3. Higgs bundles for non-constant groups

a). Non-Abelian Hodge correspondence

Non-Abelian Hodge theory sets up a correspondence

$$\left\{ \begin{array}{c} G\text{-local systems} \\ \text{with completely} \\ \text{reducible} \\ \text{monodromy} \\ \text{over } X \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{polystable } G\text{-Higgs} \\ \text{bundles with vanishing} \\ \text{Chern classes} \\ \text{over } X \end{array} \right\}$$

moduli space

$$\mathrm{Hom}(\pi_1 X; G) /\!\!/ G$$

(affine variety / $\mathbb{C}$)

: recall that $\overline{G\cdot e} = G\cdot e$ iff $\overline{\rho(\pi_1 X)}^{\mathrm{Zar}}$ is a reductive subgroup of G

X : nonsingular complex projective variety

G : complex reductive affine algebraic group

Equivalently, it is a correspondence

$$\left\{ \begin{array}{c} G\text{-bundles with} \\ \text{integrable connection} \\ \text{and completely reducible} \\ \text{monodromy over } X \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{polystable } G\text{-Higgs} \\ \text{bundles with vanishing} \\ \text{Chern classes} \\ \text{over } X \end{array} \right\}$$

$$(E, \nabla) \quad \longleftarrow\!\dashrightarrow \quad (\mathcal{E}, \theta)$$

Henceforth, we shall focus on the case $G = GL(r; \mathbb{C})$.

ppal $GL(r; \mathbb{C})$-bundle $\longleftrightarrow$ rank r | vector bundle
| locally free $\mathcal{O}_X$-module

Connection: $\nabla : E \to \Omega^1_X \otimes E$ such that

$$\nabla(f\sigma) = df \otimes \sigma + f \nabla \sigma$$

Higgs Field: $\theta : \mathcal{E} \to \Omega^1_X \otimes \mathcal{E}$ such that $\theta(f\sigma) = f\theta(\sigma)$

Observations

(i) if $rk(\mathcal{E}) = 1$, a Higgs Field is just a (holomorphic) 1-form

$$\theta \in \Omega^1_X \otimes \underbrace{End(\mathcal{E})}_{\mathcal{E}^* \otimes \mathcal{E} \rightsquigarrow \simeq \mathcal{O}_X \text{ if } \mathcal{E} \text{ is a line bundle}}$$

(ii) Rank 1 local systems of vector spaces
Flat line bundles are parameterized by

$$H^1(X; \underline{\mathbb{C}^*_X})$$

If $\dim_{\mathbb{C}} X = 1$, we have (X compact, connected)

$$H^1(X; \underline{\mathbb{C}^*_X}) \underset{\text{homeo}}{\simeq} Hom(\pi_1 X; \mathbb{C}^*) \simeq (\mathbb{C}^*)^{2g}$$

where g = genus of X.

Exponential exact sequence ($\dim_{\mathbb{C}} X = 1$)

$$f : U \to \mathbb{C}$$

$$\exp(f) : U \to \mathbb{C}^*$$
$$z \mapsto e^{f(z)}$$

$$\begin{tikzcd}
0 \arrow[r] & \underline{\mathbb{Z}}_X \arrow[r] \arrow[d, equal] & \underline{\mathbb{C}}_X \arrow[r, "\exp"] \arrow[d] & \underline{\mathbb{C}}^*_X \arrow[r] \arrow[d] & 1 \\
0 \arrow[r] & \underline{\mathbb{Z}}_X \arrow[r] & \mathcal{O}_X \arrow[r, "\exp"] & \mathcal{O}^*_X \arrow[r] & 1
\end{tikzcd}$$

induces

$$\begin{tikzcd}
0 \arrow[r] & H^1(X;\underline{\mathbb{Z}}_X) \arrow[r] \arrow[d, equal] & H^1(X;\underline{\mathbb{C}}_X) \arrow[r] \arrow[d] & \overbrace{H^1(X;\underline{\mathbb{C}}^*_X)}^{\text{Flat line bundles on } X} \arrow[r, "0"] \arrow[d] & H^2(X;\underline{\mathbb{Z}}_X) \arrow[d, equal] \\
0 \arrow[r] & H^1(X;\underline{\mathbb{Z}}_X) \arrow[r] & H^1(X;\mathcal{O}_X) \arrow[r] & \underbrace{H^1(X;\mathcal{O}^*_X)}_{\substack{\text{holomorphic} \\ \text{line bundles on } X}} \arrow[r, "\deg"] & \underbrace{H^2(X;\underline{\mathbb{Z}}_X)}_{\cong \mathbb{Z}}
\end{tikzcd}$$

