LECTURE 3: COMPUTATIONAL METHODS

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Abstract. We discuss computing quadratic Euler characteristic via Hodge cohomology and the Jacobian ring, as well as using normalizer localization to compute degrees of quadratic Euler classes.

1. Introduction

As they carry more information than the classical $\mathbb{Z}$-valued invariants, the quadratic invariants are often more difficult to compute. In this lecture, we will go over some of the computational tools that have been developed to enable such computations. The methods include the development of a calculus of characteristic classes of vector bundles with values in Witt sheaf cohomology, algebraic computations of the quadratic Euler characteristics of smooth hypersurfaces in $\mathbb{P}^n$, and localization techniques for computing Euler classes and virtual fundamental classes. As a further example we look at a quadratic count of twisted cubic curves on hypersurfaces and complete intersections in a projective space.

2. The motivic Gauss-Bonnet theorem and computations of the quadratic Euler characteristic

We need a bit a background about the motivic stable homotopy category $\mathrm{SH}(k)$ a field k. $\mathrm{SH}(k)$ is a triangulated, symmetric monoidal category, with product $\wedge$ and with translation functor $\Sigma_{S^1} := -\wedge S^1$. $\mathbb{G}_m$-suspension $\Sigma_{\mathbb{G}_m}$ is also invertible and $\mathbb{P}^1$-suspension $\Sigma_{\mathbb{P}^1}$ is the same as $\Sigma_{S^1}\Sigma_{\mathbb{G}_m} = \Sigma_{\mathbb{G}_m}\Sigma_{S^1}$. One defines the family of suspension operations

$$\Sigma^{a,b} := \Sigma_{S^1}^{a-b}\Sigma_{\mathbb{G}_m}^{b}.$$

We have the category of *pointed spaces over* k, $\mathbf{Spc}_\bullet(k)$, this being the category of pointed simplicial presheaves on $\mathbf{Sm}_k$, with the Yoneda embedding $\mathbf{Sm}_k \to \mathbf{Spc}_\bullet(k)$ sending X to the representable presheaf X_+ of sets, with an added base-point. There is a $\mathbb{P}^1$-suspension functor

$$\Sigma^\infty_{\mathbb{P}^1}(-)_+ : \mathbf{Spc}_\bullet(k) \to \mathrm{SH}(k); \quad \mathcal{X} \mapsto \Sigma^\infty_{\mathbb{P}^1}\mathcal{X}$$

in particular, we have $\Sigma^\infty_{\mathbb{P}^1}X_+ \in \mathrm{SH}(k)$ for each $X \in \mathbf{Sm}_k$, but also objects such as $\Sigma^\infty_{\mathbb{P}^1}X/X\setminus Z$ for $Z \subset X$ an arbitrary closed subset. The unit for the smash product $\wedge$ is the motivic sphere spectrum $\mathbb{S}_k := \Sigma^\infty \mathrm{Spec}\, k_+$.

Each $\mathcal{E} \in \mathrm{SH}(B)$ defines a bi-graded cohomology theory on $\mathbf{Spc}_\bullet(k)$ by setting

$$\mathcal{E}^{a,b}(\mathcal{X}) := \mathrm{Hom}_{\mathrm{SH}(B)}(\Sigma^\infty_{\mathbb{P}^1}\mathcal{X}, \Sigma^{a,b}\mathcal{E}),$$

giving the functor

$$\mathcal{E}^{a,b} : \mathbf{Spc}_\bullet(k)^{\mathrm{op}} \to \mathbf{Ab}.$$

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For $\mathcal{X} = X_+$ this is usual $\mathcal{E}$-cohomology, $\mathcal{E}^{a,b}(X)$, and for $\mathcal{X} = X/X \setminus Z$, this gives the $\mathcal{E}$-cohomology with supports $\mathcal{E}_Z^{a,b}(X)$, with the long exact sequence

$$\ldots \to \mathcal{E}_Z^{a,b}(X) \to \mathcal{E}^{a,b}(X) \to \mathcal{E}^{a,b}(X \setminus Z) \xrightarrow{\delta} \mathcal{E}_Z^{a+1,b}(X) \to \ldots$$

