal{M}_t = X(\mathbb{S}^n, t)$ be a smooth, closed and uniformly convex hypersurface.

For Case 1: $\liminf_{s\to 0^+} \frac{sG_z(s,x)}{\psi(s,x)} = \infty$, considering the following curvature flow

$$\begin{cases} \frac{\partial X}{\partial t}(x,t) &= \left(-f(\nu)\psi(u,x)r^n G_z(r,\xi)^{-1} p_\lambda^{-1}(\xi)K + \eta(t)u\right)\nu, \\ X(x,0) &= X_0(x), \end{cases} \tag{9}$$

where $\xi = \alpha^*_{\Omega_t}(x)$, and

$$\eta(t) = \frac{\int_{\mathbb{S}^n} f\psi(u,x)dx}{\int_{\mathbb{S}^n} rG_z(r,\xi)p_\lambda(\xi)d\xi}. \tag{10}$$

- The functional

$$\mathcal{J}(u) = \int_{\mathbb{S}^n} f\Psi(u,x)dx.$$

The suitably designed curvature flow

Lemma

Let $X(\cdot, t)$ be a smooth solution to the flow (9) with $t \in [0, T)$, and $\mathcal{M}_t = X(\mathbb{S}^n, t)$ be a smooth, closed and uniformly convex hypersurface. Suppose that the origin lies in the interior of the convex body Ω_t enclosed by $\mathcal{M}_t$ for all $t \in [0, T)$. Then, for any $t \in [0, T)$, one has

$$\widetilde{V}_{G,\lambda}(\Omega_t) = \widetilde{V}_{G,\lambda}(\Omega_0). \tag{11}$$

Lemma

The functional $\mathcal{J}$ is non-increasing along the flow (9). That is, $\frac{d\mathcal{J}(u(\cdot,t))}{dt} \leq 0$, with equality if and only if $\mathcal{M}_t$ satisfies the elliptic equation (6).

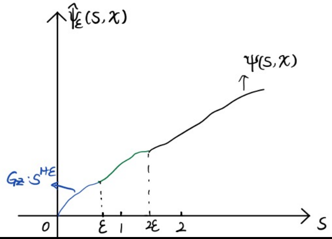
The suitably designed curvature flow

For Case 2: $\liminf\limits_{s\to 0^+} \dfrac{sG_z(s,x)}{\psi(s,x)} < \infty$, considering a suitably designed parabolic flow with the smooth function ψ replaced by the smooth function $\widehat{\psi}_\varepsilon : [0,\infty)\times\mathbb{S}^n \to [0,\infty)$, $\varepsilon\in(0,1)$, as follows:

$$\widehat{\psi}_\varepsilon(s,x) = \begin{cases} \psi(s,x), & \text{if } s \geq 2\varepsilon, \\ \\ G_z(s,\alpha^*(x))s^{1+\varepsilon}, & \text{if } 0\leq s\leq \varepsilon, \end{cases} \tag{12}$$

and $\widehat{\psi}_\varepsilon(s,x) \leq C_0$ for $(s,x)\in(\varepsilon,2\varepsilon)\times\mathbb{S}^n$ is chosen so that $\widehat{\psi}_\varepsilon$ is smooth on $[0,\infty)\times\mathbb{S}^n$ and $\widehat{\psi}_\varepsilon(s,x) > 0$ for all $(s,x)\in(0,\infty)\times\mathbb{S}^n$. Hereafter,

$$C_0 = \max\{1, \max_{(s,x)\in[0,2]\times\mathbb{S}^n} \psi(s,x)\} \tag{13}$$



The suitably designed curvature flow

For Case 2: $\liminf_{s\to 0^+} \frac{sG_z(s,x)}{\psi(s,x)} < \infty$, considering the following curvature flow:

$$\begin{cases} \frac{\partial X_\varepsilon}{\partial t}(x,t) & = \left(-f(\nu)\widehat{\psi}_\varepsilon(u_\varepsilon, x) r^n G_z(r,\xi)^{-1} p_\lambda^{-1}(\xi) K + \eta_\varepsilon(t) u_\varepsilon\right)\nu, \\ X_\varepsilon(x,0) & = X_0(x), \end{cases} \tag{14}$$

where $X_\varepsilon(\cdot,t) : \mathbb{S}^n \to \mathbb{R}^{n+1}$ parameterizes convex hypersurface $\mathcal{M}_t^\varepsilon$, u_ε denotes the support function of the convex body Ω_t^ε circumscribed by $\mathcal{M}_t^\varepsilon$, and

$$\eta_\varepsilon(t) = \frac{\int_{\mathbb{S}^n} f\widehat{\psi}_\varepsilon(u,x)dx}{\int_{\mathbb{S}^n} rG_z(r,\xi)p_\lambda(\xi)d\xi}. \tag{15}$$

