Contemporary Challenges in Trefftz Methods, from Theory to Applications (24w5218)

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1 Overview of the Field

Trefftz methods are a class of methods for the numerical approximation of solutions to partial differential equations. The study of complex physical systems is nowadays systematically assisted with numerical simulations, which are used both in academic and industrial research. The advent of supercomputers provides scientists and engineers with increasing computational resources, and efficient and accurate numerical methods are required to leverage such resources for the study of large-scale phenomena. As numerical methods for the approximation of solutions to boundary and initial value problems, Trefftz methods open the perspective of reducing the computational costs while maintaining a high degree of accuracy.

Trefftz methods belong to the class of finite element methods, yet their fundamental specificity is to rely on approximation spaces specifically selected for the problem at hand. While standard finite element methods are based use on approximation spaces made by piecewise polynomial functions, independently on the considered problem, Trefftz methods rely on approximation spaces made by function that are in the kernel of the partial differential equation to be discretized. The benefits of Trefftz methods derive from the fact that the physics of the problem is injected into the basis functions. For instance, in the field of time-harmonic wave propagation, basis functions such as plane waves or spherical waves encode an oscillating behavior. As compared to, for instance, standard finite element methods, they can better reproduce intrinsic properties of the physical solutions, and can approximate solutions within a given accuracy using less computational resources.

The main features of Trefftz methods can be summarized as follows:

- Approximation spaces made of local solutions, in comparison to classic polynomial spaces, can achieve the same high order convergence while requiring considerably fewer degrees of freedom.
- Like discontinuous finite element methods, Trefftz methods are very suitable for parallel computation.
- It is possible to formulate Trefftz methods so that the discrete problem involves only integrals over the skeleton of the mesh, thus saving on memory storage and computation time.

2 Recent Developments and Open Problems

The twofold objective of this workshop was to tackle

- contemporary challenges in the theoretical study of Trefftz methods,
- and their application to large-scale problems.

Recent theoretical developments have mainly focused on overcoming three significant limitations of Trefftz methods, namely *(i)* ill-conditioning of wave-based methods, *(ii)* restriction to constant-coefficient partial differential equations, and *(iii)* restriction to homogeneous equations (zero right-hand side).

The workshop has gathered experts who have developed theoretical analysis of Trefftz methods and tackled these new challenges, together with scientists and engineers - both from academia and industry- interested in efficient and accurate simulation of large-scale physical problems via Trefftz methods. Experts in closely related fields (such as discontinuous Galerkin or Petrov–Galerkin methods) have also participated in the workshop, creating a unique opportunity to disseminate recent breakthroughs, stimulate new interactions and shape the future developments in the theory and applications of Trefftz methods.

Challenges and recent developments

In 1997, the pioneer of finite element methods O. Zienkiewicz wrote: "it seems without doubt that in the future Trefftz type elements will frequently be encountered in general finite element codes. ... It is the author's belief that the simple Trefftz approach will in the future displace much of the boundary type analysis with singular kernels."

Even though Trefftz methods have certainly seen considerable developments since then, in particular in the field of wave propagation, and have been implemented in industry (cf. [1]), they have not yet made such a significant impact. Several reasons come to mind as limiting barriers, however, these have been addressed by recent developments.

1. Wave-based bases, such as plane or spherical wave bases, suffer from an ill-conditioning problem. The development of Trefftz virtual element methods as well as linear algebra techniques for the resolution of the discrete linear system have now proved that it is possible to free high-order Trefftz methods from the effect of this ill-conditioning, see for instance [2, 3].
2. Basis functions that solve exactly the governing differential equation are not available for many equations, even in the field of wave propagation when the propagating medium is not homogeneous. Quasi-Trefftz methods are currently being developed to allow instead for basis functions that only solve approximately the governing equation, see for instance [4].
3. The proved high order convergence properties of Trefftz methods are limited to problems governed by an equation with no sources. However, to be able to handle problems with a non-zero source, a new method called embedded Trefftz was recently proposed to replace the explicit construction of a Trefftz space by constructing a corresponding embedding, see for instance [5].

It then seems that the prediction on the impact of Trefftz methods has all chances to become true in the near future.