We usually work with commutative rings $\mathcal{E}$ in $\mathrm{SH}(k)$, with unit $u : \mathbb{S}_k \to \mathcal{E}$ and product $\mathcal{E} \wedge \mathcal{E} \to \mathcal{E}$. This makes $\mathcal{E}^{*,*}(X) := \oplus_{a,b}\mathcal{E}^{a,b}(X)$ into a bi-graded ring with unit $1_X^{\mathcal{E}} \in \mathcal{E}^{0,0}$, $1_X := p_X^*(u)$, $p_X : X \to \operatorname{Spec} k$ the structure map.

We will work with two special types of $\mathcal{E}$ in $\mathrm{SH}(k)$: the *oriented* spectra and the SL-*oriented spectra*; these "simplify" the $\mathcal{E}$-cohomology in the following way. There is a canonical isomorphism

$$\Sigma^\infty_{\mathbb{P}^1}(\mathbb{A}^r \times X/(\mathbb{A}^r \setminus \{0\}) \times X) \cong \Sigma^{2r,r}X_+$$

giving the canonical isomorphism, for $V \to X$ the trivial rank r vector bundle on X,

$$\mathcal{E}_{0_V}^{a+2r,b+r}(V) \cong \mathcal{E}^{a,b}(X)$$

If $\mathcal{E}$ is oriented, one has canonical and natural isomorphisms

$$\mathcal{E}_{0_V}^{a+2r,b+r}(V) \xrightarrow[\sim]{\phi_V} \mathcal{E}^{a,b}(X)$$

for *arbitrary* $V \to X$ ($r = \mathrm{rank}V$). If $\mathcal{E}$ is SL-oriented, one has canonical and natural isomorphisms

$$\mathcal{E}_{0_V}^{a+2r,b+r}(V) \xrightarrow[\sim]{\phi_{V,\rho}} \mathcal{E}^{a,b}(X)$$

for each isomorphism $\rho : \det V \xrightarrow{\sim} O_X$ (if such exists). An oriented theory is also SL-oriented, and the isomorphism $\phi_{V,\rho}$ is independent of ρ.

Definition 2.1. Let $\mathcal{E}$ be an SL-oriented spectrum. $L \to X$ a line bundle on $X \in \mathbf{Sm}_k$. Let $\mathcal{L}$ be the invertible sheaf of section of L. Define the $\mathcal{L}$-twisted $\mathcal{E}$-cohomology by

$$\mathcal{E}^{a,b}(X; \mathcal{L}) := \mathcal{E}_{0_L}^{a+2,b+1}(L)$$

Note that $\mathcal{E}^{a,b}(X; \mathcal{L}) = \mathcal{E}^{a,b}(X)$ if $\mathcal{E}$ is oriented.

An SL-oriented theory $\mathcal{E}$ admits proper pushforward maps similar to those we have seen for $\widetilde{\mathrm{CH}}$: given a proper morphism $f : Y \to X$ in $\mathbf{Sm}_k$, of relative dimension d, and $\mathcal{L}$ an invertible sheaf on X, we have

$$f_* : \mathcal{E}^{a,b}(Y, f^*\mathcal{L} \otimes \omega_{Y/k}) \to \mathcal{E}^{a-2d,b-d}(X, \mathcal{L})..$$

with $(gf)_* = g_*f_*$, and a projection formula if $\mathcal{E}$ is a commutative ring spectrum: $f_*(f^*(x) \cdot y) = x \cdot f_*(y)$. Thus, we also have Euler classes $e^{\mathcal{E}}(V) \in \mathcal{E}^{2r,r}(X, \det^{-1} V)$ for $V \to X$ a rank r vector bundle

$$e^{\mathcal{E}}(V) = s^*s_{0*}(1_X)$$

for $s : X \to V$ any section. For $\mathcal{E}$ oriented, we have f_* as above, without needing any twists, and in addition to the Euler class, we have all the Chern classes $c_i^{\mathcal{E}}(V) \in \mathcal{E}^{2i,i}(X)$, with $c_r^{\mathcal{E}}(V) = e^{\mathcal{E}}(V)$ for $r = \mathrm{rank}(V)$.

