(Higher) Topological Complexity of non-k-Equal Spaces

José Luis León-Medina, Jesús González

Mathematics Department

Center for Research and Advanced Studies

of the National Polytechnical Institute (CINVESTAV)

Introduction

The non-k-equal manifold $M_d^{(k)}(n)$—so named in Baryshnikov preprint [1]—is defined as the complement in $(\mathbb{R}^d)^n$ of the diagonal-subspace arrangement, $A_d^{(k)}(n)$, formed by the union of subspaces

$$A_I = \left\{(x_1, \ldots, x_n) \in \left(\mathbb{R}^d\right)^n \mid x_{i_1} = \cdots = x_{i_k}\right\},$$

where $I = \{i_1, \ldots, i_k\}$ runs through all cardinality-k subsets of the segment $\mathbf{n} = \{1, 2, \ldots, n\}$.
For the smallest possible value $k = 2$, $M_d^{(k)}(n)$ yields the classical and extensively studied configuration space of n distinct ordered points in $\mathbb{R}^d$. On the other extreme, $M_d^{(n)}(n) \simeq \mathbb{S}^{dn-d-1}$ whereas $M_d^{(k)}(n) = (\mathbb{R}^d)^n$ for $k > n$. So, the present thesis will only deal with the cases where $3 \le k < n$.

The Cohomology Ring for $d = 1$

The cohomology ring of non-k-equal manifolds with coordinates in $\mathbb{R}$ was first described by Baryshnikov in his preprint [1] and later stated by Dobrinskaya and Turchin in [3, sec. 4]. The essential combinatorial objects to consider are string preorders encoded in terms of the following definition.

Definition 1. *A string preorder is an arrangement of alternating () and []-blocks of the form*

$$(I_0)[J_1](I_1)[J_2]\cdots(I_{\ell-1})[J_\ell](I_\ell)$$

where the sets I_0, J_1, …, J_ℓ, I_ℓ are mutually disjoint and their union is the set $\mathbf{n} = \{1, 2, \ldots, n\}$. Such a string preorder determines a submanifold in $\mathbb{R}^n$ defined by the following conditions:

- $x_{k_1} = x_{k_2}$ *if $k_1, k_2 \in J_m$ for some $m = 1, \ldots, \ell$,*
- $x_i \le x_j$ *if $i \in I_m$ and $j \in J_{m+1}$ for some $m = 0, \ldots, \ell - 1$,*
- $x_j \le x_i$ *if $j \in J_m$ and $i \in I_m$.*

Hence, the first condition says that []-blocks encode collided coordinates and the second and third conditions ensure that the coordinates are ordered according to the corresponding sets from left to right.

Example 2. *In $\mathbb{R}^8$ we can consider, for example, the string preorder*

$$(1)[2,3,4](\,)[5,6](7,8),$$

and this preorder has the associated submanifold

$$\{(x_1, x_2, \ldots, x_8) \mid x_1 \le x_2 = x_3 = x_4,\ x_5 = x_6 \le x_7, x_8\}.$$

Definition 3. *A string preorder is said to be k-elementary or just elementary for short, if it has the form $(I)[J](K)$ with $|J| = k - 1$.*

Example 4. *(1)[2,3,4](5,6,7,8) and (1,2,7,4,5)[6,3,8]( ) are elementary string preorders in $M_1^{(4)}(8)$.*

The elementary string preorders are generators for Borel-Moore homology in dimension $k-2$ subject to the boundary additive relations. After dualization, the multiplicative structure of the cohomology ring is determined by the transverse intersection—intersection product—of the corresponding submanifolds. Therefore transverse intersection of manifolds associated with elementary string preorders—or, more specifically their corresponding string preorders—are the basic elements generating the cohomology ring in dimensions which are multiples of $k-2$.