Outline of proof for the Case 2

Let G, Ψ, f and p_λ be smooth positive functions, and u_0 be a positive and uniformly convex function. Suppose that $u_\varepsilon(,t)$ is positive, smooth and uniformly convex solution to the flow (14) for all $t \in [0,T)$.
Step 1: C^0 and C^1-estimates: $C_\varepsilon^{-1} \leq u_\varepsilon(\cdot,t) \leq C_1$, $|\nabla u_\varepsilon(\cdot,t)| \leq C_1$.

- Maximum principle.
  The key point is the uniform bound of $\eta_\varepsilon(t)$.(The construction idea of function $\widehat{\psi}_\varepsilon$)

Lemma

Let $u(\cdot,t)$ be a positive, smooth and uniformly convex solution to (14). Then

$$\frac{1}{C_2} \leq \eta_\varepsilon(t) \leq C_2 \quad \text{for all } t \in [0,T), \tag{16}$$

where $C_2 > 0$ is a constant depending only on f, p_λ, G, Ψ and Ω_0, but independent of ε.

Outline of proof for the Case 2

Step 2: The maximal and minimal widths of Ω are defined respectively,

$$w_\Omega^+ = \max_{x\in\mathbb{S}^n}\{u_\Omega(x) + u_\Omega(-x)\} \quad \text{and} \quad w_\Omega^- = \min_{x\in\mathbb{S}^n}\{u_\Omega(x) + u_\Omega(-x)\}.$$

Lemma

Let $u(\cdot, t)$ be a positive, smooth and uniformly convex solution to (14). Then there is a constant $C_3 > 0$ depending only on f, p_λ, G, Ψ, and Ω_0, but independent of ε, such that, for all $t \in [0, T)$,

$$1/C_3 \le w_{\Omega_t}^- \le w_{\Omega_t}^+ \le C_3.$$

Outline of proof for the Case 2

Step 3: C^2-estimates: $\overline{C}_\varepsilon^{-1} I \leq \nabla^2 u_\varepsilon(\cdot, t) + u_\varepsilon(\cdot, t) I \leq \overline{C}_\varepsilon I$.
(Tso'85, Urbas'91, Ivaki'16, Li-sheng-Wang'16......)

- Consider the following auxiliary function

$$Q = \frac{-u_t + u\eta(t)}{u - \epsilon},$$

where $\epsilon = \frac{1}{2} \inf_{\mathbb{S}^n \times [0,T)} u(x,t)$. Let $x_t \in \mathbb{S}^n$ for each $t \in [0, T)$ be such that $Q(x_t, t) = \max_{x \in \mathbb{S}^n} Q(x, t)$.
(the maximum principle)$\Longrightarrow$ $\det(\nabla^2 u_\varepsilon + u_\varepsilon I) \geq 1/C_\varepsilon$.

- Consider the following auxiliary function

$$W(x,t) = \log b(x,t) - \beta \log u(x,t) + \frac{A}{2} r^2(x,t),$$

where β and A are large constants to be decided, and

$$b(x,t) = \max \left\{ \sum b_{ij}(x,t)\zeta_i\zeta_j : \sum_i \zeta_i^2 = 1 \right\}.$$

where $b_{ij} = u_{ij} + u\delta_{ij}$. LetT'$\in (0, T)$ be an arbitrary number but fixed. Assume that W attains its maximum on $\mathbb{S}^n \times [0, T']$ at (x_0, t_0) with $t_0 > 0$.(the maximum principle)$\Longrightarrow$ $\nabla^2 u_\varepsilon + u_\varepsilon I \leq C_\varepsilon$.

Outline of proof for the Case 2

Step 4: Long time existence of solution to the flow (14), together with the monotonicity of the functional $\mathcal{J}(u(\cdot)) \Rightarrow$ there exists a subsequence of $\{u_\varepsilon(\cdot, t_i)\}$ converging to a positive and uniformly convex function $u_{\varepsilon,\infty} \in C^\infty(\mathbb{S}^n)$ satisfying that

$$u_{\varepsilon,\infty}(x) r_{\varepsilon,\infty}^{-n}(\xi) G_z\big(r_{\varepsilon,\infty}(\xi), \xi\big) p_\lambda(\xi) \det\big(\nabla^2 u_{\varepsilon,\infty}(x) + u_{\varepsilon,\infty}(x) I\big) = \gamma_\varepsilon f(x) \widehat{\psi}_\varepsilon(u_{\varepsilon,\infty},$$

where $\gamma_\varepsilon = \lim_{t_i \to \infty} \frac{1}{\eta_\varepsilon(t_i)}$. That is, $\Omega_{\varepsilon,\infty} \in \mathcal{K}_V$ with

$$\mathcal{K}_V = \Big\{ \mathbb{K} \in \mathcal{K} : \widetilde{V}_{G,\lambda}(\mathbb{K}) = \widetilde{V}_{G,\lambda}(\Omega_0) \Big\}.$$