Artificial intelligence techniques have permeated many aspects of modern numerical simulations. They often generate high computational burdens and their efficiency is not systematic, due to instability phenomena that are not well understood. Yet in some cases they have demonstrated spectacular results, making statistical methods serious candidates for the simulations of tomorrow. In the absence of proofs of convergence and stability, so dear to mathematicians, a new approach consists in assisting classical numerical methods, for which we have a well-framed theory, by neural networks whose learning is exploited to deploy a large number of simulations. In this context, the Physically Informed Neural Networks (PINNs) have demonstrated an interesting potential to solve partial differential equations. PINNs share with Trefftz methods the idea of integrating the physics of the considered phenomenon, and some preliminary results show that their coupling works well, at least for problems in one and two space-dimensions. Recent developments in this direction have been discussed in the workshop.

High-performance computing landscape and Trefftz methods

The richness of research to develop and implement advanced numerical methods to simulate real phenomena, whether natural or resulting from human activity, is no longer to be demonstrated. One of the motivations to build new numerical methods is the absolutely necessary scalability to consider the use of supercomputers that can make real what were dreams a decade ago, such as inverting the global Earth, probing the interior of the Sun, or imaging a whole human body. Considerable resources are invested all around the globe in high-performance computing capabilities. The most powerful computers in the world include Frontier in the United States of America (built-in 2022), Fugaku in Japan (built-in 2020), Lumi in Finland (built-in 2022), and Sunway TaihuLight in China (built-in 2016). Yet the resulting processing power can only be leveraged thanks to scalable computational methods. In order to stimulate fruitful collaborations towards the development of scalable methods, many initiatives, such as the EuroHPC program, encourage collaborations between applied mathematicians, computer scientists, and scientists concerned with applications.

Trefftz methods address the approximation of solutions to boundary-value or initial-value problems governed by partial differential equations. Several of their fundamental aspects are specifically amenable to scalability. The central property of Trefftz methods is to rely on function spaces of local solutions to the governing equations, as opposed to more classical polynomial spaces. As a result, *(i)* the Trefftz-weak formulation of the problem at stake in dimension d involves integrals that are restricted to domains in dimension $d-1$, and *(ii)* the number of degrees of freedom required to reach high order convergence is considerably lower than with standard polynomial bases. Hence the computational burden to compute the matrix entries of the discrete system is significantly reduced compared to classic finite element methods. Besides, Trefftz-weak formulations are perfectly suitable for domain decomposition techniques, allowing them to tackle large-scale problems as can appear in meteorology, geophysics, or astrophysics.

If we want to continue our progress in the simulation of complex phenomena, it is crucial to propose less energy-consuming numerical solutions. This issue is essential in the context of the energy crisis that the whole world is beginning to face. As they are massively parallelizable and can play a leading role in the construction of more parsimonious numerical methods for the simulation of problems governed by partial differential equations, Trefftz methods deserve a great deal of attention.

The following questions on Trefftz methods in high-performance computing have been discussed in the workshop:

- Are Trefftz methods completely ready for exascale calculations?
- How do they compare with other methods widely used in industry, such as finite differences or spectral elements, in terms of precision and efficiency?
- Can we define a set of benchmarks that can then be used to compare methods coupling Trefftz methods and machine learning with different approaches?
- Which applications should be considered in priority to answer tomorrow's societal issues through simulations? Do they all need supercomputing resources?

3 Presentation Highlights

The workshop has covered topics ranging from theoretical study to practical aspects of implementation for high-performance computing. We report on the scientific talks, on the software presentations, and on the panel discussions, which took place during the workshop.

Peter Monk, The ultra weak variational formulation of Maxwell's equations
The Ultra Weak Variational Formulation (UWVF) of Maxwell's equations was proposed in Olivier Cessenat's seminal thesis in 1996. It is a Trefftz method based, typically, on the use of plane waves to provide a local approximation of the global solution on a finite element grid. Local solutions are coupled via upwinding across element faces in a variational formulation that enforces inter-element continuity approximately. For real electromagnetic coefficients, the method is a special case of a more general interior penalty Trefftz

discontinuous Galerkin method, and this observation underlies an error analysis of the method. After a historical overview of the UWVF for Maxwell's equations including a derivation, summary of the error analysis and comments on the solution of the linear system, some numerical results concerning the choice of number of plane wave directions per element and a comparison to the finite element method were presented.

