Spectral decomposition

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The general setting for the Langlands program over global fields.

Let G be a reductive group over a global field F. We assume G is split to simplify. We denote by $\widehat{G}$ the Langlands dual group of G. It is the split reductive group whose weights and roots are the coweights and coroots of G. Examples :

G	$\widehat{G}$
GL_n	GL_n
SL_n	PGL_n
SO_{2n+1}	Sp_{2n}
Sp_{2n}	SO_{2n+1}
SO_{2n}	SO_{2n}

and if G is one of the five exceptional groups, $\widehat{G}$ is of the same type.

The locally compact ring of adèles $\mathbb{A}$ of F contains F discretely, and the goal of the Langlands program is to decompose $L^2(G(F)\backslash G(\mathbb{A}), \mathbb{C})$, as a representation of $G(\mathbb{A})$, in terms of global Langlands parameters, which are (if we impose some algebraicity conditions in the case of number fields) continuous morphisms $\mathrm{Gal}(\overline{F}/F) \to \widehat{G}(\overline{\mathbb{Q}_\ell})$, up to conjugacy. From now on we consider only function fields.

Geometric interpretation of the adelic quotient.

Let X be a smooth projective geometrically connected curve over $\mathbb{F}_q$ and F its field of rational functions.
Let $\mathbb{A}$ be the ring of adèles of F and $\mathbb{O} = \prod_v \mathcal{O}_{F_v}$ be the ring of integral adèles.
Let G be a split reductive group over F.
We have

$$G(F)\backslash G(\mathbb{A})/G(\mathbb{O}) = \mathrm{Bun}_G(\mathbb{F}_q) \tag{0.1}$$

where $\mathrm{Bun}_G(\mathbb{F}_q)$ is the set of isomorphism classes of G-principal bundles over X.
We recall that a G-principal bundle over X is defined as a morphism $Y \to X$ equipped with a simply transitive action of G on the fibers. The GL_r-principal bundles can be equivalently seen as the frame bundles of the vector bundles of rank r.
Equality (0.1) holds because any G-principal bundle over X can be trivialized over $X \setminus S$ where S is a finite set of places of X, and is then given by an element of $\prod_{v \in S} G(F_v)/G(\mathcal{O}_{F_v})$. Moreover $G(\mathbb{A})/G(\mathbb{O})$ is the union of all $\prod_{v \in S} G(F_v)/G(\mathcal{O}_{F_v})$ where S varies, and two trivializations of a G-principal bundle over $X \setminus S$ for some S are related by the action of an element of $G(F)$.

Definition of automorphic forms over function fields.

Let N be a level, i.e. a finite subscheme of X (which is the same as a finite subset of places of X with multiplicities).
Let $\mathcal{O}_N$ be the ring of functions on N. We note that $G(\mathcal{O}_N)$ is a finite group. We define $K_N = \mathrm{Ker}(G(\mathbb{O}) \to G(\mathcal{O}_N))$. It is an open compact subgroup of $G(\mathbb{A})$.
Then we have

$$G(F)\backslash G(\mathbb{A})/K_N = \mathrm{Bun}_{G,N}(\mathbb{F}_q)$$

where $\mathrm{Bun}_{G,N}(\mathbb{F}_q)$ is the set of isomorphism classes of G-principal bundles over X together with a trivialization of their restriction to N.

Definition. An automorphic form with level N is a function on $\mathrm{Bun}_{G,N}(\mathbb{F}_q)$.
In particular an automorphic form with trivial level is a function on $G(F)\backslash G(\mathbb{A})/G(\mathbb{O}) = \mathrm{Bun}_G(\mathbb{F}_q)$.

Stacks.

In fact, as G-principal bundles over X may have automorphisms, $\mathrm{Bun}_{G,N}(\mathbb{F}_q)$ is a groupoid whose elements have finite automorphism groups.
It is the groupoid of points over $\mathbb{F}_q$ of a stack $\mathrm{Bun}_{G,N}$ over $\mathbb{F}_q$. The definition of $\mathrm{Bun}_{G,N}$ is that its groupoid of "points" over a scheme S over $\mathbb{F}_q$ (by which we mean morphisms $S \to \mathrm{Bun}_{G,N}$) classify the G-principal bundles over $X \times S$ together with a trivialization of their restriction to $N \times S$.
The products $X \times S$ and $N \times S$ are products of schemes over $\mathrm{Spec}(\mathbb{F}_q)$.

