Some linked families

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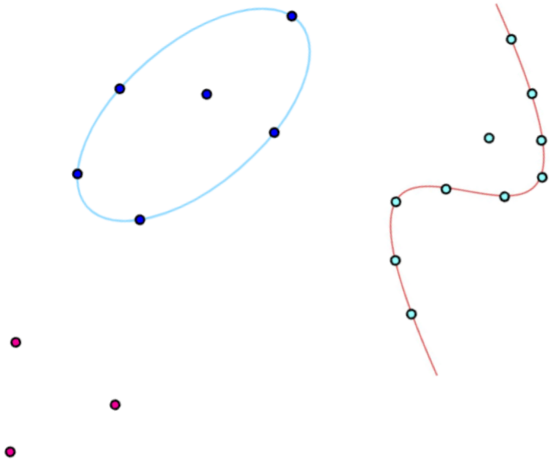
Motivation



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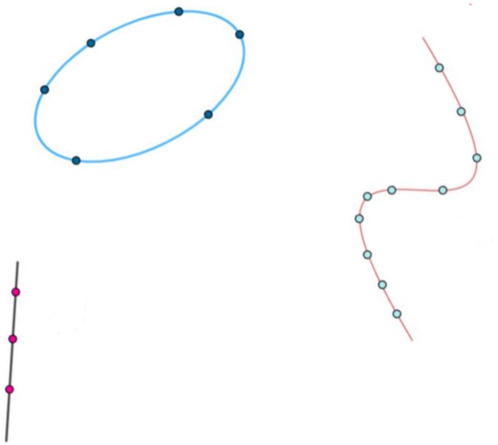

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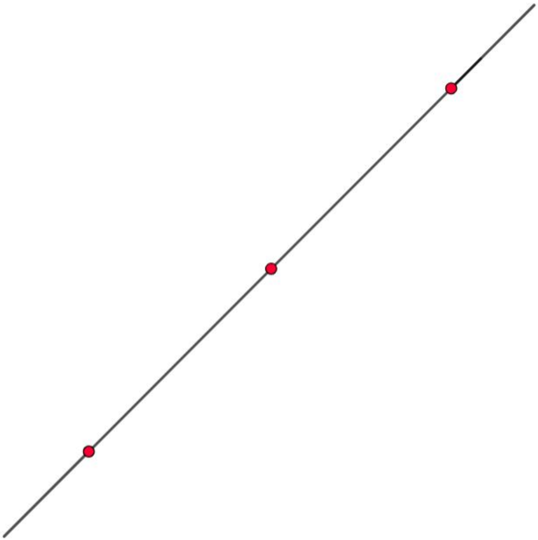

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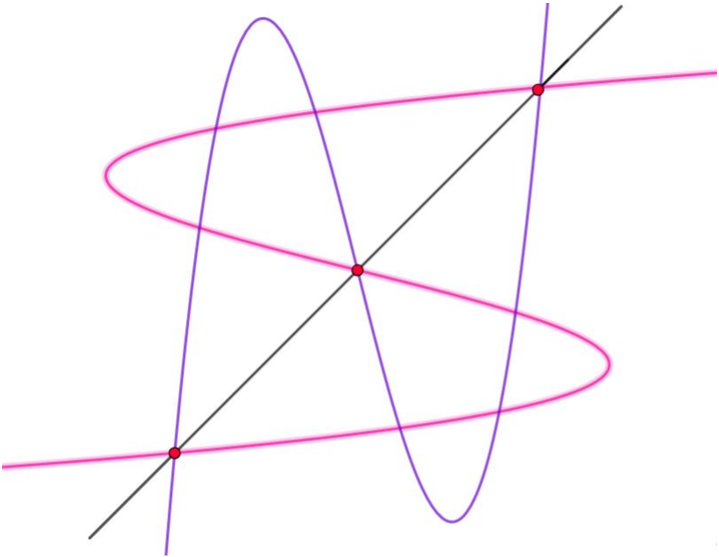
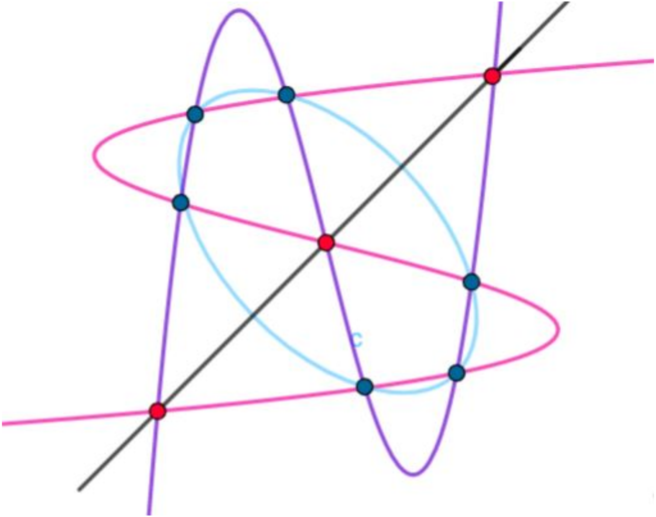