Paul Stocker, Embedded Trefftz discontinuous Galerkin methods
A new variant of Trefftz methods, the embedded Trefftz discontinuous Galerkin method, was presented. This approach is based on the Galerkin projection of an underlying discontinuous Galerkin (DG) method onto a subspace of Trefftz-type. The subspace can be described in a very general way. To obtain it no Trefftz functions have to be calculated explicitly, instead the corresponding embedding operator is constructed. In the simplest cases, the method recovers established Trefftz DG methods. However, the approach can be conveniently extended to general cases, including inhomogeneous sources and non-constant coefficient differential operators. After an introduction to the method, implementational aspects have been discussed and its potential has been explored on a set of standard PDE problems. As typical for Trefftz-DG methods, the new approach leads to a severe reduction of the globally coupled unknowns when compared to standard DG methods, reducing the corresponding computing time significantly. Moreover, for the Helmholtz problem, an improved accuracy similar to Trefftz DG methods based on plane waves, has been reported.

Igor Voulis, Trefftz discontinuous Galerkin methods for the Stokes problem
Trefftz Discontinuous Galerkin (DG) methods provide a way to reduce the computational costs of DG methods. Recently, with the introduction of embedded, weak and quasi-Trefftz DG methods, the range of applications for the Trefftz DG paradigm has increased. This presentation was concerned with the application of the embedded Trefftz methodology for solving the Stokes equations as an example for a vectorial PDE. Discrete solutions of a corresponding method fulfill the Stokes equation pointwise within each element and yield element-wise divergence-free solutions, but are not normal-continuous. Due to the Trefftz ansatz, velocity and pressure unknowns are strongly coupled on an element level. This gives rise to a special structure in the discrete Stokes saddle-point problem. The structure of the discrete formulation was presented and a full a-priori error analysis was outlined. Further, implementational aspects for the construction of Trefftz bases and the handling of inhomogeneous forcing terms on the right-hand side were addressed, together with a comparison with other non-conforming methods. Together with a numerical validation, current limitation and possible extensions were discussed.

Pedro Antunes, A well-conditioned method of fundamental solutions for Laplace equation
The method of fundamental solutions (MFS) is a numerical method for solving boundary value problems involving linear partial differential equations. It is well-known that it can be very effective assuming regularity of the domain and boundary conditions. The main drawback of the MFS is that the matrices involved are typically ill-conditioned and this may prevent the method from achieving high accuracy. A new algorithm to remove the ill-conditioning of the classical MFS in the context of the Laplace equation defined in planar domains has been discussed in this talk. The main idea is to expand the MFS basis functions in terms of harmonic polynomials. Then, using the singular value decomposition and Arnoldi orthogonalization we define well conditioned basis functions spanning the same functional space as the MFS's. Several numerical examples showed that when possible to be applied, this approach is much superior to previous approaches, such as the classical MFS or the MFS-QR.

Joseph Coyle, Preconditioning strategies for discontinuous Galerkin plane wave Trefftz methods: a single element analysis
Plane-wave Trefftz methods for the Helmholtz equation offer significant advantages over standard discretization approaches. However, a disadvantage of these methods which cannot be overlooked is the inherent poor conditioning in the resulting system matrices. The focus of this talk was to examine the conditioning of the plane-wave discontinuous Galerkin method with respect to a single physical element where the properties of the mass and stiffness matrices depend on the size and geometry of the domain. The analysis begun by considering a single disk-shaped element. In this setting, the condition of the matrices can be quantified in a straightforward way. In addition, the matrices related to these elements have characteristics that allow for easily constructed preconditioners. This was followed by numerical investigation of simple preconditioning strategies on more practical polygonal domains where we employ preconditioners with characteristics similar to the matrices for the disk-shaped element.

