Pushforwards of Intrinsic Volumes

Georg Hofstätter[1]

jointly with Thomas Wannerer[2]

[1]Tel Aviv University, [2]Friedrich-Schiller University Jena

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Valuations on Manifolds

$M^n \ldots$ oriented smooth manifold of dimension n

Definition

A (smooth) valuation on M is a map $\mu : \mathcal{P}(M) \to \mathbb{R}$ given by $\omega \in \Omega^{n-1}(SM)$ and $\phi \in \Omega^n(M)$, s.t.

$$\mu(P) = \int_{N(P)} \omega + \int_P \phi, \quad P \in \mathcal{P}(M).$$

The space of valuations is denoted by $\mathcal{V}(M)$.

$\mathcal{P}(M)$	$\ldots$	compact submanifolds with corners
SM	$\ldots$	cosphere bundle
$N(P) \subseteq SM$	$\ldots$	normal cycle/bundle

Examples

$M = \mathbb{R}^n$

- Intrinsic Volumes μ_k, $k = 0, \ldots, n$, defined by

$$\mathrm{vol}_{\mathbb{R}^n}(K_r) = \mathrm{vol}_{\mathbb{R}^n}(K + rB^n) = \sum_{k=0}^{n} r^{n-k}\omega_{n-k}\mu_k(K), \quad r > 0,$$

  where $K \subset \mathbb{R}^n$ is convex and compact.
  - $\mu_0 \equiv 1$
  - $\mu_{n-1} \propto$ Surface area
  - $\mu_n \ldots$ Volume

B^n	$\ldots$	Unit ball
ω_n	$=$	$\mathrm{vol}_{\mathbb{R}^n}(B^n)$

Pullbacks and Pushforwards of Valuations (Alesker 2010)

Let M, N be oriented smooth manifolds and $f : M \to N$ smooth.

Definition

If f is an immersion, then the *pullback* of $\mu \in \mathcal{V}(N)$ is defined (locally) by

$$(f^*\mu)(P) = \mu(f(P)), \quad P \in \mathcal{P}(M).$$

Definition

If f is a proper submersion, then the *pushforward* of $\mu \in \mathcal{V}(M)$ is defined by

$$(f_*\mu)(P) = \mu(f^{-1}(P)), \quad P \in \mathcal{P}(N).$$

Note: Pullback and pushforward can be fully described on the level of differential forms.

Lipschitz–Killing valuations

(M^n, g) Riemannian manifold, $e : M \hookrightarrow \mathbb{R}^N$ isometric embedding.

Definition

The Lipschitz–Killing valuations $\mu_k^M \in \mathcal{V}(M)$ are defined by

$$\mu_k^M = e^* \mu_k, \quad k = 0, \ldots, n.$$

Note:

- μ_k^M does not depend on e (Weyl's principle)!
- $\mu_0^M = \chi \ldots$ Euler characteristic
- $\mu_n^M = \mathrm{vol}_M \ldots$ Riemannian volume measure

Theorem (Fu & Wannerer 2019)

The subspace generated by the Lipschitz–Killing valuations μ_k^M is characterised among a (narrow) natural family of valuations on M by invariance under pullback w.r.t. isometric immersions.

Main Question

Let $f : M \to N$ be a proper Riemannian submersion, i.e.

- f is a proper submersion,
- $df|_{(\mathrm{Ker}\, df)^\perp} : (\mathrm{Ker}\, df)^\perp \to TN$ is an isometry.

Problem (Fu)

Understand the pushforwards $f_*\mu_k^M$ for a Riemannian submersion f.

Example: Hopf fibration $\pi : S^{2n+1} \subseteq \mathbb{C}^{n+1} \to \mathbb{C}P^n$

$$(z_1, \ldots, z_{n+1}) \mapsto [z_1, \ldots, z_{n+1}] \in \left(\mathbb{C}^{n+1} \backslash \{0\}\right) / \mathbb{C}^\times = \mathbb{C}P^n$$

Fiber $\pi^{-1}(p) \cong S^1$

Pushforward of $\mu_0^M = \chi$ and $\mu_n^M = \mathrm{vol}_M$

Let $f : M \to N$ be a proper Riemannian submersion, N connected.

Proposition

There exist $c \in \mathbb{R}$, s.t.

- $f_*\chi = c \cdot \chi$
- $(f_* \mathrm{vol}_M)(P) = \int_P \mathrm{vol}(f^{-1}(\cdot)) \cdot \mathrm{vol}_N$

Proof.

- By Ehresmann's fibration theorem, $f : M \to N$ is a fibration (admits local trivializations), denote the fiber by F.

$$\implies \chi(M) = \chi(F)\chi(N)$$

- By fiber integration of differential forms. □

Variations of Valuations

Idea: Take variation (derivative) of a valuation

Definition (Bernig & Fu 2011)

Let $\mu \in \mathcal{V}(M)$ be given by

$$\mu(P) = \int_{N(P)} \omega + \int_P \phi, \quad P \in \mathcal{P}(M).$$

Then the variation $\widetilde{\delta}\mu \in \mathcal{V}(M)$ of μ is defined by

$$\left(\widetilde{\delta}\mu\right)(P) = \int_{N(P)} i_T(D\omega + p_M^*\phi), \quad P \in \mathcal{P}(M).$$

T	...	Reeb vector field on M
D	...	Rumin differential
$p_M : SM \to M$	...	projection

Variations of Valuations

Note: It holds

$$\left(\widetilde{\delta}\mu\right)(P) = \left.\frac{d}{dr}\right|_{r=0} \mu(P_r),$$

where P_r is the r-neighborhood of $P \in \mathcal{P}(M)$.