A stack is like a scheme whose points may have algebraic automorphism groups.
Examples of stacks are given by quotients of schemes by actions of affine smooth group schemes.

Cuspidal automorphic forms over function fields.

Let ℓ be a prime number not dividing q.
We write

$$\mathrm{Funct}_c^{\mathrm{cusp}}(\mathrm{Bun}_{G,N}(\mathbb{F}_q),\mathbb{Q}_\ell)\subset \mathrm{Funct}_c(\mathrm{Bun}_{G,N}(\mathbb{F}_q),\mathbb{Q}_\ell)$$

the $\mathbb{Q}_\ell$-vector subspace formed by "cuspidal" functions, inside the vector space of all functions with finite support on $\mathrm{Bun}_{G,N}(\mathbb{F}_q)$. The cuspidal automorphic forms are the "elementary bricks" to build all automorphic forms.
We can define cuspidal automorphic forms with coefficients in $\mathbb{Q}$. We take them with coefficients in $\mathbb{Q}_\ell$ because the ℓ-adic cohomology we need to use and the Langlands parameters we want to construct are both with coefficients in $\mathbb{Q}_\ell$.

Definition of the unramified Hecke operators.

We assume first that N is empty. Let v be a closed point of X.
If $\mathcal{G}$ and $\mathcal{G}'$ are two G-principal bundles over X we say that $\mathcal{G}'$ is a modification of $\mathcal{G}$ at v if we are given an isomorphism between their restrictions to $X \setminus v$. Then their relative position $[\mathcal{G}' : \mathcal{G}]$ at v is a dominant coweight λ of G (when $G = GL_n$ it is the n-uple of the elementary divisors). We introduce the unramified Hecke operator

$$\begin{aligned} T_{\lambda,v} : \mathrm{Funct}_c(\mathsf{Bun}_G(\mathbb{F}_q), \mathbb{Q}_\ell) &\to \mathrm{Funct}_c(\mathsf{Bun}_G(\mathbb{F}_q), \mathbb{Q}_\ell) \\ f &\mapsto [\mathcal{G} \mapsto \sum_{\mathcal{G}', [\mathcal{G}':\mathcal{G}]=\lambda} f(\mathcal{G}')] \end{aligned}$$

The sum, taken over the modifications $\mathcal{G}'$ of $\mathcal{G}$ at v with relative position λ, is finite.

More generally, with a level N, for any closed point v in $X \setminus N$, and any coweight λ, we have an operator $T_{\lambda,v}$ acting on $\mathrm{Funct}_c(\mathsf{Bun}_{G,N}(\mathbb{F}_q), \mathbb{Q}_\ell)$.
When λ varies the operators $T_{\lambda,v}$ span the unramified Hecke algebra $\mathcal{H}_v$ which is commutative and acts on $\mathrm{Funct}_c(\mathsf{Bun}_{G,N}(\mathbb{F}_q), \mathbb{Q}_\ell)$. Its action preserves $\mathrm{Funct}_c^{\mathrm{cusp}}(\mathsf{Bun}_{G,N}(\mathbb{F}_q), \mathbb{Q}_\ell)$.

Definition of the global Langlands parameters.

Let $\overline{F}$ be an algebraic closure of the function field F.
For any open subscheme $U \subset X$ (the complement of a finite number of closed points), we denote by $\overline{F}^U \subset \overline{F}$ the subfield generated by all finite extensions of F associated to unramified coverings of U. Then $\mathrm{Gal}(\overline{F}/F)$ acts on $\overline{F}^U$ by a quotient denoted by $\pi_1(U)$ (with base point $\mathrm{Spec}\,\overline{F}$).
We have $F \supset \mathbb{F}_q$ (the subfield of constant functions on the curve) and $\overline{F} \supset \overline{F}^U \supset \overline{\mathbb{F}_q}$.
This gives a short exact sequence

$$1 \to \pi_1^{\mathrm{geom}}(U) \to \pi_1(U) \to \mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q) \to 1.$$

We have $\mathrm{Gal}(\overline{\mathbb{F}_q}/\mathbb{F}_q) = \widehat{\mathbb{Z}}$ (with generator $x \mapsto x^q$) and we define $\mathrm{Weil}(U)$ by the short exact sequence

$$1 \to \pi_1^{\mathrm{geom}}(U) \to \mathrm{Weil}(U) \to \mathbb{Z} \to 1.$$

Definition. A global Langlands parameter is a $\widehat{G}(\overline{\mathbb{Q}_\ell})$-conjugacy class of continuous and semisimple morphisms $\sigma : \mathrm{Weil}(X \setminus N) \to \widehat{G}(\overline{\mathbb{Q}_\ell})$.