Theorem 5 (Baryshnikov [1, Theorem 1], Dobrinskaya-Turchin [3, Section 4])**.** *For $k \ge 3$, the cohomology ring $H^*(M_1^{(k)}(n))$ is isomorphic to the (anti)commutative free exterior algebra generated in dimension $k-2$ by the elementary preorders subject to the following relations:*

1. $\sum_{\iota \in I}(-1)^{g(\iota)}(I-\iota)[J+\iota](K) = \sum_{\kappa \in K}(-1)^{g(\kappa)}(I)[J+\kappa](K+\kappa)$ *whenever $\mathbf{n}$ can be written as a disjoint union $\mathbf{n} = I \coprod J \coprod K$ with $\mathrm{card}(J) = k-2$.*
2. $(I)[J](K) \cdot (I')[J'](K') = 0$, *for elementary preorders $(I)[J](K)$ and $(I')[J'](K')$ whose intersection has a [ ]-block of cardinality larger than $k-1$.*

References

[1] Yuliy Baryshnikov. On the cohomology ring of no k-equal manifolds. *Preprint 1997. Available from* `https://publish.illinois.edu/ymb/home/papers/`, 1997.

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[3] Natalya Dobrinskaya and Victor Turchin. Homology of non k-Overlapping discs. *Homology, Homotopy & Applications*, 17(2), 2015.

[4] Michael Farber. Topological complexity of motion planning. *Discrete & Computational Geometry*, 29(2):211–221, 2003.

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The Cohomology Ring for $d > 1$

The combinatorial description of the cohomology ring of $M_d^{(k)}(n)$ given in [3] is similar to the case $d = 1$. In this case the cohomology is encoded by combinatorial objects called admissible k-forests.

Definition 6. *A k-forest on $\mathbf{n}$ (or simply a k-forest) is an acyclic simple graph with two types of vertices, square ones and round ones, each containing a certain subset of $\mathbf{n}$. A square vertex must contain $k-1$ elements of $\mathbf{n}$, and cannot be an isolated vertex; in fact the set of immediate neighbors of a square vertex must contain a round vertex. A round vertex must contain a single element of $\mathbf{n}$, and must be either an isolated vertex or have valency 1, in which case it must be connected to a square vertex. We require that the subsets of integers inside the various vertices of a k-forest form a disjoint partition of $\mathbf{n}$. Square vertices are declared to have degree $d(k-2)$, while edges are declared to have degree $d-1$. The degree of a k-forest is then defined as the sum of the degrees of its square vertices and edges. An orientation for a k-forest consists of three ingredients:*

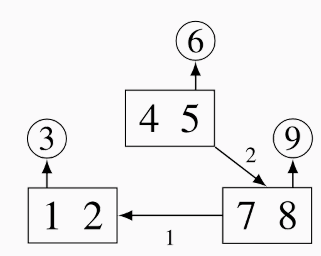
(a) An orientation for each edge;

(b) A total ordering for the elements inside each square vertex;

(c) A total ordering for the orientation set, *i. e., the set consisting of all edges and all square vertices.*

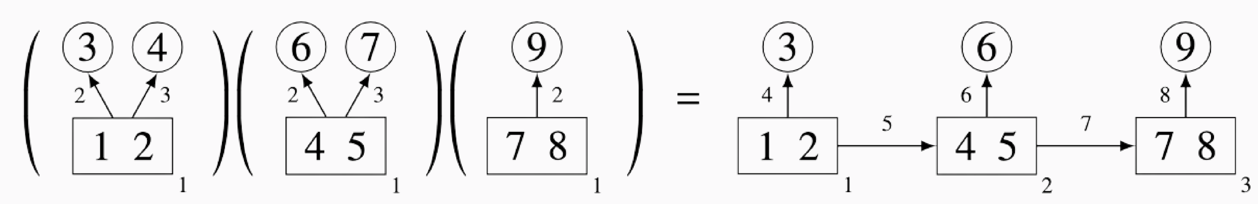
The idea of a k-forest is to encode elemental cells or manifolds in $M_d^{(k)}(n)$ by keeping track of the sub-indexes corresponding to elements in common in the square vertices, and denoting inequalities between the first coordinates of the corresponding coordinates with edges.