We can now state a version of the motivic Gauß-Bonnet theorem. Recall that $\chi(X/k) \in \mathrm{GW}(k)$ is defined by taking the categorical Euler characteristic

$$\chi_{\mathrm{SH}(k)}(\Sigma^\infty_{\mathbb{P}^1}X_+) \in \mathrm{End}_{\mathrm{SH}(k)}(\mathbb{S}_k)$$

for the dualizable object $\Sigma^\infty_{\mathbb{P}^1}X_+$ of the symmetric monoidal category $\mathrm{SH}(k)$, and then using Morel's theorem, giving the isomorphism $\mathrm{GW}(k) \cong \mathrm{End}_{\mathrm{SH}(k)}(\mathbb{S}_k)$.

Theorem 2.2. *Let $\mathcal{E}$ be an* SL*-oriented ring spectrum with unit $u : \mathbb{S} \to \mathcal{E}$, and let $p_X : X \to \operatorname{Spec} k$ be a smooth proper k-scheme. Applying u to $\chi(X/k) \in \mathrm{End}_{\mathrm{SH}(k)}(\mathbb{S}_k)$ gives $u_*(\chi(X/k)) \in \mathcal{E}^{0,0}(k) = \mathrm{Hom}_{\mathrm{SH}(k)}(\mathbb{S}_k, \mathcal{E})$. Then*

$$u_*(\chi(X/k)) = p_{X*}(e^{\mathcal{E}}(T_{X/k}))$$

Examples 2.3. Take $p_X : X \to k$ smooth and proper of dimension n.

1. $\mathcal{E} = H\mathbb{Z}$ representing motivic cohomology. $H\mathbb{Z}$ is an oriented ring spectrum and $H\mathbb{Z}^{2n,n}(X) = \mathrm{CH}^n(X)$. The unit map $u_{H\mathbb{Z}} : \mathrm{End}(\mathbb{S}_k) \to H\mathbb{Z}^{0,0}(k)$ is the rank map $\mathrm{rank} : \mathrm{GW}(k) \to \mathbb{Z}$, and we thus have

$$\mathrm{rank}(\chi(X/k)) = u_{H\mathbb{Z}*}(\chi(X/k)) = p_{X*}(e^{\mathrm{CH}}(T_{X/k})) = \deg_k(c_n^{\mathrm{CH}}(T_{X/k}))$$

in other words, $\mathrm{rank}(\chi(X/k)) = \chi^{\mathrm{top}}(X)$.

2. $\mathcal{E} = \widetilde{H\mathbb{Z}}$ representing "Milnor-Witt motivic cohomology", $\widetilde{H\mathbb{Z}}$ is an SL-oriented ring spectrum and $\widetilde{H\mathbb{Z}}^{2n,n}(X, \mathcal{L}) = \tilde{\mathrm{CH}}^n(X; \mathcal{L})$. $u_{\widetilde{H\mathbb{Z}}}$ induces the identity map $\mathrm{GW}(k) = \mathrm{End}(\mathbb{S}_k) \to \widetilde{H\mathbb{Z}}^{0,0}(k) = \tilde{\mathrm{CH}}^n(k) = \mathrm{GW}(k)$, so

$$\chi(X/k) = u_{\widetilde{H\mathbb{Z}}*}(\chi(X/k)) = \tilde{\deg}_k(e^{CW}(T_{X/k}))$$

3. $H^*(-, \mathcal{W})$ is represented by the SL-oriented ring spectrum $\mathrm{EM}(\mathcal{W}_*)$ via

$$\mathrm{EM}(\mathcal{W}_*)^{a,b}(X; \mathcal{L}) = H^{a-b}(X, \mathcal{W}(\mathcal{L}))$$

and we thus have

$$\pi(\chi(X/k)) = u_{\mathrm{EM}(\mathcal{W}_*)*}(\chi(X/k)) = \tilde{\deg}_k(e^{\mathcal{W}}(T_{X/k}))$$

where $\pi : \mathrm{GW}(k) \to W(k)$ is the canonical surjection.