$\Omega_{\varepsilon,\infty}$ solves the following optimization problem:

$$\inf \Big\{ \int_{\mathbb{S}^n} f(x) \widehat{\Psi}_\varepsilon(u_{\mathbb{K}}(x), x)\, dx : \mathbb{K} \in \mathcal{K}_V \Big\}.$$

Outline of proof for the case 2

Step 5: Recall that $\frac{1}{C} \le w^-_{\Omega_{\varepsilon,\infty}} \le w^+_{\Omega_{\varepsilon,\infty}} \le C$ and $\frac{1}{C} \le \gamma_\varepsilon \le C$, where C is independent of ε.

Proposition

Let $G \in \mathcal{G}^0_I$ and λ be a nonzero finite Borel measure on $\mathbb{S}^n$ which is absolutely continuous with respect to $d\xi$. Then the measure $\widetilde{C}_{G,\lambda}(\cdot,\cdot)$ is weakly convergent on $\mathcal{K}$, namely, if $\Omega_i \in \mathcal{K}$ for all $i \in \mathbb{N}$ and Ω_i converges to $\Omega \in \mathcal{K}$ in the Hausdorff metric, then $\widetilde{C}_{G,\lambda}(\Omega_i,\cdot) \to \widetilde{C}_{G,\lambda}(\Omega,\cdot)$ weakly.

Hence, a constant $\gamma_0 > 0$ and a sequence $\varepsilon_i \to 0$ can be found so that $\gamma_{\varepsilon_i} \to \gamma_0$. For each Borel set $\omega \subseteq \mathbb{S}^n$,

$$\gamma_0 \int_\omega \psi(u_\infty, x)\, d\mu(x) = \int_{\alpha^*_{\Omega_\infty}(\omega)} r_\infty(\xi) G_z(r_\infty(\xi), \xi) p_\lambda(\xi)\, d\xi = \int_\omega d\widetilde{C}_{G,\lambda}(\Omega_\infty, \xi$$

Moreover, Ω_∞ satisfies

$$\int_{\mathbb{S}^n} f\Psi(u_{\Omega_\infty}, x) dx = \inf\left\{ \int_{\mathbb{S}^n} f(x)\Psi(u_{\mathbb{K}}, x) dx : \mathbb{K} \in \mathcal{K}_V \right\}.$$

Outline of proof for the Case 2

Lemma

Let $G \in \mathcal{G}_I^0$ be a smooth function. Suppose that $d\mu(\xi) = f(\xi)\, d\xi$ and $d\lambda(\xi) = p_\lambda(\xi)\, d\xi$ with f and p_λ being smooth and strictly positive on $\mathbb{S}^n$. Let $\Psi \in \mathcal{G}_I^0$ be a smooth function satisfying (7). The following statements hold.
(i) *If G and ψ satisfy the conditions in Case 1, then one can find an $\Omega \in \mathcal{K}_0^{n+1}$ such that (4) holds;*
(ii) *If G and ψ satisfy the conditions in Case 2, then one can find an $\Omega \in \mathcal{K}^{n+1}$ such that (2) holds.*

Outline of proof for the Case 2

Using the standard approximations for the functions G, p_λ and Ψ in Main Theorem.

Corollary

Let G, p_λ and Ψ be as in Main Theorem and f be a smooth positive function on $\mathbb{S}^n$, then there exist $\gamma > 0$ and $\Omega \in \mathcal{K}_V$ such that Ω satisfies

$$\int_{\alpha^*_\Omega(\omega)} rG_z(r,\xi)p_\lambda(\xi)d\xi = \gamma \int_\omega f\psi(u,x)dx, \ \forall \ Borel \ set \ \omega \subseteq \mathbb{S}^n$$

and

$$\int_{\mathbb{S}^n} f\Psi(u_\Omega, x)dx = \inf\Big\{ \int_{\mathbb{S}^n} f\Psi(u_{\mathbb{K}}, x)dx : \mathbb{K} \in \mathcal{K}_V \Big\}.$$

Thank you for your attention!