Example 7. *The next figure is a forest in $M_2^3(9)$ is an oriented 3-forest of degree 11. That 3-forest corresponds to a manifold in $M_2^3(9)$ as it is indicated in the right-hand side.*



$$\Rightarrow \quad \begin{array}{ll} x_1 = x_2, & x_1^{(1)} = x_2^{(1)} \le x_3^{(1)} \\ x_4 = x_5, & x_4^{(1)} = x_5^{(1)} \le x_6^{(1)} \\ x_7 = x_8, & x_7^{(1)} = x_8^{(1)} \le x_9^{(1)} \end{array}$$

Hence, since the k-forests encode manifolds that in turn are cycles in Borel-Moore homology we have nice cup products dictated in terms of the intersection of the corresponding submanifolds or superposition of their graphs. An example of such a product is the following:



Main Results

The LS category and (higher) topological complexity of non-k-equal arrangements with $d = 1$ were found in [6].

Theorem 8. *Summarizing, the Lusternik-Schnirelmann and (higher) topological complexity for $M_1^{(k)}(n)$ are*

$$\mathrm{cat}(M_1^{(k)}(n)) = \left\lfloor \frac{n}{k} \right\rfloor, \quad \mathrm{TC}(M_1^{(k)}(n)) = 2\left\lfloor \frac{n}{k} \right\rfloor, \quad \mathrm{TC}_s(M_1^{(k)}(n)) = s\left\lfloor \frac{n}{k} \right\rfloor.$$

The invariants were fully determined by the well known upper and lower bounds for LS category and TC:

Lemma 9 ([4, Theorem 7] and [2, Theorem 3.9])**.** *Let X be a c-connected space having the homotopy type of a CW complex, then*

$$\mathrm{zcl}_s(X) \le \mathrm{TC}_s(X) \le \frac{s\,\mathrm{hdim}(X)}{c+1}.$$

Here $\mathrm{hdim}(X)$ stands for the cellular homotopy dimension of X, $\mathrm{zcl}_0(X) = \mathrm{cl}(X)$ is the cup-length of X, and $\mathrm{zcl}_s(X)$ is the length of s-th zero-divisors for X.
Both bounds coincide for the case $d = 1$ but, unfortunately, do not coincide for the case $d > 1$. The inequalities determined by Lemma 9 for non-k-equal spaces for $d > 2$ are:

Corollary 10 ([5], Theorem 3.3)**.** *The LS category and TC_s for $M_d^{(k)}(n)$ is bounded by*

$$s\left\lfloor \frac{n}{k} \right\rfloor \le \mathrm{TC}_s(M_d^{(k)}(n)) \le s\left(\left\lfloor \frac{n}{k} \right\rfloor + \left\lfloor \frac{\left(\left\lfloor \frac{n}{k} \right\rfloor + b - 1\right)(d-1)}{a} \right\rfloor\right).$$

where $a = d(k-1) - 1$ and $b = n - k\left\lfloor \frac{n}{k} \right\rfloor$ (so $0 \le b < k$).

Where we identify TC_0 with the LS category, TC_1 is the reduced topological complexity and TC_s is the reduced higher topological complexity. However, a small improvement could be done by applying usual methods of obstruction theory. Hence, the best bounds obtained so far are the following:

Theorem 11 ([5], Theorem 3.3)**.** *For $s \ge 1$,*

$$s\left\lfloor \frac{n}{k} \right\rfloor \le \mathrm{TC}_s(M_d^{(k)}(n)) \le s\left(\left\lfloor \frac{n}{k} \right\rfloor + \left\lceil \frac{\left(\left\lfloor \frac{n}{k} \right\rfloor + b - 1\right)(d-1)}{a} - 1 \right\rceil\right),$$

where $\lceil \ell \rceil$ stands for the ceiling function at ℓ.