Statement of the theorem.

To simplify we assume from now on that G is semisimple, i.e. its center is finite. Then $\mathrm{Funct}_c^{\mathrm{cusp}}(\mathrm{Bun}_{G,N}(\mathbb{F}_q), \mathbb{Q}_\ell)$ is a $\mathbb{Q}_\ell$-vector space of finite dimension.

Theorem. We have a canonical decomposition $\mathrm{Funct}_c^{\mathrm{cusp}}(\mathrm{Bun}_{G,N}(\mathbb{F}_q), \overline{\mathbb{Q}_\ell}) = \bigoplus_\sigma \mathfrak{H}_\sigma$ indexed by global Langlands parameters $\sigma : \mathrm{Weil}(X \setminus N) \to \widehat{G}(\overline{\mathbb{Q}_\ell})$. It is respected by all Hecke operators. We have a compatibility with the Satake isomorphism at all closed points of $X \setminus N$ (the meaning will be explained later).

In the case where $G = GL_n$ this was already known from the works of Drinfeld for $n = 2$ and of Laurent Lafforgue for arbitrary n, with different methods.

The proof of the theorem above uses

- general stacks of shtukas (introduced by Drinfeld and Varshavsky)
- the geometric Satake equivalence (due to Lusztig, Drinfeld, Ginzburg, and Mirkovic–Vilonen).

Meaning of the compatibility with the Satake isomorphism stated in the theorem.

The Satake isomorphism is a canonical isomorphism

$$[V] \mapsto T_{V,v}$$

from the $\overline{\mathbb{Q}_\ell}$-algebra of algebraic representations of $\widehat{G}$ (where product is tensor product) to the unramifed Hecke algebra $\mathcal{H}_v$. If V is an irreducible representation of $\widehat{G}$, $T_{V,v}$ is a combination of the $T_{\lambda,v}$ for λ a weight of V.
We have $\pi_1(v) := \mathrm{Gal}(\overline{k(v)}/k(v)) = \widehat{\mathbb{Z}}$ with generator $\mathrm{Frob}_v : x \mapsto x^{q^d}$ where d is the degree of v (and the cardinal of $k(v)$ is q^d).
We still denote by $\mathrm{Frob}_v \in \pi_1(X \setminus N)$ the image of $\mathrm{Frob}_v \in \pi_1(v)$ by the morphism $\pi_1(v) \to \pi_1(X \setminus N)$ (indeed $v \subset X \setminus N$ and any morphism of schemes $Y \to Z$ gives a morphism of groups $\pi_1(Y) \to \pi_1(Z)$). We have $\mathrm{Frob}_v \in \mathrm{Weil}(X \setminus N)$ and it is well defined up to conjugation.
Then the compatibility of the decomposition $\mathrm{Funct}_c^{\mathrm{cusp}}(\mathrm{Bun}_{G,N}(\mathbb{F}_q), \overline{\mathbb{Q}_\ell}) = \bigoplus_\sigma \mathfrak{H}_\sigma$ with the Satake isomorphism means that for any closed point v of $X \setminus N$ and for any representation V of $\widehat{G}$, $T_{V,v}$ preserves this decomposition and acts on $\mathfrak{H}_\sigma$ by multiplication by $\mathrm{Tr}_V(\sigma(\mathrm{Frob}_v))$.

In the next slides we explain the idea of the proof of the theorem.
We will construct a commutative algebra $\mathcal{B}$ of "excursion operators" containing all the $\mathcal{H}_v$ and such that

- $\mathcal{B}$ acts on $\mathrm{Funct}_c(\mathrm{Bun}_{G,N}(\mathbb{F}_q), \mathbb{Q}_\ell)$ and preserves $\mathrm{Funct}_c^{\mathrm{cusp}}(\mathrm{Bun}_{G,N}(\mathbb{F}_q), \mathbb{Q}_\ell)$
- each character $\nu : \mathcal{B} \to \overline{\mathbb{Q}_\ell}$ corresponds in a unique way to a Langlands parameter σ.