Line bundles of degree 0

The exactness of

$$0 \to H^1(X; \underline{\mathbb{Z}}_X) \to H^1(X; \mathcal{O}_X) \to H^1(X; \mathcal{O}_X^*) \xrightarrow{\deg} H^2(X; \underline{\mathbb{Z}}_X)$$

implies that

$$\mathrm{Jac}(X) := \deg^{-1}(0) \simeq \frac{H^1(X; \mathcal{O}_X)}{H^1(X; \underline{\mathbb{Z}}_X)} \simeq \frac{\mathbb{C}^g}{\mathbb{Z}^{2g}}$$

Likewise, $$H^1(X; \underline{\mathbb{C}}_X^*) \simeq \frac{H^1(X; \underline{\mathbb{C}}_X)}{H^1(X; \underline{\mathbb{Z}}_X)}.$$

Harmonic Forms

We seek to understand the map

$$\begin{array}{c} H^1(X; \underline{\mathbb{C}_X}) \\ \downarrow \\ H^1(X; \mathcal{O}_X) \end{array}$$

$$H^1(X; \underbrace{\underline{\mathbb{C}_X}}_{\text{locally constant}}) \underset{\text{DeRham}}{\simeq} \overbrace{H^1_{dR}(X;\mathbb{C})}^{\frac{\{d\text{-closed } 1\text{-forms}\}}{\{d\text{-exact } 1\text{-forms}\}}} \underset{\text{Hodge}}{\simeq} \mathrm{Harm}^1_{\Delta_d}(X) = \{\omega \in \Omega^1_{C^\infty}(X;\mathbb{C}) \mid \Delta_d \omega = 0\}$$

$\Delta_d \mapsto dd^* + d^*d$

$$H^1(X; \underbrace{\mathcal{O}_X}_{\text{coherent}}) \underset{\text{Dolbeault}}{\simeq} \overbrace{H^{0,1}_{Dol}(X)}^{\frac{\{\bar\partial\text{-closed } (0,1)\text{-forms}\}}{\{\bar\partial\text{-exact } (0,1)\text{-forms}\}}} \underset{\text{Hodge}}{\simeq} \mathrm{Harm}^{0,1}_{\Delta_{\bar\partial}}(X) = \{\beta \in \Omega^{0,1}_{C^\infty}(X;\mathbb{C}) \mid \Delta_{\bar\partial}\beta = 0\}$$

$\Delta_{\bar\partial} \mapsto \bar\partial\bar\partial^* + \bar\partial^*\bar\partial$

Kähler identities and the Hodge decomposition theorem

Recall that $d = \partial + \bar{\partial} := d^{1,0} + d^{0,1}$ (for 1-forms)

and that $\Omega^1_{cpx}(X;\mathbb{C}) = \Omega^{1,0}_{cpx}(X;\mathbb{C}) \oplus \Omega^{0,1}_{cpx}(X;\mathbb{C})$.

The Kähler identities on X imply that

$$\Delta_d = 2\,\Delta_{\bar{\partial}}$$

In particular $\omega = \underset{(1,0)}{\alpha} + \underset{(0,1)}{\beta}$ satisfies $\Delta_d = 0$ iff

$\Delta_{\bar{\partial}}\alpha = 0$ and

$\Delta_{\bar{\partial}}\beta = 0$

Hodge decomposition:

$$H^1(X;\underline{\mathbb{C}}_X) \simeq H^1_{dR}(X;\mathbb{C}) \overset{X \text{ Kähler}}{\simeq} H^{1,0}(X) \oplus H^{0,1}(X)$$

$$\simeq H^0(X;\Omega^1_X) \oplus H^1(X;\mathcal{O}_X)$$

$$H^1(X;\underline{\mathbb{C}}_X) \xrightarrow{\text{projection}} H^1(X;\mathcal{O}_X)$$