Finally, Theorem 11 could be stated in more concise terms for $d = 2$:

Corollary 12 ([5], Corollary 4.3)**.** *If $3 \le k < n \le k^2 + k - 2$ and $s \ge 1$, then*

$$\mathrm{TC}_s(M_2^{(k)}(n)) = s\left\lfloor \frac{n}{k} \right\rfloor.$$

Some TC values for $d = 2$

$n\backslash k$	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
$k+1$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+2$	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+3$	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+4$	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+5$	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+6$	3	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+7$	3	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+8$	?	3	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+9$	4	3	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1	1
$k+10$	?	3	3	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1	1
$k+11$	?	3	3	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1	1
$k+12$	?	4	3	3	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1	1
$k+13$	?	4	3	3	2	2	2	2	2	2	2	1	1	1	1	1	1	1	1	1
$k+14$	?	4	3	3	3	2	2	2	2	2	2	2	1	1	1	1	1	1	1	1
$k+15$	?	?	4	3	3	2	2	2	2	2	2	2	2	1	1	1	1	1	1	1
$k+16$	?	5	4	3	3	3	2	2	2	2	2	2	2	2	1	1	1	1	1	1
$k+17$	?	5	4	3	3	3	2	2	2	2	2	2	2	2	2	1	1	1	1	1
$k+18$	?	?	4	4	3	3	3	2	2	2	2	2	2	2	2	2	1	1	1	1
$k+19$	?	?	4	4	3	3	3	2	2	2	2	2	2	2	2	2	2	1	1	1
$k+20$	?	6	5	4	3	3	3	3	2	2	2	2	2	2	2	2	2	2	1	1
$k+21$	?	?	5	4	4	3	3	3	2	2	2	2	2	2	2	2	2	2	2	1
$k+22$	?	?	5	4	4	3	3	3	3	2	2	2	2	2	2	2	2	2	2	2
$k+23$	?	?	5	4	4	3	3	3	3	2	2	2	2	2	2	2	2	2	2	2
$k+24$	?	?	?	5	4	4	3	3	3	3	2	2	2	2	2	2	2	2	2	2
$k+25$	?	?	6	5	4	4	3	3	3	3	2	2	2	2	2	2	2	2	2	2
$k+26$	?	?	6	5	4	4	3	3	3	3	3	2	2	2	2	2	2	2	2	2
$k+27$	?	?	6	5	4	4	4	3	3	3	3	2	2	2	2	2	2	2	2	2
$k+28$	?	?	?	5	5	4	4	3	3	3	3	3	2	2	2	2	2	2	2	2
$k+29$	?	?	?	5	5	4	4	3	3	3	3	3	2	2	2	2	2	2	2	2
$k+30$	?	?	7	6	5	4	4	4	3	3	3	3	3	2	2	2	2	2	2	2
$k+31$	?	?	7	6	5	4	4	4	3	3	3	3	3	2	2	2	2	2	2	2
$k+32$	?	?	?	6	5	5	4	4	3	3	3	3	3	3	2	2	2	2	2	2
$k+33$	?	?	?	6	5	5	4	4	4	3	3	3	3	3	2	2	2	2	2	2
$k+34$	?	?	?	6	5	5	4	4	4	3	3	3	3	3	3	2	2	2	2	2
$k+35$	?	?	8	?	6	5	4	4	4	3	3	3	3	3	3	2	2	2	2	2
$k+36$	?	?	?	7	6	5	5	4	4	4	3	3	3	3	3	3	2	2	2	2
$k+37$	?	?	?	7	6	5	5	4	4	4	3	3	3	3	3	3	2	2	2	2
$k+38$	?	?	?	7	6	5	5	4	4	4	3	3	3	3	3	3	3	2	2	2
$k+39$	?	?	?	7	6	5	5	4	4	4	4	3	3	3	3	3	3	2	2	2
