Intrinsic volumes and the Weyl tube theorem in normed spaces

(based on joint work with D. Faifman)

1. Intrinsic volumes in $\mathbb{R}^n$

$K \subset \mathbb{R}^n$ convex body

Thm (Steiner)

$$\mathrm{vol}_n(K + rB^n) = \sum_{i=0}^{n} \mu_i(K) \cdot \omega_{n-i}\, r^{n-i}, \qquad r \geq 0$$

$\omega_k := \mathrm{vol}_k(B^k)$

Def $\mu_0(K), \mu_1(K), \ldots, \mu_n(K)$ are the intrinsic volumes of K.

$\mu_n(K) = \mathrm{vol}_n(K)$

$\mu_{n-1}(K) = \frac{1}{2} \mathrm{vol}_{n-1}(\partial K)$

$\mu_0(K) = 1$

Properties of the intrinsic volumes

- $K \subset E \subset \mathbb{R}^n \implies \boxed{\mu_i^E(K) = \mu_i^{\mathbb{R}^n}(K)}$

  In particular, if $\dim K = i$, then $\mu_i(K) = \mathrm{vol}_i(K)$

$$\left.\begin{array}{l} \bullet\ \mu_i(K \cup L) = \mu_i(K) + \mu_i(L) - \mu_i(K \cap L) \text{ if } K \cup L \text{ is convex} \\[1em] \bullet\ \mu_i(rK) = r^i \mu_i(K) \\[1em] \bullet\ \mu_i(K + x) = \mu_i(K) \\[1em] \bullet\ \mu_i(-K) = \mu_i(K) \\[1em] \bullet\ \text{continuous} \end{array}\right] \mu_i \in \mathrm{Val}_i^+(\mathbb{R}^n)$$

<u>Thm</u> (Klain) Let $\phi \in \mathrm{Val}_i^+$. If $\phi(K) = 0$ if $\dim K = i$, then $\phi = 0$.

<u>Cor</u> Let $\phi \in \mathrm{Val}_i^+$. If $\phi(K) = \mathrm{vol}_i(K)$ if $\dim K = i$, then $\phi = \mu_i$

<u>Proof</u> Consider $\phi - \mu_i$ $\blacksquare$

2. The Holmes-Thompson intrinsic volumes

(V, F) finite-dim normed real vector space

$B_F = \{x : F(x) \leq 1\}$ has smooth and strictly pos. curved boundary

Goal: Define $\mu_0^F, \dots, \mu_n^F$ s.t.

$$K \subset E \subset V \implies \mu_i^{F|_E}(K) = \mu_i^F(K)$$

In particular, if $\dim E = i$, then

$$\mu_i^F\Big|_E = \text{volume in } (E, F|_E)$$

How should one normalize the Haar measure on a normed space?

Holmes-Thompson definition of volume:

Let vol_{2n} be the symplectic volume on $V \times V^*$ and define

$$\mathrm{vol}_F^{HT}(A) = \frac{1}{\omega_n} \mathrm{vol}_{2n}(A \times B_F^*)$$

Thm (Álvarez Paiva-Fernandes '98)

There exists $\mu_i^F \in \mathrm{Val}_i^+(V)$ s.t.

$$\mu_i^F(K) = \mathrm{vol}_{F|_E}^{HT}(K) \text{ if } K \subset E, \dim E = i$$

Remark Schneider '97, Schneider-Wieacker '97: B_F^* is zonoid

Thm (Álvarez Paiva – Fernandes '98, Bernig '07)

$$\mu_i^F \cdot \mu_j^F = \binom{i+j}{i} \frac{\omega_{i+j}}{\omega_i \omega_j} \mu_{i+j}^F$$

$\hookrightarrow$ Alesker product

3. The Weyl tube theorem

Thm (Weyl) Let $A \subset \mathbb{R}^n$ be a compact smooth submanifold with corners. Then

(a) $\mathrm{vol}_n(A_r) = \sum_{i=0}^{m} \mu_i(A)\, \omega_{n-i}\, r^{n-i}$, $\quad 0 \leq r \ll 1$ and $m = \dim A$.

(b) $\mu_i(A)$ depends only on the induced Riem. geometry of A.

Conj (Fu) Let $A \subset V$ be a compact submanifold with corners.
Then $\mu_i^F(A)$ depends only on the induced Finslerian geometry of A.

Remark Burago-Ivanov '11.