The Abelian case of the NAHC (in dimension 1)

From the Hodge decomposition, one gets:

$$H^1(X; \underline{\mathbb{C}}^*_X) \simeq \frac{H^1(X; \mathbb{C}_X)}{H^1(X; \underline{\mathbb{Z}_X})} \underset{\text{homeo}}{\simeq} H^0(X; \Omega^1_X) \oplus \frac{H^1(X; \mathcal{O}_X)}{H^1(X; \underline{\mathbb{Z}_X})}$$

$$\simeq \mathrm{Jac}(X) \times H^0(X; \Omega^1_X)$$

holomorphic 1-form

i.e. a correspondence

$$\underbrace{\{\text{Flat line bundles } (L, \nabla)\}}_{\underset{\text{homeo}}{\simeq} (\mathbb{C}^*)^{2g}} \longleftrightarrow \underbrace{\left\{\begin{array}{l}\text{rank 1 Higgs bundles } (\mathcal{L}, \theta)\\ \text{degree } 0\end{array}\right\}}_{\frac{\mathbb{C}^g}{\mathbb{Z}^{2g}} \times \mathbb{C}^g}$$

$(\mathbb{C}^*)^{2g}$, $\mathbb{C}^* \simeq S_1 \times \mathbb{R}$ (polar decomposition)

$$\frac{\mathbb{C}^g}{\mathbb{Z}^{2g}} \times \mathbb{C}^g \underset{\text{homeo}}{\simeq} S_1^{2g} \times \mathbb{R}^{2g}$$

Observation

The rank 1 case of the NAHT of a curve X yields a homeomorphism:

$$\underbrace{(\mathbb{C}^*)^{2g} \simeq \mathrm{Hom}(\pi_1 X; \mathbb{C}^*)}_{\substack{\text{affine algebraic variety} \\ \text{(Betti space)}}} \underset{\text{homeo}}{\simeq} \left\{ \begin{array}{c} \text{rank 1 degree 0} \\ \text{Higgs bundles} \end{array} (\mathcal{L}, \theta) \right\}$$

$$\simeq \underbrace{\mathrm{Jac}(X) \times H^0(X; \Omega^1_X)}_{\substack{\text{this is the cotangent bundle} \\ \text{of } \mathrm{Jac}(X) \\ \text{(Dolbeault space)}}}$$

$\dashrightarrow$ these two moduli spaces are homeomorphic but not isomorphic as algebraic varieties or complex analytic manifolds

The higher rank case

Building on work by Kobayashi, $\underbrace{\text{Hitchin, Donaldson}}_{r=2,\ \dim_{\mathbb{C}} X = 1}$, $\underbrace{\text{Corlette and Simpson}}_{r,\ \dim X \text{ arbitrary}}$ have generalized Hodge theory to the higher rank case.

$$\mathbb{C}^* \underset{\text{replaced by}}{\rightsquigarrow} GL(r;\mathbb{C})$$

$$\left\{ \begin{array}{c} \text{Flat vector bundles} \\ \text{of rank } r \end{array} \right\} \overset{?}{\longleftarrow\text{-}\text{-}\text{-}\longrightarrow} \left\{ \begin{array}{c} \text{semistable Higgs bundles} \\ \text{of rank } r \text{ \& degree } 0 \end{array} \right\}$$

Harmonic bundles

A harmonic bundle is a quadruple (E, h, A, φ)

where:

- $E \to X$ is a $\mathcal{C}^\infty$ complex vector bundle
- h is a Hermitian metric on E
- A is a unitary connection on (E, h)
- $\varphi \in \Omega^1_{\mathcal{C}^\infty}(X; \mathrm{Herm}(E, h))$

such that

Hitchin equations

$$\begin{cases} F_A + \frac{1}{2}[\varphi \wedge \varphi] = 0 \\ d_A \varphi = 0 \\ d_A^* \varphi = 0 \end{cases}$$