Andrea Moiola, Stable approximation of Helmholtz solution with evanescent plane waves
One of the prototypical applications of Trefftz methods is the approximation of solutions to the Helmholtz equation $\Delta u + k^2 u = 0$, with positive wavenumber k, by propagative plane waves (PPWs) $\mathbf{x} \mapsto e^{ik\mathbf{x}\cdot\mathbf{d}}$. However the representation of Helmholtz solutions by linear combinations of PPWs is notoriously unstable: to approximate some smooth solutions, linear combinations of PPWs require huge coefficients. In computer arithmetics, this leads to numerical cancellation and prevents any accuracy. This can be shown rigorously for the unit ball in 2D and 3D. This instability is often described in terms of ill-conditioning. A remedy to such instability is the use of evanescent plane waves (EPWs): plane waves with complex propagation vectors $\mathbf{d}$. It was shown that any Helmholtz solution u on a ball can be written as a continuous superposition of EPWs, and that the coefficient density is bounded by the H^1 norm of u. A discretization strategy of this representation, based on modern sampling techniques, was proposed to approximate any solution u with a finite combination of EPWs with bounded coefficients. The theory presented has been supported by numerical experiments on the ball and on convex shapes, including the application to Trefftz discontinuous Galerkin schemes.

Christoph Lehrenfeld, The Trefftz approach for unfitted finite element methods
In recent years, the Trefftz Discontinuous Galerkin (DG) method has emerged as a promising alternative to standard DG methods, particularly in the realm of body-fitted finite element method (FEM) discretizations. The potential of Trefftz methods has been investigated, in this talk, within the context of unfitted discretizations. One notable advantage of Trefftz DG lies in its compatibility with major stabilization concepts for unfitted FEM, such as the Ghost penalty method, contrasting with the conflict that hybridization poses in this regard. This compatibility was illustrated through the lens of a fictitious domain problem, highlighting the efficacy of the Trefftz approach in such scenarios. Moreover, in surface PDEs within unfitted FEM, also referred to as TraceFEM, the background finite element space operates within a higher-dimensional domain, namely the relevant part of the background mesh. This characteristic often translates into heightened computational costs compared to body-fitted discretizations. Here, the Trefftz method offers a solution by enabling a reduction in the dimensionality of the underlying finite element space, thereby mitigating computational overhead. The conceptual framework and analysis of the Trefftz DG method were discussed for the two unfitted cases, and an outlook on further potential benefits and applications of Trefftz DG methods in this area was given.

Sergio Gomez, Space-time ultra-weak discontinuous Galerkin method for the Schrödinger equation
For time-dependent problems, the natural setting for Trefftz methods is the space-time framework. In this talk, a space-time ultra-weak discontinuous Galerkin formulation was presented for the linear time-dependent Schrödinger equation. After proving that the method is well-posed and quasi-optimal in mesh-dependent norms for very general discrete spaces, four different choices of discrete spaces were discussed: *(i)* a non-polynomial Trefftz space of complex wave functions, *(ii)* the full polynomial space, *(iii)* a quasi-Trefftz polynomial space, and *(iv)* a polynomial Trefftz space. The accuracy and the advantages of the proposed method were validated in several numerical experiments.

Andrés Prieto Aneiros, A modal-basis partition of unity finite element method for frequency-dependent layered time-harmonic wave propagation problems
This talk focused on a novel Trefftz method to approximate accurately the solution of time-harmonic wave motion problems in acoustics and structural dynamics involving layered materials with frequency-dependent physical properties, particularly at middle and high frequencies. Classical finite element methods (FEM) based on polynomials (even at high-order) are computationally intensive and suffer from phase leaks and pollution phenomena at high-frequency regimes. The proposed numerical method is a modal-based partition of unity finite element method (PUFEM), which utilizes a set of closed-form eigenfunctions as part of the modal basis for a related auxiliary time-harmonic wave motion problem, which can be computed off-line, usually without taking into account all the complexities of the geometrical and physical information of the original time-harmonic wave propagation problem. This combination of a model basis and the partition of unity allows for an accurate representation of the solution at the middle and high-frequency contributions at a reduced computational cost. The method is particularly efficient for problems with a known modal basis in simple geometries and homogeneous isotropic materials. Some numerical results were shown to illustrate the robustness of the proposed method. An industrial application where this numerical methodology is used

to detect cracks in layered materials was discussed.