Def (Alesker) M smooth oriented mfd, $\mathcal{P}(M) = \{$cpt submfds with corners$\}$
A smooth valuation is a function $\phi : \mathcal{P}(M) \to \mathbb{R}$ of the form

$$\phi(A) = \int_A \Theta + \int_{\boxed{N(A)}} \omega, \qquad \Theta \in \Omega^n(M),\ \omega \in \Omega^{n-1}(SM)$$

normal cycle of A

$V^\infty(M) := \{$smooth valuations on $M\}$

If $e : M \to N$ is an embedding and $\phi \in V^\infty(N)$, then

$$(e^*\phi)(A) := \phi(e(A)), \qquad A \in \mathcal{P}(M)$$

the pullback of ϕ

This definition can be extended to immersions.

Conj (Fu) Let $e_j : M \to V_j$ $(j = 1, 2)$ be embeddings of a smooth mfd M into normed spaces (V_j, F_j). If $e_1^* F_1 = e_2^* F_2$, then

$$e_1^* \mu_i^{F_1} = e_2^* \mu_i^{F_2}.$$

4. Main results

Def. Let $M \subset V$. We say that M satisfies the weak Weyl principle (WWP) at $p \in M$, if whenever F, F' are norms on V s.t.

$$F|_{TU} = F'|_{TU}$$

for some neighborhood $p \in U \subset M$, then $\mu_i^F|_U = \mu_i^{F'}|_U$.

Thm A (Faifman-W.) Let $m = \dim M$, $n = \dim V$

(a) If $n \leq m+2$, then WWP holds for any immersion $M \to V$

(b) If $m+3 \leq n \leq 2m$, then WWP holds for a dense residual subset of immersions $M \to V$.

Def Let $\Theta: SM \to S^{n-1}$, $\Theta(p,v) = v$. M is called directionally regular at (p,v) if $d\Theta_{(p,v)}: T_{(p,v)}SM \to T_v S^{n-1}$ has full rank.

Def
- The 2nd osculating space $O_p^2 M \subset T_pV$ is spanned by the velocity and acceleration vectors of curves through p.
- The 1st normal space $N_p^1 M :=$ ortho complement of T_pM inside $O_p^2 M$.

Thm B (Faifman-W.) Assume $m \geq 3$ and $M \subset V$ directionally regular at $(p,v) \in SM$. Then WWP fails at p if $\dim O_p^2 M > 2m$

In particular, since $\dim O_p^2 M = m + \min\left(\frac{m(m+1)}{2}, \operatorname{codim} M\right)$ generically, WWP fails generically when $\operatorname{codim} M > m$.

Thm C (Faifman-W.) Let $M \subset V$. Then $\mu_i^F|_M$ is determined by $\frac{\partial^j F}{\partial v^j}(u)$ for all $(p,u) \in SM$ and $v \in N_p^1 M$ for all $j \leq m-1$

Problem Does the full Weyl principle in codimensions 1 and 2 ?

5. Proofs

Let $\phi \in \mathcal{V}^\infty(M)$, $\phi(A) = \int_A \Theta + \int_{N(A)} \omega$

Rumin differential

Thm (Bernig-Bröcker '07) $\phi \equiv 0 \iff$ (1) $\underbrace{D\omega + \pi^*\Theta}_{=:\Delta_\phi \in \Omega^n(SM)} = 0$

(2) $\underset{\substack{\| \\ \mathcal{F}_\phi \in C^\infty(M)}}{\phi(\{p\})} = 0 \quad \forall p \in M$

Prop $\mathcal{F}_{\mu_1^F}|_M = 0$ and $\Delta_{\mu_1^F}|_M$ at $(p,u) \in SM$ is determined by the restriction to S_pM of all derivatives $\frac{\partial^j F}{\partial v^j}$, $v \in \mathcal{N}_p^u M$, $j \leq m-1$.

Lemma Assume $M \subset V$ and $\Theta : SM \to S^{n-1}$ is submersive in a dense subset of SM. Then $\frac{\partial^i F}{\partial v^i}$ is determined by $F|_{TM}$

Proof If Θ is submersive at (p,u), then $\Theta(SM)$ contains a neighborhood of u in S^{n-1}!

$\boxtimes$

Lemma Let $W \subset C^\infty(M,V)$ be the set of immersions. Then there is dense residual subset $W' \subset W$ s.t. for all $f \in W_1'$ f is directionally regular on a dense open subset of SM.

$\Rightarrow$ Proves (b) of Thm A.

codim $M = 2$:

Lemma Let $h : \mathrm{Sym}^2 V \to W$ be linear. If $\dim \mathrm{im}(h) \leq 2 \leq \dim V$, then there exists a dense open subset $U \subset V$ s.t. for every $u \in U$ the map $V \to W$, $v \mapsto h(u,v)$ is onto $\mathrm{im}\, h$

Lemma $d\Theta_{(p,u)} = T_u\, \Theta(S_p M) \oplus \mathrm{im}(h_p(u, \cdot))$