From harmonic bundles to Higgs and Flat bundles

$$
\begin{tikzcd}
 & \left\{ \begin{array}{c} \text{harmonic bundles} \\ (E, h, A, \psi) \end{array} \right\} \arrow[dl, bend right, "D = A + \psi"'] \arrow[dr, bend left, "\begin{array}{c} \mathcal{E} = (E, d_A^{0,1}) \\ \theta = \psi^{1,0} \end{array}"] & \\
\left\{ \begin{array}{c} \text{Flat vector bundles} \\ (E, D) \end{array} \right\} & & \left\{ \begin{array}{c} \text{Higgs bundles with} \\ \text{vanishing Chern classes} \\ (\mathcal{E}, \theta) \end{array} \right\}
\end{tikzcd}
$$

$$
\left. \begin{array}{r} F_A + \frac{1}{2}[\psi, \psi] = 0 \\ d_A \psi = 0 \end{array} \right\} \Leftrightarrow F_D = 0
\qquad\qquad
\left. \begin{array}{r} d_A \psi = 0 \\ d_A^* \psi = 0 \end{array} \right\} \Leftrightarrow \left\{ \begin{array}{l} d_A^{0,1} \theta = 0 \\ (\theta \wedge \theta = 0) \end{array} \right.
$$

$$
d_A^{*h} \psi = 0 \Longrightarrow (E, D) \text{ has } \underline{\text{completely}}\ \underline{\text{reducible}}\ \underline{\text{holonomy}}
$$

$$
\left. \begin{array}{l} \underbrace{F_A + \frac{1}{2}[\psi \wedge \psi] = 0} \\ \Leftrightarrow F_A + [\theta \wedge \theta^{*h}] = 0 \end{array} \right\} \Rightarrow (\mathcal{E}, \theta) \text{ is } \underline{\text{polystable}}
$$

Harmonic metrics

(i) on a flat bundle (E, ∇):

a metric h induces a decomposition

$\nabla = A + \psi$ with A h-unitary and ψ Hermitian

(satisfying $F_A + \frac{1}{2}[\psi \wedge \psi] = 0$ and $d_A \psi = 0$)

<u>Theorem</u> (Donaldson – Corlette)

If ∇ has completely reducible holonomy, then (E, ∇) admits h such that $d_A^{*_h} \psi = 0$.

<u>i.e.</u> (E, ∇) comes from a harmonic bundle.

Harmonic metrics

(ii) on a Higgs bundle $(\mathcal{E}, \theta)$:

Since $\mathcal{E}$ is holomorphic, a metric h determines a Chern connection A, which is unitary, and a Hermitian 1-form $\varphi := \theta + \theta^{*_h}$ (satisfying $d_A \varphi = 0$ and $d_A^{*_h} \varphi = 0$).

Theorem (Hitchin–Simpson)

If $(\mathcal{E}, \theta) \simeq (\mathcal{E}_1, \theta_1) \oplus \cdots \oplus (\mathcal{E}_k, \theta_k)$ with each $(\mathcal{E}_i, \theta_i)$ stable with vanishing Chern classes, then $(\mathcal{E}, \theta)$ admits h such that $F_A + [\theta \wedge \theta^{*_h}] = 0$

i.e. $(\mathcal{E}, \theta)$ comes from a harmonic bundle.

Stability for degree 0 Higgs bundles

($\dim_{\mathbb{C}} X = 1$)

Let $(\mathcal{E}, \theta)$ be a degree 0 Higgs bundle.

Then $(\mathcal{E}, \theta)$ is called stable if

$\forall (0 \subsetneq \mathcal{F} \subsetneq \mathcal{E})$ such that $\theta(\mathcal{F}) \subset \Omega^1_X \otimes \mathcal{F}$,

one has $\deg \mathcal{F} < 0$.