4. $\mathcal{E} = \mathrm{KGL}$, representing algebraic K-theory $\mathrm{KGL}^{a,b}(X) = K_{2b-a}(X)$. KGL is oriented and $u_{\mathrm{KGL}*}$ induces the rank map $\mathrm{GW}(k) \to \mathbb{Z}$, so

$$\chi^{\mathrm{top}}(X) = \mathrm{rank}(\chi(X/k)) = u_{\mathrm{KGL}*}(\chi(X/k)) = p_{X*}(e^K(T_{X/k}))$$

The pushforward in K_0 is defined by taking the derived pushforward of coherent sheaves, then taking a resolution by locally free sheaves. For $p : V \to X$ a rank r vector bundle, with 0-section $s_0 : X \to V$, we have $s_{0*}(1_X) = s_{0*}(\mathcal{O}_X)$, which has the Koszul resolution

$$0 \to \Lambda^r p^*\mathcal{V}^\vee \to \ldots \to \Lambda^j p^*\mathcal{V}^\vee \to \ldots \to p^*\mathcal{V}^\vee \to s_{0*}(\mathcal{O}_X) \to 0$$

where $\mathcal{V}$ is the sheaf of sections of V, so

$$e^K(V) = \sum_{j=0}^{r} (-1)^j [\Lambda^j \mathcal{V}^\vee]$$

and

$$p_{X*}(e^K(T_{X/k})) = \sum_{i,j=0}^{\dim X} (-1)^{i+j} \dim_k H^i(X, \Omega^j_{X/k})$$

that is

$$\chi^{\mathrm{top}}(X) = \sum_{i,j=0}^{\dim X} (-1)^{i+j} \dim_k H^i(X, \Omega^j_{X/k})$$

Let $n = \dim_k X$. We have the quadratic form q^{hdg} on $\oplus_{i,j} H^i(X, \Omega^j_{X/k})[j-i]$ defined by composing the product

$$H^i(X, \Omega^j_{X/k})[j-i] \otimes_k H^{n-i}(X, \Omega^{n-j}_{X/k})[i-j] \to H^n(X, \Omega^n_{X/k})$$

with the canonical trace map

$$\mathrm{Tr}_{X/k} : H^n(X, \Omega^n_{X/k}) \to k$$

Theorem 2.4 (L.-Raksit)**.** $\chi(X/k) = [q^{hdg}] \in \mathrm{GW}(k)$

Proof. We apply the motivic Gauß-Bonnet formula to $\mathcal{E} = KQ$, the ring spectrum representing hermitian K-theory (K-theory of quadratic forms). The unit map induces the identity

$$\mathrm{GW}(k) = \mathrm{End}_{\mathrm{SH}(k)}(\mathbb{S}_k) \to KQ^{0,0}(k) = \mathrm{GW}(k)$$

We then use the explicit known expressions for s_{0*}, s_0^* and p_{X*} □

Remark 2.5. q^{hdg} is a sum of hyperbolic forms for $i \neq j$, $i+j < n$ and $i < j$, $i + j = n$,

$$q^{hdg}_{i,j} : H^i(X, \Omega^j_{X/k})[j-i] \oplus H^{n-i}(X, \Omega^{n-j}_{X/k})[i-j] \to k$$

and in addition, in case $n = 2m$, the form

$$q^{hdg}_{m,m} : H^m(X, \Omega^m_{X/k}) \to k$$

Thus, applying $\pi : \mathrm{GW}(k) \to W(k)$, we have

$$\pi(\chi(X/k)) = \begin{cases} 0 \in W(k) & \text{if } n \text{ is odd} \\ [q_{m,m}] \in W(k) & \text{if } n = 2m \text{ is even} \end{cases}$$

3. Explicit computations for a hypersurface

Let $X \subset \mathbb{P}^{n+1}$ be a hypersurface of degree d, defined by a homogeneous polynomial $F \in k[X_0, \ldots, X_{n+1}]$. We assume that d is prime to $\mathrm{char} k$ if $\mathrm{char} k > 0$. We have already seen the Jacobian ring

$$J(F) := k[X_0, \ldots, X_{n+1}]/(\partial F/\partial X_0, \ldots, \partial F/\partial X_{n+1})$$

