Intersection Betti numbers of the GIT quotient of quartic plane curves

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Introduction

Plan of the talk: Kirwan's techniques for cohomology of GIT quotients.

- Space of quartic plane curves
- An equivariantly perfect stratification (HKKN)
- Cohomology of quotients (equivariant cohomology)
- Partial desingularization (Kirwan blow-up)
- Intersection Betti numbers

Space of quartic plane curves

Let $X = \mathbb{P}(\mathbb{C}[x,y,z]_4)$ and $G = \mathrm{SL}_3(\mathbb{C})$. Then consider

$$G \times X \to X$$
$$(g, F(x,y,z)) \mapsto F(g^{-1}(x,y,z)).$$

- We are interested in the GIT quotient

$$X//G = \mathrm{Proj}\, R(X)^G.$$

- This is an irreducible projective variety of dimension 6.

GIT of quartic plane curves

- $R(X)^G$ is generated by 13 homogeneous invariant polynomials.
- To construct the GIT quotient we first eliminate a closed subset X^{un} from X.
- A point $x \in X$ is unstable if $f(x) = 0$ for every $f \in R(X)^G$.
- $X \setminus X^{un} := X^{ss}$ is the open subset of semistable points. The GIT quotient is

$$X^{ss} \to X//G := \operatorname{Proj} R(X)^G.$$

Stability of quartic plane curves

We can characterize all the quartics according to their stability.

- Semistable quartics:
  - Smooth quartics (stable)
  - Quartics with ordinary double points (stable)
  - Tacnodal quartics (strictly semistable)
  - $\{\mathtt{O}^2,\ \mathtt{OO},\ \mathtt{OX}\}$ (strictly semistable)
- Unstable quartics:
  - Quartics with a triple point or a product of 4 concurrent lines.

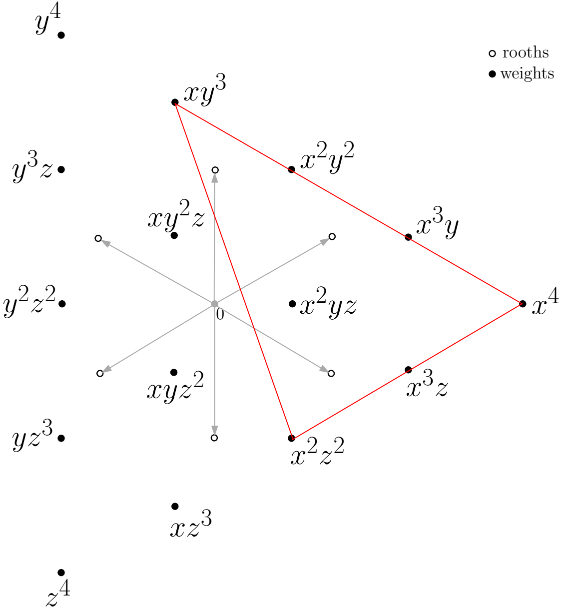


Figure: Diagram of weights and unstable set

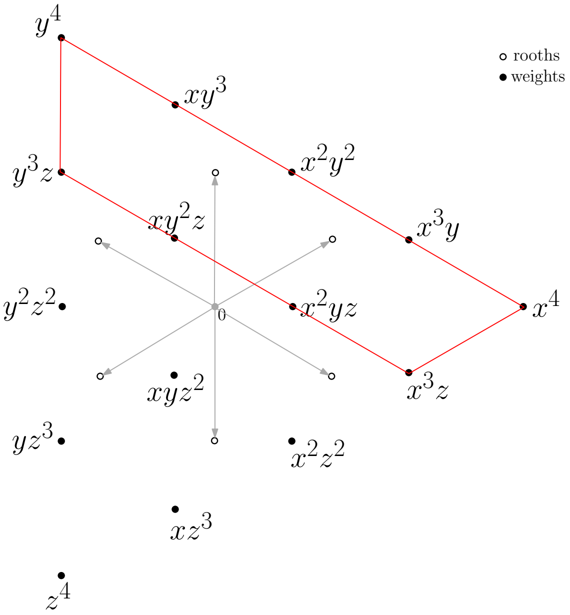


Figure: Diagram of weights and unstable set

Kirwan's Stratification

Theorem (Kirwan)

There exists a stratification $\{S_\beta : \beta \in \mathcal{B}\}$ of X such that

- *The unique open stratum is $S_0 = X^{ss}$.*
- *S_β is a non singular, locally closed subvariety for every $0 \neq \beta \in \mathcal{B}$.*
- *For $\beta \neq 0$, $\overline{S_\beta} \subseteq \bigcup_{\beta \leq \beta'} S_{\beta'}$.*
- *This stratification is perfectly equivariant.*