Examples (i) rank 1 Higgs bundles $(\mathcal{L}, \theta)$

(ii) $\mathcal{E} = \Omega_X^{-1/2} \oplus \Omega_X^{1/2}$, $\theta = \begin{pmatrix} 0 & \boxed{q_2} \\ 1 & 0 \end{pmatrix}$

$\theta : \Omega_X^{1/2} \oplus \boxed{\Omega_X^{-1/2}} \longrightarrow \boxed{\Omega_X^{3/2}} \oplus \Omega_X^{3/2}$ $\qquad q_2 \in \Gamma(\underbrace{\mathrm{Hom}(\Omega_X^{-1/2}, \Omega_X^{3/2})}_{\simeq \Omega_X^{\otimes 2}})$

b). The non-constant case

G : complex reductive group

A principal G-Higgs bundle is a pair $(\mathcal{E}, \theta)$ where:

- $\mathcal{E}$ is a principal G-bundle
- $\theta \in H^0(X; \Omega^1_X \otimes \mathrm{ad}(\mathcal{E}))$ such that $[\theta \wedge \theta] = 0$

$$\left(\mathrm{ad}(\mathcal{E}) = \left(\mathcal{E} \times \mathrm{Lie}(G)\right) / G \right)$$

<u>Example</u> $G = GL(n; \mathbb{C})$ $\quad \mathrm{ad}(\mathcal{E}) \leftrightarrow \mathrm{End}(V)$

$\mathcal{E} \leftrightarrow V$ $\dashrightarrow$ classical Higgs bundle

$\mathcal{G} \to X$ a group bundle on X

A $\mathcal{G}$-Higgs torsor is a pair $(\mathcal{E}, \theta)$ where:

- $\mathcal{E}$ is a $\mathcal{G}$-torsor
- $\theta \in H^0(X; \Omega^1_X \otimes \mathrm{ad}(\mathcal{E}))$ such that $[\theta \wedge \theta] = 0$

$$\left(\mathrm{ad}(\mathcal{E}) = \left(\mathcal{E} \times_X \mathrm{Lie}(\mathcal{G})\right) / \mathcal{G} \right)$$

<u>Example</u> $\mathcal{G} = X \times G$ $\dashrightarrow$ principal G-Higgs bundles

Relation to equivariant principal bundles

Assume that $\mathcal{G} = \mathcal{G}_\varphi := (\tilde{X} \times G)/\pi_1 X$ for $\varphi : \pi_1 X \to \mathrm{Aut}(G)$.

[The only case in which we can have a correspondence with twisted local systems]

Define a cover $\underbrace{X_\varphi \xrightarrow{p} X}_{\text{Galois cover, finite if } F := \underbrace{\mathrm{Im}\,\varphi}_{\cong \mathrm{Aut}_X(X_\varphi)} \text{ is finite}}$ by $\pi_1 X_\varphi := \ker \varphi \triangleleft \pi_1 X$.

$$p^* \mathcal{G}_\varphi = G_{X_\varphi} = X_\varphi \times G$$

because $\pi_1 X_\varphi$ acts trivially on G

$\mathcal{G}_\varphi$-torsors (on X) $\longleftrightarrow$ F-equivariant principal G-bundles

Equivariant Higgs Fields

$(\mathcal{E},\theta)$: a $\mathcal{G}_\varphi$ - Higgs torsor

$E := p^*\mathcal{E}$ has an F-equivariant structure:

$$\begin{tikzcd} E \arrow[r, "\tau_f"] \arrow[d] & E \arrow[d] \\ X_\varphi \arrow[r, "f"] & X_\varphi \end{tikzcd}$$

$$\tau_{1_F} = id_E$$

$$\tau_{f_1 f_2} = \tau_{f_1}\tau_{f_2}$$

$$\tau_f(m\cdot g) = \tau_f(m)\cdot\varphi_f(g)$$

$\theta \in H^0(X, \Omega^1_X \otimes ad(\mathcal{E}))$

$\updownarrow$ 12

$\tilde{\theta} \in Fix_F H^0(X_\varphi; \Omega^1_{X_\varphi} \otimes ad(E))$

$$\begin{tikzcd} TX_\varphi \arrow[r, "\tilde{\theta}"] \arrow[d, "f"] & ad(E) = (E \times Lie(G))/G \arrow[d, "\tau_f"] & \ni [m,\xi] \arrow[d, mapsto] \\ TX_\varphi \arrow[r, "\tilde{\theta}"] & ad(E) = (E \times Lie(G))/G & \ni [\tau_f(m), \varphi_f(\xi)] \end{tikzcd}$$