Chiara Perinati, A quasi-Trefftz DG method for the diffusion-advection-reaction equation with piecewise-smooth coefficients
Trefftz schemes are high-order Galerkin methods whose discrete functions are elementwise exact solutions of the underlying Partial Differential Equation (PDE). Since a family of local exact solutions is needed, Trefftz basis functions are usually restricted to PDEs that are linear, homogeneous and with piecewise-constant coefficients. If the equation has varying coefficients construction of suitable discrete Trefftz spaces is usually out of reach. Quasi-Trefftz methods have been introduced to overcome this limitation, relying on discrete functions that are elementwise "approximate solutions" of the PDE, in the sense of Taylor polynomials. The main advantage of Trefftz and quasi-Trefftz schemes over more classical ones is the higher accuracy for comparable numbers of degrees of freedom. The focus of this talk was to present polynomial quasi-Trefftz spaces for general linear PDEs with smooth coefficients, to describe their optimal approximation properties and to provide a simple algorithm to compute the basis functions, based on the Taylor expansion of the PDE's coefficients. Then, a quasi-Trefftz DG method for the diffusion-advection-reaction equation with varying coefficients was presented, showing stability and high-order convergence of the scheme. The method was also extended to non-homogeneous problems with piecewise-smooth source term, constructing a local quasi-Trefftz particular solution and then solving for the difference. Numerical experiments in 2 and 3 space dimensions highlighted the excellent properties in terms of approximation and convergence rate.

Andrea Lagardere, Quasi-Trefftz method for solving aeroacoustic problems
Variational Trefftz methods are discontinuous Galerkin methods whose basis functions are local solutions of the PDE under consideration. In the context of homogeneous problems, analytical solutions, such as plane waves or Bessel functions, are available. Aeroacoustic models involve equations whose physical characteristics depend on the spatial variables. In general, this PDE system cannot be solved analytically. A natural idea is to resort to basis functions that are approximate solutions of the considered PDE. In this talk, the simplified model $\Delta u + k^2(\mathbf{x})u = 0$ was considered. A strategy to build two families of generalized plane wave bases was presented. The first one is called phase based and takes the following form: $\exp(P(x, y))$, where $P(x, y) = ik_x x + ik_y y + \mathcal{O}(x^2 + y^2)$ is a complex polynomial functions, with $k_x^2 + k_y^2 = k^2(x_0, y_0)$. The second family is called amplitude based: $Q(x, y) \exp(ik_x x + ik_y y)$, where Q is a complex polynomial such that $Q(0, 0) = 1$. The approximation properties of these functions were illustrated with numerical results. Then, a variational formulation for the boundary value problem, based on a hyperbolic system formulation was introduced. It leads to the principle of reciprocity and provides a formulation similar to the classical Ultra Weak Variational Formulation by Cessenat & Després, when the coefficient k^2 is constant.

Bernardo Cockburn, Transforming stabilizations into spaces
A new technique, which allows the transformation of stabilizations into spaces was described and few applications sketched: 1. How to recast any mixed method for second-order elliptic equations as an hybridizable discontinuous Galerkin (HDG) method in order to improve its efficiency. 2. How to show that the discretization of the time derivative by the continuous and discontinuous Galerkin methods for ODEs is exactly the same. And immediately obtain superconvergence points of the DG method obtained. 3. How to define the Turbo Post-Processing (TPP) to transform oscillations of the approximation error around zero into new enhanced accuracy approximations for DG methods for convection and for CG methods for diffusion. As the technique can be applied to any numerical method and any PDE, it was argued it should work for Trefftz methods too.

Lise-Marie Imbert-Gerard, Quasi-Trefftz functions and Quasi-Trefftz spaces
Given a governing equation, the Trefftz property for a function refers to the function satisfying exactly the equation. This talk is concerned with the notion of Taylor-based quasi-Trefftz property: instead of the image of the function by the partial differential operator to be zero, what is zero is the Taylor polynomial of a given order of this image. In the talk, three types of quasi-Trefftz functions were discussed: two types of generalized plane waves and the polynomial type. The questions of existence of quasi-Trefftz functions was studied, under a general condition on the governing equation. As for the question of (non-)uniqueness of these quasi-Trefftz functions, it was addressed in relation to the notion of quasi-Trefftz spaces.

