Hodge cycles on some fibered varieties

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Hodge cycles

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Recall that one has a cycle map

$$\square^p : CH^p(X) \to H^{2p}(X)$$

The image is the space of algebraic cycles.

Hodge cycles

There is a natural constraint on algebraic cycles

$$\operatorname{im}\square^p \subseteq \underbrace{H^{2p}(X)\cap H^{pp}}_{\text{Hodge cycles}}$$

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The Hodge conjecture asserts that equality holds.

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- There is a variant of these conjectures due to Jannsen which makes sense of quasiprojective varieties.

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Suppose that $f : X \to Y$ is a surjective morphism. Let $U \subset Y$ the complement of the discriminant, and $V = f^{-1}U$.

(This notation is fixed for the rest of the talk.)

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The Leray filtration L is a filtration on cohomology of V such that

$$Gr_L^b H^a(V) \cong H^b(U, R^{a-b} f_* \mathbb{Q})$$

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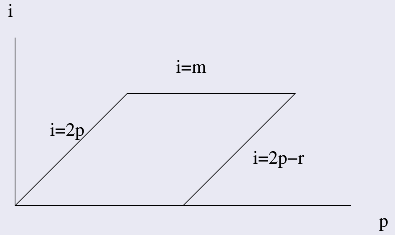
There is a similar map $\square_\ell^{p,i}$ for ℓ-adic cohomology.

Main Criterion

Theorem (A, 2022)

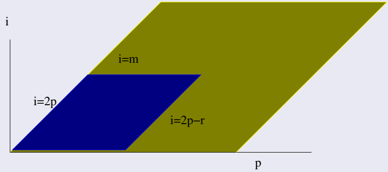
Let $n = \dim X$, $m = \dim Y$, $r = n - m$. The Hodge (Tate) conjecture holds for V if $\square^{p,i}$ ($\square_\ell^{p,i}$) is surjective for (p, i) in the closed parallelogram



Main Criterion

Proof.

Use the Leray spectral sequence to write down the obvious region where theorem holds (in yellow).



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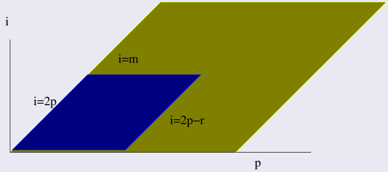

Then use Hard Lefcshetz twice to cut size to blue region. □

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For the rest of the talk, I want to turn to examples.

Families of genus 2 curves

Theorem (A 2019)

The Hodge (and Tate) conjecture holds for the universal genus 2 curve with level structure.

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Let $f : X \to Y$ be the universal genus 2 curve. Then $n = 4$. □

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Therefore $\square^{2,3}$ is trivially surjective, so HC follows. (Tate requires more.) $\square$

Families of abelian varieties

Suppose that D is a quaternion division algebra over a totally real field, split at exactly one real place. Let $G = D_1^*$ be the units of norm 1. Fix a torsion free arithmetic group $\Gamma \subset G(\mathbb{Q})$. It acts on $\mathbb{H}$ through the natural representation of $G(\mathbb{R}) \to SL_2(\mathbb{R})$.

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The Shimura curve $Y = \mathbb{H}/\Gamma$ can be viewed as a moduli space of abelian varieties with some extra structure. In particular, Y carries a natural family of abelian varietes $\mathcal{A} \to Y$.

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One needs to check that the main criterion applies.
In half the cases, I check this using invariant theory.
In the remaining cases, I check this with the help of a vanishing theorem deduced from work of Viehweg and Zuo.

□

Elliptic surfaces

Given a semistable elliptic surface $\mathcal{E} \to Y$, the fibre product $\mathcal{E} \times_Y \ldots \times_Y \mathcal{E}$ has toroidal singularities. Fix a toroidal resolution X. We want to understand when the Hodge conjecture holds for X.

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The nondegeneracy condition above is that a certain Kodaira-Spencer map

$$\kappa_\lambda : Gr_F^1 H^1(U_\lambda, R^1 f_{\lambda *}\mathbb{C}) \to Gr_F^0 H^1(U_\lambda, R^1 f_{\lambda *}\mathbb{C}) \otimes \Omega^1_\lambda$$

is injective for some λ.

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  In this case, the vanishing theorem is deduced from the nondegeneracy condition.