Since $\mathcal{B}$ is commutative and $\mathrm{Funct}_c^{\mathrm{cusp}}(\mathrm{Bun}_{G,N}(\mathbb{F}_q), \mathbb{Q}_\ell)$ is of finite dimension we will obtain a canonical spectral decomposition in generalized eigenspaces

$$\mathrm{Funct}_c^{\mathrm{cusp}}(\mathrm{Bun}_{G,N}(\mathbb{F}_q), \overline{\mathbb{Q}_\ell}) = \bigoplus_{\nu} \mathfrak{H}_\nu$$

where the sum is taken over the characters $\nu : \mathcal{B} \to \overline{\mathbb{Q}_\ell}$. By associating to every ν a Langlands parameter σ we will deduce the decomposition of the theorem

$$\mathrm{Funct}_c^{\mathrm{cusp}}(\mathrm{Bun}_{G,N}(\mathbb{F}_q), \overline{\mathbb{Q}_\ell}) = \bigoplus_{\sigma} \mathfrak{H}_\sigma.$$

Definition of the stack of shtukas (for N empty, to simplify) .

For any scheme S over $\mathbb{F}_q$ we recall that $\mathrm{Frob}_S : S \to S$ is the morphism acting on functions by $\mathrm{Frob}_S^*(f) = f^q$.

Let I be a finite set. We define Sht_I as the stack over X^I whose "points" over a scheme S over $\mathbb{F}_q$ (by which we mean morphisms $S \to \mathrm{Sht}_I$) classify shtukas, namely

- points $(x_i)_{i\in I} : S \to X^I$, called the legs of the shtuka,
- a G-principal bundle $\mathcal{G}$ over $X \times S$,
- an isomorphism

$$\phi : \mathcal{G}\big|_{(X\times S)\smallsetminus(\bigcup_{i\in I}\Gamma_{x_i})} \xrightarrow{\sim} (\mathrm{Id}_X \times \mathrm{Frob}_S)^*(\mathcal{G})\big|_{(X\times S)\smallsetminus(\bigcup_{i\in I}\Gamma_{x_i})}$$

  where $\Gamma_{x_i} \subset X \times S$ denotes the graph of x_i.

It is a Deligne-Mumford stack (i.e. the automorphism groups of points are finite étale).

The stack of shtukas without legs $\mathrm{Sht}_\emptyset$ is equal to the groupoid $\mathrm{Bun}_G(\mathbb{F}_q)$.

Remark. Shtukas do not have analogues over number fields in general because nobody knows what $(\mathrm{Spec}(\mathbb{Z}))^I$ should be for $\sharp I > 1$. The work of Fargues and Scholze uses an analogue of local shtukas over $\mathbb{Q}_p$.

The geometric Satake equivalence.

We define $\mathcal{M}_I$ as the stack over X^I whose points over a scheme S over $\mathbb{F}_q$ classify

- points $(x_i)_{i\in I} : S \to X^I$,
- G-principal bundles $\mathcal{G}$ and $\mathcal{G}'$ over the formal completion $\widehat{X \times S}$ of $X \times S$ along the union of the Γ_{x_i},
- an isomorphism
$$\phi : \mathcal{G}|_{\widehat{X\times S} \smallsetminus (\bigcup_{i\in I} \Gamma_{x_i})} \xrightarrow{\sim} \mathcal{G}'|_{\widehat{X\times S} \smallsetminus (\bigcup_{i\in I} \Gamma_{x_i})}$$
where Γ_{x_i} denotes the graph of x_i.

Thus $\mathcal{M}_I(S)$ depends only on $\bigcup_{i\in I} \Gamma_{x_i}$.
Fusion of legs is what happens when some x_i become equal.
The geometric Satake equivalence associates to any finite set I and any algebraic finite dimensional $\mathbb{Q}_\ell$-linear representation W of $\widehat{G}^I$ a perverse sheaf $\mathcal{S}_{I,W}$ on $\mathcal{M}_I$, which is functorial in W and compatible with the fusion of legs.

The obvious forgetful morphism $\alpha : \mathrm{Sht}_I \to \mathcal{M}_I$ is smooth.
We define a perverse sheaf $\mathcal{F}_{I,W}$ on Sht_I as the pull-back $\alpha^*(\mathcal{S}_{I,W})$.

The ℓ-adic cohomology of the stacks of shtukas.