Example: Variations of $\mathrm{vol}_{\mathbb{R}^n}$ via Steiner formula

$$\widetilde{\delta}\,\mathrm{vol}_{\mathbb{R}^n}(K) = \left.\frac{d}{dr}\right|_{r=0} \mathrm{vol}_{\mathbb{R}^n}(K_r) = \left.\frac{d}{dr}\right|_{r=0} \sum_{k=0}^{n} r^{n-k}\omega_{n-k}\mu_k(K), \quad r > 0$$

$$\implies \widetilde{\delta}^i\,\mathrm{vol}_{\mathbb{R}^n} = \omega_i\, i!\, \mu_{n-i}$$

Variations and Pushforwards

Let $f : M \to N$ be a proper Riemannian submersion.

Theorem (H. & Wannerer 2022+)

Suppose that $\mu \in \mathcal{V}(M)$. Then

$$f_*(\widetilde{\delta}\mu) = \widetilde{\delta}(f_*\mu).$$

Consequently, if $f^{-1}(p)$ has constant volume for all $p \in N$, then there exists $c' \in \mathbb{R}$, s.t.

$$f_*\left(\widetilde{\delta}^k \operatorname{vol}_M\right) = c' \cdot \widetilde{\delta}^k \operatorname{vol}_N, \quad k \in \mathbb{N}.$$

Problem: Write μ_k^M in terms of $\widetilde{\delta}^i \operatorname{vol}_M$? Not possible in general!

Hopf Fibration – Steiner Formula on S^{2n+1}

Theorem (Glasauer 1995)

There exist (explicit) constants $c_{n,j} \in \mathbb{R}$, s.t.

$$\mathrm{vol}_{S^{2n+1}}(P_r) = \sum_{j=0}^{2n} c_{n,j} \left(\int_0^r \sin(s)^{2n-j} \cos(s)^j \, ds \right) \tau_j(P) + c_{n,2n+1} \tau_{2n+1}(P).$$

$\tau_0, \ldots, \tau_{2n+1} \in \mathcal{V}(S^{2n+1})$ is a (natural) basis of $\mathcal{V}(S^{2n+1})^{\mathrm{SO}(2n+2)}$, s.t.

$$\tau_k(S^j) = \delta_{j,k} 2^{k+1}.$$

In particular, $\mu_k^{S^{2n+1}} \in \langle \tau_0, \ldots, \tau_{2n+1} \rangle$ (with explicit formulas).

Hopf Fibration – Steiner Formula on $\mathbb{C}P^n$

Theorem (Bernig, Fu, Solanes 2014)

There exist (explicit) constants $\tilde{c}_{n,k} \in \mathbb{R}$, s.t.

$$\mathrm{vol}_{\mathbb{C}P^n}(P_r) = \sum_{k=0}^{2n} \tilde{c}_{n,k} \sin(r)^{2n-k} \cos(r)^k \tau_{k,0}(P).$$

Here,

$$\tau_{k,0} = \sum_{q=\max\{0,k-n\}}^{\lfloor \frac{k}{2} \rfloor} \mu_{k,q},$$

where $\mu_{k,q}$ is the Hermitian intrinsic volume (Bernig & Fu 2011).

Pushforwards by the Hopf Fibration

$$\begin{array}{ccccc} \mu_k^{S^{2n+1}} & \Longleftrightarrow & \tau_k & \Longleftrightarrow & \widetilde{\delta}^k \operatorname{vol}_{S^{2n+1}} \\ & & & & \downarrow \pi_* \\ & & \tau_{k,0} & \Longleftrightarrow & \widetilde{\delta}^k \operatorname{vol}_{\mathbb{C}P^n} \end{array}$$

$\implies$ Get $\pi_* \mu_k^{S^{2n+1}}$ in terms of $\tau_{k,0}$

Note: $\pi_* \mu_k^{S^{2n+1}} \notin \langle \mu_j^{\mathbb{C}P^n} \rangle$

Pushforwards by the Hopf Fibration

Theorem (H. & Wannerer 2022+)

Let $\pi : S^{2n+1} \to \mathbb{C}P^n$ be the Hopf fibration. Then

$$\pi_* \operatorname{vol}_{S^{2n+1}} = 2\pi \operatorname{vol}_{\mathbb{C}P^n}$$

and

$$\pi_* \tau_j = \frac{2^{j+1}}{\omega_{j-1}} \tau_{j-1,0} - \frac{2^{j+1}}{\omega_{j+1}} \tau_{j+1,0}, \quad j = 0, \dots, 2n.$$

(with the convention $\tau_{-1,0} = 0 = \tau_{2n+1,0}$)

Check: $\chi = \sum_{j \geq 0} \left(\frac{1}{4}\right)^j \tau_{2j} \implies \pi_* \chi = 0.$

Conclusion

Results

- Variation and pushforward commutes
- Complete answer for Hopf fibration

Future Questions

- General case
- Understand image of pushforward operation
- Integral geometric interpretations
- Relation to algebraic structures on valuations
- ...

Thank you for your attention!