$J(F)$ is a graded ring with maximal non-zero degree $(d-2)(n+2)$. Write

$$\partial F/\partial X_i = \sum_{j=0}^{n+1} a_{ij} X_j$$

and let $e_{SS}(F)$ be the image in $J(F)_{(d-2)(n+2)}$ of $\det(a_{ij})$; in fact $J(F)_{(d-2)(n+2)} = k \cdot e_{SS}(F) \neq \{0\}$. Let $\ell : J(F) \to k$ be the projection on $J(F)_{(d-2)(n+2)}$ followed by the isomorphism $J(F)_{(d-2)(n+2)} \cong k$ sending $e_{SS}(F)$ to 1. This gives us the non-degenerate quadratic form on $J(F)$

$$q_{SS} : J(F) \to k; \quad q_{SS}(x) = \ell(x^2).$$

Carlson-Griffiths (and others) have defined isomorphisms

$$\psi_q : H^q(X, \Omega^{n-q}_X)_{prim} \to J(F)_{d(q+1)-n-2}$$

where $H^q(X, \Omega^{n-q}_X)_{prim} = H^q(X, \Omega^{n-q}_X)$ if $2q \neq n$, and if $2q = n$,

$$H^q(X, \Omega^q_{X/k})_{prim} = (c_1^{hdg}(O_X(1))^q)^\perp = \{x \in H^q(X, \Omega^q_{X/k}) \mid x \cdot (c_1^{hdg}(O_X(1))^q) = 0\}$$

Here $c_1^{hdg}(O_X(1)) \in H^1(X, \Omega^1_{X/k})$ is the 1st Chern class, which can be defined by applying the dlog map $\mathcal{O}_X^\times \to \Omega^1$ to $[O_X(1)] \in H^1(X, \mathcal{O}_X^\times)$. Let

$$q_{SS}^{hdg}(F) : \oplus_{q=0}^n J(F)_{d(q+1)-n-2} \to k$$

be the restriction of q_{SS}.

Theorem 3.1 (L.-Pepin Lehalleur- Srinivas)**.** *Let X be a smooth hypersurface of degree d in $\mathbb{P}^{n+1}$, with d prime to* char$k$ *if* char$k > 0$. *Then $\chi(X/k)$ is represented by the quadratic form $\langle -d\rangle \cdot q_{SS}^{hdg} + (n+1/2)H$ if n is odd and by $\langle d\rangle + \langle -d\rangle \cdot q_{SS}^{hdg} + (n/2)H$ if n is even.*

Exercises 1. Show again that $\chi(X/k)$ is hyperbolic if X is smooth and proper over k of odd dimension, using Theorem 2.4.

2. Compute $\chi(X/k)$ for $X = V(F) \subset \mathbb{P}^{n+1}$, $F = \sum_{i=0}^{n+1} a_i X_i^d$, using Theorem 3.1.

4. Localization in Witt-sheaf cohomology

Torus localization is a powerful technique for computing degrees of characteristic classes. The basic idea is to endow a (smooth) k-scheme X with an action by a torus $T = \mathbb{G}_m^n$ and apply the Atiyah-Bott localization theorem (in this setting proven by Edidin-Graham). First one needs to define the T-equivariant Chow groups. This is done using an algebraic approximation of a contractible space ET on which T acts freely, and then defining $\mathrm{CH}^*_T(X) := \mathrm{CH}^*(X \times ET/T)$ (roughly speaking). Each T-equivariant vector bundle $V \to X$ defines a vector bundle $V \times ET/T \to X \times ET/T$ and thus has Chern classes

$$c_i^T(V) \in \mathrm{CH}^*_T(X)$$

Taking $X = \mathrm{Spec}\, k$, a T-equivariant vector bundle is just a representation $\rho : T \to \mathrm{Aut}_k(V)$ on some k-vector space V. Letting $x_i = c_1^T(\pi_i)$, where $\pi_i : T \to \mathbb{G}_m = \mathrm{Aut}_k(k)$, we have

$$\mathrm{CH}^*(BT) := \mathrm{CH}^*_T(\mathrm{Spec}\, k) = \mathbb{Z}[x_1, \ldots, x_n]$$

One can also define $\mathrm{CH}_n^T(X) = \mathrm{CH}_T^{\dim X - n}(X)$.