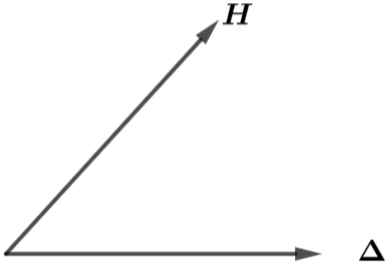


Motivation

3 points

$\mathbb{P}^{2[3]}$ $\rightarrow$ parametrizes sets of three points in the projective plane.

$$N^1_{\mathbb{R}}(\mathbb{P}^{2[3]})$$



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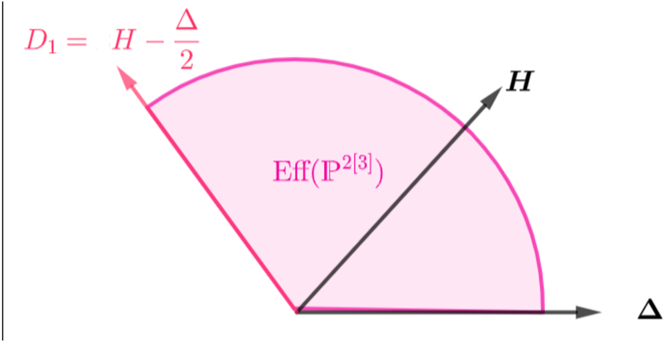

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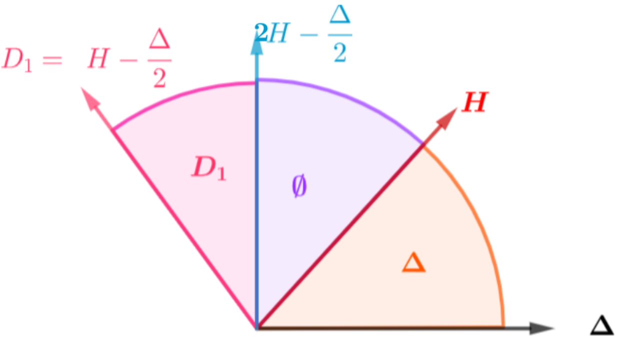
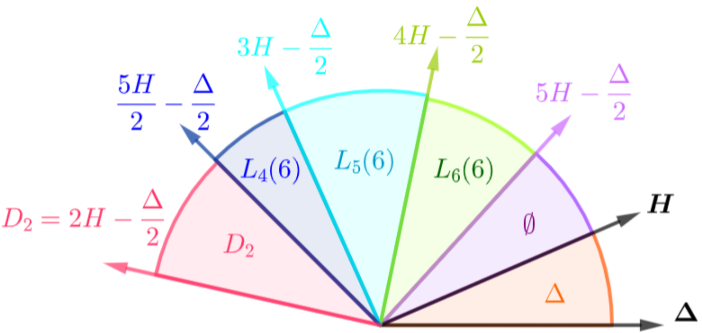


Motivation

6 points

$\mathbb{P}^{2[6]}$ $\rightarrow$ parametrizes sets of six points in the projective plane.