Stratification of unstable quartics

Stratum	Dim	Characterization
S_1	2	l^4: line of multiplicity 4
S_2	4	$l_1^3 l_2$: product of a triple line and other line
S_3	6	products $l_1 l_2 l_3 l_4$, $l_1^2 l_2 l_3$, $l_1^2 l_2^2$ of concurrent lines
S_4	6	product of an irreducible conic and a double tangent line
S_5	7	product of a cuspidal cubic and a tangent line at the cusp
S_6	8	irreducible quartic with a simple cusp of multiplicity 3

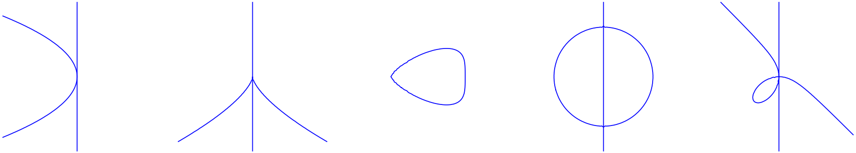
Table: Classification of unstable quartic plane curves

Stratification of unstable quartics

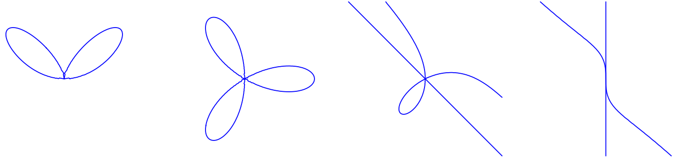
Stratum	Dim	Characterization
S_7	7	product of a conic and a non tangent double line
S_8	8	product of a nodal cubic and a tangent line at the node
S_9	9	quartic with a triple point with two branches meeting transversely, one of them is smooth and the other one is a cuspidal curve at the point
S_{10}	10	quartic plane curve with an ordinary triple point
S_{11}	9	product of a non-singular cubic with a tangent line at a flex point

Table: Classification of unstable quartic plane curves

Stratification of unstable quartics



(a) S_4 of dim. 6 (b) S_5 of dim. 7 (c) S_6 of dim. 8 (d) S_7 of dim. 7 (e) S_8 of dim. 8



(f) S_9 of dim. 9 (g) S_{10} of dim. 10 (h) S_{10} (i) S_{11} of dim. 9

Figure: Some unstable strata

Cohomology of GIT quotients

Theorem (Kirwan)

If for every point $x \in X^{ss}$ *the stabilizer of* x *is finite, then* $H^*(X//G;\mathbb{Q}) \cong H^*_G(X^{ss};\mathbb{Q})$, *and*

$$P_t^G(X^{ss}) = P_t^G(X) - \sum_{0 \neq \beta \in \mathcal{B}} t^{2d(\beta)} P_t^G(S_\beta),$$

where $d(\beta) = codim(S_\beta)$. *Moreover,*

- $P_t^G(X) = P_t(X)P_t(BG)$ if G is connected.
- $P_t^G(S_\beta) = P_t^{\mathrm{Stab}(\beta)}(Z_\beta^{ss})$ for a closed subvariety Z_β of X.

The GIT quotient is singular

Unfortunately, for quartic plane curves, $P_t(X//G) \neq P_t^G(X^{ss})$.

$$P_t^G(X^{ss}) = 1 + t^2 + 3t^4 + 5t^6 + 5t^8 + 4t^{10} + 2t^{12}.$$

There exists quartic plane curves with stabilizer of positive dimension. The GIT quotient is singular at the following places:

- $\{(y^2 - xz)^2\}$ with stabilizer $SO(3)$.
- $\mathbb{P}\{ax^2z^2 + bxy^2z + cy^4\}$ with stabilizer $T = \{(t, 1, t^{-1}) \mid t \in \mathbb{C}^*\}$.

Partial desingularization of the GIT quotient

We can solve these singularities by a sequence of blow-ups over X^{ss} (Kirwan blow-up, [Kir85])

$$X_1 \xrightarrow{\pi_1} X_2^{ss} \xrightarrow{\pi_2} \cdots \xrightarrow{\pi_{r-1}} X_r^{ss} \xrightarrow{\pi_r} X^{ss},$$

which induces a sequence of blow-ups over the GIT quotient

$$X_1//G \xrightarrow{\pi_1} X_2//G \xrightarrow{\pi_2} \cdots \xrightarrow{\pi_{r-1}} X_r//G \xrightarrow{\pi_r} X//G \ ,$$

such that the last blow-up is a partial desingularization of $X//G$.

Kirwan blow-up

To construct a Kirwan blow-up, consider the following:

- Take $R := (\mathrm{Stab}(x))_0$ for some $x \in X^{ss}$.
- R is a reductive subgroup of G.
- Define $Z_R^{ss} := \{x \in X^{ss} \mid x \text{ is fixed by } R\}$.
- GZ_R^{ss} is a G-invariant, non-singular, closed subvariety of X^{ss}.