Theorem 4.1. *Let $i : X^T \to X$ be the inclusion of the fixed points. Then there is a non-zero homogeneous polynomial $P \in \mathbb{Z}[x_1, \ldots, x_n]_d$ for some $d > 0$ such that*

$$i_* : \mathrm{CH}_*^T(X^T) \to CH_*^T(X)$$

is an isomorphism after inverting P.

Allied with this is the Bott residue theorem, which says, for an equivariant vector bundle $V \to X$, we have

$$i_*(c_i^T(i^*V)/c_m(N_i)) = c_i^T(V)$$

after inverting perhaps a larger P. Here m is the codimenison of X^T in X and N_i is the normal bundle.

We would like to apply this to computations in equivariant Witt sheaf cohomology, but there is a problem: the equivariant Euler classes $e^T(\pi_i)$ are all zero, so

$H^*(BT, \mathcal{W}) = W(k)$ concentrated in degree 0. Instead, we use a small enlargement of $\mathbb{G}_m$, namely, let $N \subset \mathrm{SL}_2$ be the normalizer of the torus

$$\mathbb{G}_m = \{\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}\}$$

N is generated by this $\mathbb{G}_m$, together with an additional element

$$\sigma := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Let $e \in H^2(BN, \mathcal{W})$ be the Euler class of the rank two vector bundle associated to the representation $N \subset \mathrm{SL}_2 \subset \mathrm{GL}_2$. Then

$$H^*(BN, \mathcal{W})[1/e] = W(k)[e, 1/e]$$

In fact $H^*(BN, \mathcal{W})$ is almost $W(k)[e]$, except there is one extra element $q \in H^0(BN, \mathcal{W})$, which we won't care about.

Replacing T with N^n, we have a nearly direct analog of the Atiyah-Bott localization theorem and the Bott residue formula. Unfortunately, the localization will in general kill the (very interesting) two-primary torsion in $W(k)$, but will at least let us get at the signature information coming from total orderings on k.

With Sabrina Pauli, we have applied this to compute the quadratic counts for twisted cubics on hypersurfaces and complete intersections in a $\mathbb{P}^n$. One has the closure H_n of the locus of smooth twisted cubics in a suitable Hilbert scheme. H_n is a smooth projective variety of dimension $4n$, with universal bundle $p : \mathcal{C}_n \to H_n$ with map $q : \mathcal{C}_n \to \mathbb{P}^n$. As for lines, we have the locally free sheaf $\mathcal{E}_{m,n} = p_* q^* \mathcal{O}_{\mathbb{P}^n}(m)$, whose Euler class counts the twisted cubics on a hypersurface of degree m. Since $\mathcal{E}_{m,n}$ has rank $3m + 1$, the condition for finiteness is

$$3m + 1 = 4n$$

for example a quintic in $\mathbb{P}^4$. There is an additional orientation condition, namely n must be even and $m \equiv 1 \mod 4$; there are similar numerical and orientation conditions for complete intersections of multi-degree $(m_1, \ldots, m_r)$. Using the equivariant machinery, we developed an algorithm for computing the signature of $\tilde{\deg}_k(e^{\mathcal{W}}(\mathcal{E}_{m,n}))$, which yields for example the following table

n	degree(s)	signature	rank
4	(5)	765	317206375
5	(3,3)	90	6424326
10	(13)	768328170191602020	794950563369917462703511361114326425387076
11	(3,11)	4407109540744680	311908449683213824455028807369870409 16
11	(5,9)	313563865853700	1634858783493329027386903535388009 00
11	(7,7)	136498002303600	31226586782010349970656128100205356
12	(3,3,9)	43033957366680	35502236537604625191071472539252 04
12	(3,5,7)	5860412510400	6794415721803210746415212176 8900
12	(5,5,5)	1833366298500	68075954259605149177418598 12500

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