$$N^1_{\mathbb{R}}(\mathbb{P}^{2[6]})$$

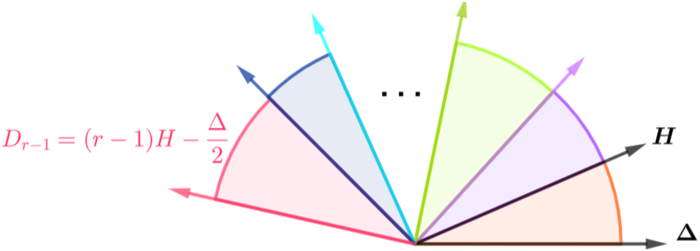


Motivation

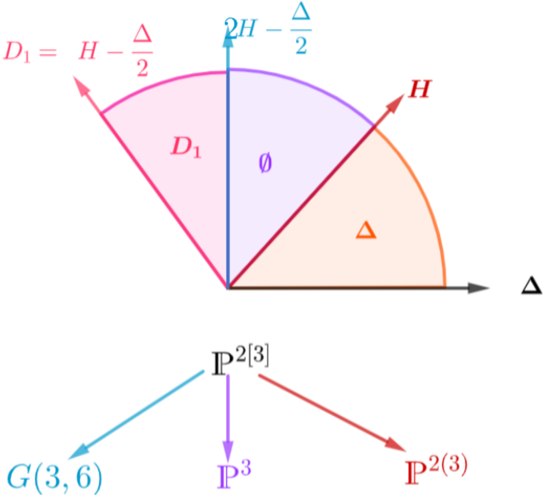
$\frac{r(r+1)}{2}$ points

$\mathbb{P}^{2[\frac{r(r+1)}{2}]} \rightarrow$ parametrizes sets of $\frac{r(r+1)}{2}$ points in the projective plane.

$$N^1_{\mathbb{R}}(\mathbb{P}^{2[\frac{r(r+1)}{2}]})$$



Motivación



$D_1 = H - \frac{\Delta}{2}$
$2H - \frac{\Delta}{2}$
H
D_1
$\emptyset$
Δ
Δ
$\mathbb{P}^{2[3]}$
$G(3,6)$
$\mathbb{P}^3$
$\mathbb{P}^{2(3)}$

Motivation

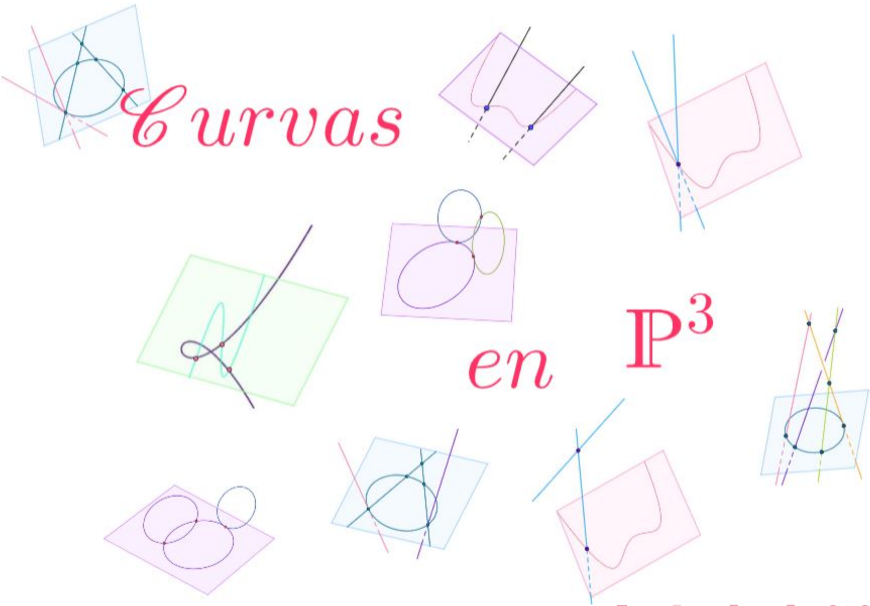
$\mathbb{P}^2$ ← ambient space

$Z \subset \mathbb{P}^2$ ← object

“residual points” ← tool

¨Points in an extremal divisor are residual to points in an extremal divisor ¨

Curvas en $\mathbb{P}^3$



We only consider locally Cohen-Macaulay (lcm) pure one-dimensional subschemes of $\mathbb{P}^3_k$. That means, we consider curves that may be singular, reducible, or not reduced, but they must not have embedded or isolated points.