Let Y be the blow-up of X^{ss} over GZ_R^{ss}. There exists a G-action on Y such that R doesn't occur as a stabilizer in Y.

Kirwan blow-up

The cohomology of the blow-up $Y \to X^{ss}$ of X^{ss} along GZ_R^{ss} is given by

$$H_G^*(Y;\mathbb{Q}) \cong H_G^*(X^{ss};\mathbb{Q}) \oplus H_G^*(E;\mathbb{Q})/H_G^*(GZ_R^{ss};\mathbb{Q}).$$

Lemma (Kirwan)

The GIT quotient

$$Y^{ss} \to Y//G$$

is the blow-up of $X//G$ *over* $GZ_R^{ss}//G := Z_R//N$.

Cohomology of the desingularization

For quartic plane curves, we have the following:

$$X_1 \xrightarrow{\pi_1} X_3^{ss} \xrightarrow{\pi_3} X^{ss} \ .$$

- X_3 is the blow-up of X^{ss} over $GZ^{ss}_{SO(3)}$.
- X_1 is the blow-up of X_3^{ss} over $G\tilde{Z}^{ss}_T$.

This induces a sequence of blow-ups

$$X_1//G \xrightarrow{\pi_1} X_3//G \xrightarrow{\pi_3} X//G \ .$$

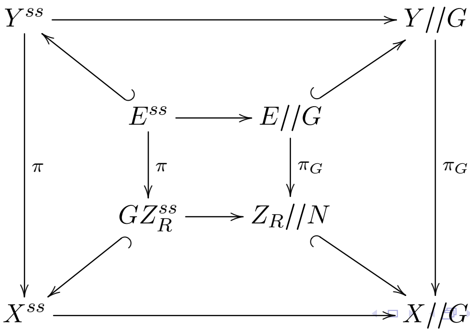
Every semistable point in X_1 has finite stabilizer and

$$P_t(X_1//G) = P_t^G(X_1^{ss})$$

Intersection cohomology of the GIT quotient

The intersection Betti numbers of the GIT quotients $X//G$ and $Y//G$ are related by ([Kir86])

$$\dim(IH^i(X//G;\mathbb{Q})) = \dim(IH^i(Y//G;\mathbb{Q}))$$
$$- \sum_{p+q=i} \dim[H^p(Z_R//N(R)_0;\mathbb{Q}) \otimes IH^{t(q)}(\mathbb{P}\mathcal{N}_x//R;\mathbb{Q})]^{\pi_0 N(R)}.$$



Intersection Betti numbers of the GIT quotient

For the blow-ups on the GIT quotient of quartic plane curve, the intersection Betti numbers are given by:

$$\begin{array}{ll} X_1//G & IP_t(X_1//G) = 1 + 3t^2 + 5t^4 + 6t^6 + 5t^8 + 3t^{10} + t^{12}. \\ \big\downarrow \pi_1 & \\ X_3//G & IP_t(X_3//G) = 1 + 2t^2 + 4t^4 + 4t^6 + 4t^8 + 2t^{10} + t^{12}. \\ \big\downarrow \pi_3 & \\ X//G & IP_t(X//G) = 1 + t^2 + 2t^4 + 2t^6 + 2t^8 + t^{10} + t^{12}. \end{array}$$

Intersection Betti numbers of the GIT quotient

Theorem (– 2022)

The Betti numbers of the partial desingularization $X_1//G$ of the GIT quotient of quartic plane curves are given by

$$P_t(X_1//G) = 1 + 3t^2 + 5t^4 + 6t^6 + 5t^8 + 3t^{10} + t^{12}.$$

Theorem (– 2022)

The intersection Betti numbers of the GIT quotient $X//G$ of quartic plane curves are given by

$$IP_t(X//G) = 1 + t^2 + 2t^4 + 2t^6 + 2t^8 + t^{10} + t^{12}.$$

Intersection Betti numbers of the GIT quotient

Let $\mathcal{F}_2$ be the space of holomorphic foliatitions on $\mathbb{P}^2$ of degree 2.

$\mathcal{F}_2$ is a projective variety of dimension 14.

Theorem (C. Reynoso, – 2022)

The intersection Betti numbers of the GIT quotient $\mathcal{F}_2//\mathrm{SL}_3(\mathbb{C})$ *are given by*

$$IP_t(\mathcal{F}_2//G) = 1 + t^2 + 2t^4 + 2t^6 + 2t^8 + t^{10} + t^{12}.$$

There is a birational morphism $\mathcal{F}_2 \to X$ [Esteves and Marchisio].

How are the GIT quotients $\mathcal{F}_2//G$ and $X//G$ related?

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Sicarú zedatu guidxi Lula' (Welcome to Oaxaca)

Xquixepe laatu (Thank you)