We denote by $H_{I,W}$ the $\mathbb{Q}_\ell$-vector space of ℓ-adic cohomology with compact support of the fiber of Sht_I over a geometric generic point of X^I (or, a posteriori equivalently, over a geometric generic point of the diagonal $X \subset X^I$) with coefficients in $\mathcal{F}_{I,W}$.
We no more assume N empty. We can define a stack Sht_I of shtukas with level N, and construct $\mathcal{F}_{I,W}$ and $H_{I,W}$ in the same way.
We note that what matters is not the total space Sht_I but the morphism

$$\begin{array}{c} \mathrm{Sht}_I \\ \downarrow \\ (X \setminus N)^I \end{array}$$

which associates to a shtuka the I-uple of its legs.
The $\mathbb{Q}_\ell$-vector space $H_{I,W}$ is equipped with a continuous action of $(\mathrm{Weil}(X \setminus N))^I$ (thanks to partial Frobenius morphisms introduced by Drinfeld, my work, and the work of Cong Xue). Here continuous means it is the union of subspaces of finite dimension with continuous action of $(\pi_1^{\mathrm{geom}}(X \setminus N))^I$.

The strategy to construct the algebra $\mathcal{B}$ of excursion operators.

When $I = \emptyset$ and $W = \mathbf{1}$ (the trivial one dimensional representation), we have isomorphisms

$$H_{\emptyset,\mathbf{1}} \simeq \mathrm{Funct}_c(\mathrm{Bun}_{G,N}(\mathbb{F}_q), \mathbb{Q}_\ell)$$

because $\mathrm{Sht}_\emptyset = \mathrm{Bun}_{G,N}(\mathbb{F}_q)$, and $\mathcal{F}_{\emptyset,\mathbf{1}}$ is the constant sheaf $\mathbb{Q}_\ell$.

We will use the $H_{I,W}$ to construct an algebra $\mathcal{B}$ of excursion operators acting on $H_{\emptyset,\mathbf{1}} \simeq \mathrm{Funct}_c(\mathrm{Bun}_{G,N}(\mathbb{F}_q), \mathbb{Q}_\ell)$. This action will preserve $\mathrm{Funct}_c^{\mathrm{cusp}}(\mathrm{Bun}_{G,N}(\mathbb{F}_q), \mathbb{Q}_\ell)$.

Properties of the $H_{I,W}$

a) Functoriality of $H_{I,W}$ in W

For any finite set I,

$$W \mapsto H_{I,W}$$

is a $\mathbb{Q}_\ell$-linear functor from the category of representations of $\widehat{G}^I$ to the category of continuous representations of $(\mathrm{Weil}(X \setminus N))^I$.
This means that for any morphism

$$u : W \to W'$$

of representations of $\widehat{G}^I$, we have a morphism

$$\mathcal{H}(u) : H_{I,W} \to H_{I,W'}$$

of continuous representations of $(\mathrm{Weil}(X \setminus N))^I$.

b) Fusion for the $H_{I,W}$

Fusion can be associated to any map $\zeta : I \to J$ but we consider here only the case where J is a singleton, which we denote by $\{0\}$.

For any representation W of $\widehat{G}^I$, we have a fusion isomorphism, functorial in W,

$$H_{I,W} \xrightarrow{\sim} H_{\{0\},W_{\mathrm{diag}}}$$

where W_{diag} denotes the representation of $\widehat{G}$ on W obtained by composition with the diagonal morphism $\widehat{G} \to \widehat{G}^I$.

Two examples of the fusion isomorphism of the previous slide.

- If W_1 and W_2 are two representations of $\widehat{G}$, we have the fusion isomorphism

$$H_{\{1,2\},W_1 \boxtimes W_2} \xrightarrow{\sim} H_{\{0\},W_1 \otimes W_2}$$

associated to the obvious map $\{1,2\} \to \{0\}$. We note the difference between $W_1 \boxtimes W_2$ which is a representation of $(\widehat{G})^2$ and $W_1 \otimes W_2$ which is a representation of $\widehat{G}$.

- We have the fusion isomorphism

$$H_{\emptyset,\mathbf{1}} \xrightarrow{\sim} H_{\{0\},\mathbf{1}}$$

associated to the obvious map $\emptyset \to \{0\}$ (the idea is that $H_{\emptyset,\mathbf{1}}$, *resp.* $H_{\{0\},\mathbf{1}}$ is the cohomology of the stack of shtukas without legs, *resp.* with an inactive leg and that they are identical).

Construction of the algebra $\mathcal{B}$ of excursion operators.