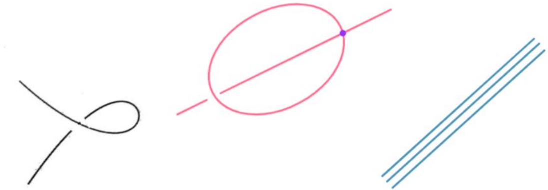


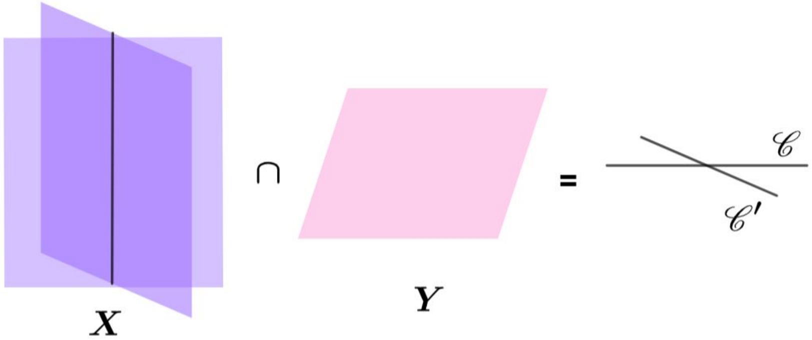
Figure: locally Cohen-Macaulay curves

Liaison Theory

Definition

Two curves C and C' in $\mathbb{P}^3$ are linked by a complete intersection of two surfaces $X \cap Y$ if $\mathscr{C}' = X \cap Y - \mathscr{C}$ as divisors on X

*The curves C and C' belong in the same linkage class if there exists a finite family of curves $C_0, \ldots, C_n$ such that C_i is linked with C_{i+1} for all i, with $C_0 = C$ and $C_n = C'$.



Liaison Theory

Definition

The Hartshorne-Rao module or deficiency module of a curve C in $\mathbb{P}^3$ is the module

$$M_C := \bigoplus_{n \in \mathbb{Z}} H^1(\mathbb{P}^3, \mathcal{I}_C(n))$$

Theorem (P. Rao (1979))

- *Two curves C and C' are in the same linkage class if and only if their Hartshorne-Rao-modules are isomorphic (except for a degree translation)*
- *For every $S = \mathbb{C}[x_0, x_1, x_2, x_3]$-module of finite length M, there exists a no singular irreducible curve $C \subseteq \mathbb{P}^3$, with Hartshorne-Rao module isomorphic to M (except for a degree translation).*

ACM curves

Proposition (F. Gaeta)

A curve C is an ACM (aritmetically Cohen-Macaulay) curve if and only if its Hartshorne-Rao-module is trivial ($M_C = 0$).

Why are ACM curves relevant?

- An ACM curve is always:

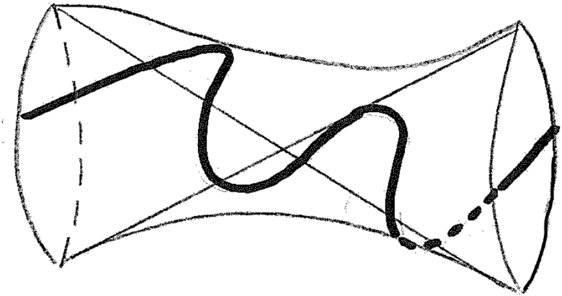
A determinantal curve

A smooth point on its Hilbert Scheme

ACM curves

Example

A curve C of type (a, b) on a nonsingular quadric surface in $\mathbb{P}^3_k$ is ACM if and only if $|a - b| \leq 1$.



Triangular curves

For every positive integer r, fix:

$$d_r := \frac{r(r+1)}{2} \qquad \text{y} \qquad g_r := \frac{r(r+1)(2r-5)}{6} + 1$$

Let $\mathscr{H}_r^{lcm}$ the set of locally Cohen-Macaulay curves of degree d_r and genus g_r and define:

$$\mathscr{H}_r := \overline{\mathscr{H}_r^{lcm}} = Hilb^3_{d_r t + (1-g_r)}$$

Theorem (–)

The Hilbert scheme $\mathscr{H}_r$ has an unique component that parametrizes ACM curves. Furthermore, this component is generically smooth of dimension $2r(r+1)$

Let $\mathscr{C}_r$ be the family of ACM curves in $\mathscr{H}_r$.