Ralf Hiptmair, Coupling finite elements and Trefftz approximations
As in [D. Casati and R. Hiptmair, Coupling finite elements and auxiliary sources, Comput. Math. Appl.,

77 (2019), pp. 1513-1526], [D. Casati and R. Hiptmair, Coupling FEM with a multiple-subdomain Trefftz method, J. Sci. Comput., 82 (2020), Paper No. 74] and [D. Casati, R. Hiptmair, and J.Smajic, Coupling finite elements and auxiliary sources for electromagnetic wave propagation, International Journal of Numerical Modelling: Electronic Networks, Devices and Fields, 33 (2020), p. E2752], and, in an abstract framework in [D. Casati, L. Codecasa, R. Hiptmair, and F. Moro, Trefftz co-chain calculus, Z. Angew. Math. Phys., 73 (2022), Paper No. 43], scalar and electromagnetic wave propagation in frequency domain on unbounded domains, partly filled with inhomogeneous media, were considered. A discretization that relies on a Trefftz approximation by multipole auxiliary sources in some parts of the domain and on standard mesh-based primal Lagrangian finite elements in other parts was proposed. Several approaches were developed and, based on variational saddle point theory, rigorously analyzed to couple both discretizations across the common interface: 1. Least-squares-based coupling using techniques from PDE-constrained optimization. 2. Coupling through Dirichlet-to-Neumann operators. 3. Three-field variational formulation in the spirit of mortar finite element methods. These approaches were compared in a series of numerical experiments.

Bruno Després, New plane wave basis with strong orthogonality properties
Starting from recent ideas by Parolin-Huybrechs-Moiola, the approximation properties of propagative and non propagative plane waves in various domains were presented. If the domain is a square, it was shown that one can define new families which are Hilbertian with respect to the D-trace scalar product on the boundary (or the N-trace scalar product on the boundary). This opens the possibility to improve on the condition number of the matrices generated by Trefftz methods.

Timo Lähivaara, Electromagnetic wave simulation with ultra-weak variational formulation
The Ultra-Weak Variational Formulation (UWVF) is a Trefftz discontinuous Galerkin method, utilizing superpositions of plane waves for localized solutions on a finite element grid. The UWVF was applied to the time-harmonic Maxwell's equations, with focus on the parallel implementation of UWVF in the software named ParMax, and highlighting its efficacy in the simulation of electromagnetic wave problems. Recent enhancements such as the support for various element types, curved elements, and a low memory strategy have a major impact on the software's applicability in industrial problems. Here, the applicability of the simulation software was examined using various numerical examples. Leveraging large curved elements and the low-memory strategy, X-band frequency scattering from an aircraft could be effectively simulated, underscoring the practical utility of ParMax for industrial applications. Potential development directions to overcome some current challenges and speed up the computation were also discussed.

Guosheng Fu, HDG for diffusion
The concept of an M-decomposition was introduced, showing how to use it to systematically construct hybridizable discontinuous Galerkin and mixed methods for steady-state diffusion methods with superconvergence properties on unstructured meshes.

Igor Tsukerman, Trefftz approximations of fields in complex media
Trefftz functions, by definition, satisfy (locally) the underlying differential equation of a given problem, along with the relevant interface boundary conditions. Simple examples include harmonic polynomials for the Laplace equation; plane waves, cylindrical or spherical harmonics for wave problems; exponential functions for the linearized Poisson-Boltzmann equation. At the same time, more complex cases are of great theoretical and practical interest: e.g., waves in disordered structures and Bloch modes in periodic media. In the first part, this presentation was skewed toward numerical techniques rather than physical phenomena; the second part had the opposite slant. Discussed topics were:
1. Numerical applications of Trefftz bases: Finite difference Trefftz schemes "FLAME", the Flexible Local Approximation MEthod), with a variety of tutorial-style examples; Trefftz difference schemes for the Poisson-Boltzmann equation; Trefftz schemes as radiation boundary conditions; Trefftz difference schemes for waves in disordered structures; Trefftz difference schemes for the computation of Bloch bands.
2. Physical applications of Trefftz bases: Non-asymptotic/nonlocal homogenization of periodic structures; Topologically protected boundary modes in electrodynamics.