$k+40$	?	?	?	?	6	6	5	5	4	4	4	3	3	3	3	3	3	3	2	2
$k+41$	?	?	?	?	6	6	5	5	4	4	4	3	3	3	3	3	3	3	2	2
$k+42$	?	?	?	8	7	6	5	5	4	4	4	4	3	3	3	3	3	3	3	2
$k+43$	?	?	?	8	7	6	5	5	4	4	4	4	3	3	3	3	3	3	3	2
$k+44$	?	?	?	8	7	6	5	5	5	4	4	4	3	3	3	3	3	3	3	3
$k+45$	?	?	?	?	7	6	6	5	5	4	4	4	4	3	3	3	3	3	3	3
$k+46$	?	?	?	?	7	6	6	5	5	4	4	4	4	3	3	3	3	3	3	3
$k+47$	?	?	?	?	7	6	6	5	5	4	4	4	4	3	3	3	3	3	3	3
$k+48$	?	?	?	9	?	7	6	5	5	5	4	4	4	4	3	3	3	3	3	3
$k+49$	?	?	?	9	8	7	6	5	5	5	4	4	4	4	3	3	3	3	3	3
$k+50$	?	?	?	?	8	7	6	6	5	5	4	4	4	4	3	3	3	3	3	3
$k+51$	?	?	?	?	8	7	6	6	5	5	4	4	4	4	4	3	3	3	3	3
$k+52$	?	?	?	?	8	7	6	6	5	5	5	4	4	4	4	3	3	3	3	3
$k+53$	?	?	?	?	8	7	6	6	5	5	5	4	4	4	4	3	3	3	3	3
$k+54$	?	?	?	10	?	7	7	6	5	5	5	4	4	4	4	4	3	3	3	3
$k+55$	?	?	?	?	?	7	7	6	6	5	5	4	4	4	4	4	3	3	3	3
$k+56$	?	?	?	?	9	8	7	6	6	5	5	5	4	4	4	4	3	3	3	3
$k+57$	?	?	?	?	9	8	7	6	6	5	5	5	4	4	4	4	4	3	3	3
$k+58$	?	?	?	?	9	8	7	6	6	5	5	5	4	4	4	4	4	3	3	3
$k+59$	?	?	?	?	9	8	7	6	6	5	5	5	4	4	4	4	4	3	3	3
$k+60$	?	?	?	?	?	8	7	7	6	6	5	5	5	4	4	4	4	4	3	3
$k+61$	?	?	?	?	?	8	7	7	6	6	5	5	5	4	4	4	4	4	3	3
$k+62$	?	?	?	?	?	8	7	7	6	6	5	5	5	4	4	4	4	4	3	3
$k+63$	?	?	?	?	10	?	8	7	6	6	5	5	5	4	4	4	4	4	4	3
$k+64$	?	?	?	?	10	9	8	7	6	6	5	5	5	5	4	4	4	4	4	3
$k+65$	?	?	?	?	10	9	8	7	6	6	6	5	5	5	4	4	4	4	4	3

Conclusion

Computing the LS category and (higher) topological complexity of these manifolds is a highly non-trivial problem with potential applications to motion planning problems in robotics. We hope that the developments in this work can be successfully applied or generalized in the future. For instance, an interesting open problem that would benefit from our contribution is the design of reasonably efficient motion planning algorithms of automated guided particles that are allowed to interact (collision) among them in an organized and controlled way.
Our results show that, unlike the case of ordinary configuration spaces on $\mathbb{R}^d$, the parity of the dimension of the ambient Euclidean space does not seem to be a decisive parameter for the actual value of the topological complexity of collision-controlled motion planning of particles in $\mathbb{R}^d$. While the 1-dimensional case is as difficult as it can get, there is the (intuition-compatible) possibility that the higher dimensional situation exhibits lower TC values that could actually be independent of (the parity of) d.