Examples

- $r = 1$

$$d_1 = 1 \qquad g_1 = 0$$

Lines in $\mathbb{P}^3$

$$\overline{\mathscr{C}_1} = \mathscr{H}_1 \equiv \mathbb{G}(1,3) \qquad rank(Pic(\mathscr{H}_1) \otimes \mathbb{Q}) = 1$$

Is minimal

- $r = 2$

$$d_2 = 3 \qquad g_2 = 0$$

twisted cubics in $\mathbb{P}^3$

$\overline{\mathscr{C}_2} = \mathscr{H}_2$ is a smooth irreducible scheme of dimension 12

$$rank(Pic(\mathscr{H}_2) \otimes \mathbb{Q}) = 2$$

(Dawei Chen, Mori's program for the Kontsevich moduli space $\overline{M}_{0,0}(\mathbb{P}^3, 3)$)

The case $\mathscr{H}_3$

We know that $\mathscr{H}_3$ is reducible and we found three components:

$\overline{\mathscr{C}_3}$:
- It is an irreducible component of dimension 24.
- The generic element is an ACM curve.
- It is generically smooth.

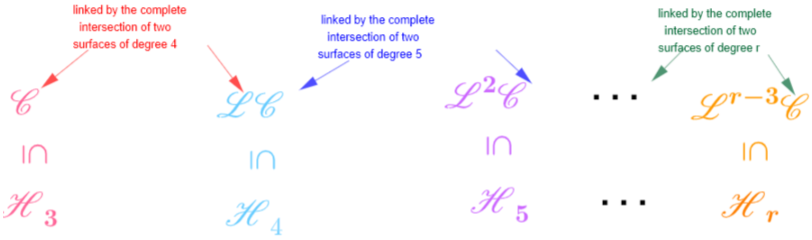
$\overline{\mathscr{R}_3}$:
- It is an irreducible component of dimension 24.
- The generic element is the union of a conic and a plane quartic.
- It is generically smooth.

$\mathscr{E}_3$:
- It is a irreducible component of dimension 30.
- The generic element is an extremal curve.
- It is generically no reduced.

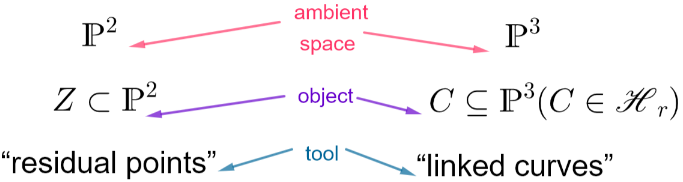
Notation

Notation

Given a family of curves $\mathscr{C}$ on $\mathscr{H}_r$, we denote by $\mathscr{L}_{r+1}\mathscr{C}$ to the family of curves liked to the elements of $\mathscr{C}$ by the complete intersection of two surfaces of degree $r+1$.



Let D_{r-1} be the family of curves in $\mathscr{H}_r$ that lies in a surface of degree $r-1$.



$$\mathscr{L} D_{r-1} = D_r?$$

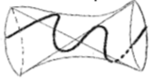
False! The elements on $\mathscr{L} D_{r-1}$ do not lie on a surface of degree r.

Special families

$\overline{\mathscr{C}_3}$ ← Closure of ACM curves

$\mathscr{C}^h = \Bigg\{$ Hyperelliptic curves

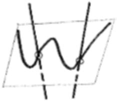
(lies on a smooth quadric surface)



$\Bigg\} = D_2$

$\mathscr{A} = \Bigg\{$ “Antenitas”

(2 skew lines + plane quartic)



$\Bigg\}$

Results

Lemma (–)

- *If r is an odd number, then:*

$$\mathscr{L}^{r-3}\mathscr{C}^{h} = D_{r-1}$$

- *If r is an even number, then:*

$$\mathscr{L}^{r-3}\mathscr{A} = D_{r-1}$$

In particular

$$\mathscr{L}^{2}D_{r-1} = D_{r+1}$$

Results

Proposition (–)

- *If r is an odd number, then:*

$$\overline{\mathscr{L}^{r-3}\mathscr{C}^h} \subseteq \overline{\mathscr{C}_r}$$

- *If r is an even number, then:*

$$\overline{\mathscr{L}^{r-3}\mathscr{A}} \subseteq \overline{\mathscr{C}_r}$$

Results

Theorem (–)

The classes $\overline{\mathscr{A}}$ *and* $\overline{\mathscr{C}^h}$ *on* $N^1_{\mathbb{R}}(\overline{\mathscr{C}_3})$ *are linearly independent and generate a face of the effective cone:*

$$Eff(\overline{\mathscr{C}_3}) \subseteq N^1_{\mathbb{R}}(\overline{\mathscr{C}_3}).$$

Corolary (–)

The dimension of the vector space $N^1_{\mathbb{R}}(\overline{\mathscr{C}_3})$ *is 3.*

Results

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¡Thank you!