Reza Abedi, ParaSDG: a parallel-adaptive spacetime solver for hyperbolic and parabolic systems
ParaSDG, a parallel-adaptive causal space-time Discontinuous Galerkin (cSDG) solver for hyperbolic PDEs, a generalization of Tent Pitching, was presented. Adaptive meshing and the numerical solution localize to

patches- clusters of spacetime simplex cells such that all patch-boundary facets are space-like. In lieu of traditional domain decomposition, patches act as the unit of parallel execution within an asynchronous, task-based distributed software architecture. This structure supports probabilistic procedures for extremely dynamic data and load balancing. Causal adaptive meshing in up to $3D \times$ time was described and it was demonstrated how propagating cracks are tracked. Numerical examples were drawn from seismology, fracture mechanics, and electromagnetics. A cSDG variant was explored, in which single spacetime polytope elements cover entire patch domains. This reduces the number of degrees of freedom per patch (vs. simplex elements) and eliminates the need for Riemann solutions in many problems. Although not yet implemented, the potential for a cSDG solver using Trefftz basis functions defined over polytope elements was discussed. Another connection was made with Trefftz methods for simple $1D \times$ time problems wherein the solution from the inflow facets is directly mapped to the outflow facets using precomputed transfer matrices. Schemes that extend cSDG concepts or methods beyond hyperbolic systems were also reviewed. This includes a parabolic-system solver that uses a variant of cSDG spacetime meshing in which stability, rather than the causality constraint limits local time advance. A second scheme uses the cSDG method to find a hyperbolic system's steady or harmonic-state solution.

Sebastien Tordeux, Trefftz variational iterative methods for solving linear hyperbolic systems
Trefftz variational methods, originally introduced by Cessenat and Després, are discontinuous Galerkin numerical methods whose basis functions are solutions of the underlying partial differential equation that we want to solve numerically. They benefit from a solid theoretical framework that ensures their convergence. These methods can be solved iteratively and define a domain decomposition method. This drastically reduce their memory cost avoiding to resort to a LU decomposition. However, they are polluted by rounding errors that have severely limited their use in 3D. A new perspective to these methods was discussed in the context of linear hyperbolic problems. This includes a large variety of PDE systems like heterogeneous and anisotropic acoustic, elastic, and Maxwell systems. Recalling the theory of Friedrichs and Rauch, it was explained how to define general boundary conditions for this type of boundary value problems. It was also explained how these variational methods can be modified to limit the impact of rounding errors. Two alternative techniques were presented: a filtering method and a modification of the basis functions known as quasi-Trefftz. Concrete illustrations on very large computational scenes were discussed.

In addition to these talks, Paul Stocker gave two tutorials on the finite element library NGSolve and on its NGSTrefftz package, respectively.

Paul Stocker, Exploring the finite element library NGSolve: A user's perspective
NGSolve is a versatile finite element solver that offers efficient numerical treatment of partial differential equations. The software's computational core, coded in C++, ensures high performance, while its flexible Python interface enhances user accessibility. An overview of some of NGSolve's functionality was given, underscoring its practicality and adaptability for researchers. In a hands-on demonstration, Paul led the audience into walk through the process of setting up and solving a simple PDE problem using the Python interface. Some of the advanced features of NGSolve, and show how to work with the C++ code were also discussed.

Paul Stocker, NGSTrefftz: Add-on to NGSolve for Trefftz methods The package NGSTrefftz was designed to incorporate Trefftz finite element spaces into NGSolve. This package offers various Trefftz spaces, including harmonic polynomials, plane waves, and caloric polynomials, among others. Moreover, the package introduces unique functionalities, including a quasi-Trefftz space that mimics Trefftz properties for PDEs with smoothcoefficients, space-time Trefftz methods on tent pitched meshes, and a general framework for implicit generation of Trefftz spaces through the embedded Trefftz method. In this second tutorial, it was demonstrated how to set up and utilize these features, showing multiple examples. It was also illustrated how to extend the package, giving insight into its C++ core.