Equivariant NAHC

$$\left\{\begin{array}{c}\text{F-equivariant}\\ \text{harmonic bundles}\\ (E, h, A, \psi, \tau)\end{array}\right\}$$

$\nabla = A + \psi$

$\mathcal{E} = (E, d_A^{0,1})$

$\theta = \psi^{1,0}$

$$\left\{\begin{array}{c}\text{F-equivariant}\\ \text{flat vector bundles}\\ (E, \nabla, \tau)\end{array}\right\} \qquad \left\{\begin{array}{c}\text{F-equivariant}\\ \text{Higgs bundles with}\\ \text{vanishing Chern classes}\\ (\mathcal{E}, \theta, \tau)\end{array}\right\}$$

$\rightsquigarrow$ study the existence of F-invariant solutions to Hitchin's equations

Stability

(à la Ramanathan, when $\dim_{\mathbb{C}} X = 1$)

An F-equivariant principal G-Higgs bundle (E, θ, τ) or X_φ is called stable if, for all F-invariant parabolic subgroup $P \subset G$ and all F-equivariant reduction of structure group $s : X_\varphi \to E/P$ such that $\theta(T_{X_\varphi}) \subset \mathrm{ad}(\underbrace{s^*E}_{=: E_P})$, one has

$$\deg \mathrm{ad}(E_P) < 0 .$$

OK in all our examples

Example (no stability issue here)

Recall the case of a Riemann surface with involution (Y, σ) : σ acts on $\mathbb{C}^*$ via $z \mapsto z^{-1}$ and we can look at twisted local systems for the non-constant group $\mathcal{G}_\sigma := (Y \times \mathbb{C}^*) / \langle\sigma\rangle$ on $X := Y/\langle\sigma\rangle$.

[anti-invariant rank 1 local systems]

<u>NAHC on Y</u>: $\left\{ \text{Flat } \mathbb{C}^* \text{-bundles on } Y \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{rank 1 line bundles of degree 0} \\ \text{with a holomorphic 1-form} \end{array} \right\}$

$$\cup \qquad\qquad\qquad\qquad \cup$$

<u>equivariant version</u>: $\underbrace{\left\{ \begin{array}{l} \text{anti-invariant rank 1} \\ \text{local systems on } Y \end{array} \right\}}_{\sigma\text{-twisted } \mathbb{C}^*\text{-local systems on } X} \longleftrightarrow \underbrace{\left\{ \begin{array}{l} \text{anti-invariant} \\ \text{line bundle} \\ (\sigma^* \mathcal{L})^* \simeq \mathcal{L} \end{array} \right\} \times \left\{ \begin{array}{l} \text{1-form } \omega \\ \text{such that} \\ \sigma^* \omega = -\omega \end{array} \right\}}_{\mathcal{G}_\sigma\text{-Higgs torsors (on } X)}$

Observations

(i) $\mathfrak{g}_\sigma$ - Higgs torsors are not classical Higgs bundles on X (as opposed to the case of invariant local systems on Y, giving rise to local systems and classical Higgs bundles on X)

(ii) The NAHC on $X/\langle\sigma\rangle$ yields a homeomorphism

$$\left[\mathrm{Hom}_\sigma\left(\pi_1 X; \mathbb{C}^* \rtimes \langle\sigma\rangle\right) \big/ \mathbb{C}^* \right] \simeq T^* \mathrm{Bun}_{\mathfrak{g}_\sigma} = T^* \mathrm{Jac}(X) \big/ \langle\sigma\rangle$$

which is a twisted version of

$$\mathrm{Hom}(\pi_1 X; \mathbb{C}^*) \simeq T^* \mathrm{Jac}(X).$$

(iii) When the space X is allowed to be an orbifold, then the total space of the cover $X_\rho \to X$ is not a manifold in general. So, to apply the equivariant version of the NAHC, one needs to pass to a higher degree cover, which is not canonical in general.

Applications of this point of view include the study of Hitchin components for cocompact Fuchsian groups $\Gamma \subset PGL(2, \mathbb{R})$.