Finally, two panels took place during the workshop:

- one research panel, moderated by H. Barucq; this panel, run as an open discussion, touched on topics like code development, ill conditioning and applications;

- one professional development panel; the discussion between moderator, L.-M. Imbert-Gérard, and panelists A. Nicholopolous and S. Tordeux, revolved around the theme "Mathematics and Industry".

4 Schedule of the workshop

	Monday	Tuesday	Wednesday	Thursday	Friday
9	Opening	Andrea Moiola	Bernardo Cockburn	Ralf Hiptmair	Reza Abedi
9.30	Peter Monk	Andrea Moiola	Bernardo Cockburn	Ralf Hiptmair	Reza Abedi
10	Peter Monk	Chris. Lehrenfeld	L.-M. Imbert-Gérard	Hélène Barucq	Séb. Tordeux
10.30	Coffee break	Coffee break	Coffee break	Coffee break	Coffee break
11	Paul Stocker	Sergio Gomez	Pro. dev. panel	Bruno Despres	Res. panel
11.30	Igor Voulis	Andrés Prieto	Pro. dev. panel	Bruno Després	Res. panel
12	Discussions	Discussions	Lunch	Discussions	Res. panel
12.30	Pedro Antunes	Chiara Perinati	Lunch	Timo Lähivaara	Lunch
13	Joe Coyle	Andrea Lagardere	*Off*	Guosheng Fu	Lunch
13.30	Lunch	Lunch	*Off*	Lunch	——–
15	Discussions	Discussions	*Off*	Discussions	——–
16	Coffee break	Coffee break	*Off*	Coffee break	——–
16.30	NG Solve(I)	NG Solve (II)	*Off*	Igor Tsukerman	——–
18.30	Adjourn	Adjourn	Adjourn	Adjourn	——–

5 Outcome of the Meeting

The workshop came to a close on Friday morning, and to conclude we decided to organize an interactive session during which participants had the opportunity to discuss the medium- and long-term future of Trefftz methods. An initial discussion focused on the impact of Trefftz methods in the industrial sector. Indeed, theory suggests that Trefftz methods should dethrone polynomial methods, but why isn't this the case today? Progress is being made, but it is slower than expected. Quasi-Trefftz methods, which use approximate basis functions, open up many prospects for progress, but the treatment of variable physical parameters remains a real challenge. Progress is being made, but it is slower than expected. Quasi-Trefftz methods, which use approximate basis functions, open up many prospects for progress, but the treatment of variable physical parameters remains a real challenge. We also discussed the question of using Trefftz methods to solve time-dependent problems. One of the presentations (by Reza Abedi) showed their potential, but the idea of building 4D meshes was often rebutted. Our conclusion was that this topic illustrates the value of working closely with experts in HPC and mesh generation. Another important aspect of this workshop was the very strong involvement of young researchers, both PhD students and postdocs. They were very active throughout the week, and we have the feeling that a group has been created. More generally, we all expressed a desire to repeat the workshop experience, as we were brought together with the aim of contributing to the same methodology, but for different applications and therefore with different constraints too. For example, in geophysics, the media are very large, characterized by highly variable parameters, and wave physics in particular translates into ill-conditioned Trefftz matrices. We have then decided to organize a second event on Trefftz methods with the objective of creating an international research group on Trefftz methods.

References

[1] https://www.hypercomp.net/HDPhysicsF.html

[2] H. Barucq, A. Bendali, J. Diaz and S. Tordeux, Local strategies for improving the conditioning of the plane-wave ultra-weak variational formulation, *Journal of Computational Physics* **441** (2021), 110449.

[3] S. Congreve, J. Gedicke and I. Perugia, Numerical investigation of the conditioning for plane wave discontinuous Galerkin methods. In *European Conference on Numerical Mathematics and Advanced Applications*, 493–500, Springer, 2017.

[4] L.M. Imbert-Gerard, Amplitude-based generalized plane waves: New quasi-Trefftz functions for scalar equations in two dimensions, *SIAM Journal on Numerical Analysis* **59**(3) (2021), 1663–1686.

[5] C. Lehrenfeld and P. Stocker, Embedded Trefftz discontinuous Galerkin methods, *International Journal for Numerical Methods in Engineering* **124**(17) (2022), 3637–3661.