For any algebraic function f on $\widehat{G}\backslash\widehat{G}^I/\widehat{G}$ we can find a representation W of $\widehat{G}^I$ and $x \in W$ and $\xi \in W^*$ invariant by the diagonal action of $\widehat{G}$ such that

$$f((g_i)_{i\in I}) = \langle \xi, (g_i)_{i\in I} \cdot x\rangle. \tag{0.2}$$

Let $(\gamma_i)_{i\in I} \in (\mathrm{Weil}(X \setminus N))^I$. The excursion operator $S_{I,f,(\gamma_i)_{i\in I}}$ of $H_{\{0\},\mathbf{1}}$ is defined as the composition

$$H_{\{0\},\mathbf{1}} \xrightarrow{\mathcal{H}(x)} H_{\{0\},W_{\mathrm{diag}}} \overset{\text{fusion}}{\overset{\sim}{\to}} H_{I,W} \xrightarrow{(\gamma_i)_{i\in I}} H_{I,W} \overset{\text{fusion}}{\overset{\sim}{\to}} H_{\{0\},W_{\mathrm{diag}}} \xrightarrow{\mathcal{H}(\xi)} H_{\{0\},\mathbf{1}}$$

where W_{diag} is the diagonal representation of $\widehat{G}$ on W, and $x : \mathbf{1} \to W_{\mathrm{diag}}$ and $\xi : W_{\mathrm{diag}} \to \mathbf{1}$ are considered here as morphisms of representations of $\widehat{G}$.
The construction above does not depend on the choice of W, x, ξ satisfying (0.2).

Construction of the decomposition of the theorem.

Thanks to the properties of the $H_{I,W}$ explained in the previous slides we show that

1) the algebra $\mathcal{B}$ of endomorphisms of $H_{\{0\},\mathbf{1}}$ generated by the $S_{I,f,(\gamma_i)_{i\in I}}$ when I, f and $(\gamma_i)_{i\in I}$ vary is commutative,

2) for any character $\nu : \mathcal{B} \to \overline{\mathbb{Q}_\ell}$ there is a unique Langlands parameter σ such that for any I, f and $(\gamma_i)_{i\in I}$,

$$\nu(S_{I,f,(\gamma_i)_{i\in I}}) = f((\sigma(\gamma_i))_{i\in I}).$$

Since $\mathcal{B}$ is commutative and its action preserves the finite dimensional subspace $\mathrm{Funct}_c^{\mathrm{cusp}}(\mathrm{Bun}_G(\mathbb{F}_q), \mathbb{Q}_\ell) \subset \mathrm{Funct}_c(\mathrm{Bun}_G(\mathbb{F}_q), \mathbb{Q}_\ell) = H_{\emptyset,\mathbf{1}} = H_{\{0\},\mathbf{1}}$ we have a canonical spectral decomposition $\mathrm{Funct}_c^{\mathrm{cusp}}(\mathrm{Bun}_G(\mathbb{F}_q), \overline{\mathbb{Q}_\ell}) = \bigoplus_\nu \mathfrak{H}_\nu$ where the sum is taken over characters ν of $\mathcal{B}$ (in other words $\mathfrak{H}_\nu$ is a generalized eigenspace for the elements of $\mathcal{B}$). Associating to ν a unique Langlands parameter σ as in 2) we deduce the decomposition $\mathrm{Funct}_c^{\mathrm{cusp}}(\mathrm{Bun}_G(\mathbb{F}_q), \overline{\mathbb{Q}_\ell}) = \bigoplus_\sigma \mathfrak{H}_\sigma$ of the theorem.

Compatibility with the Satake isomorphism and end of the proof of the theorem.

The unramified Hecke operators are particular cases of excursion operators. Indeed let V be an irreducible representation of $\widehat{G}$. We take

$$I = \{1,2\} \text{ and } f : (g_1, g_2) \mapsto \mathrm{Tr}_V(g_1 g_2^{-1}) \text{ as a function on } \widehat{G}\backslash\widehat{G}^I/\widehat{G}.$$

By a geometric argument (computing the intersection of algebraic cycles in the stack of shtukas) we show that for any closed point v,

$$T_{V,v} = S_{\{1,2\},f,(\mathrm{Frob}_v,1)}.$$

This equality plays an important role in technical arguments (the Eichler-Shimura relations), and it justifies the compatibility of the decomposition with Satake isomorphism at closed points v of X.

Some open questions.

1) We hope that all Langlands parameters σ which appear in this decomposition come from elliptic Arthur parameters. This would imply the Ramanujan-Petersson conjecture for all reductive groups over function fields.
2) We hope that the decomposition

$$\mathrm{Funct}_c^{\mathrm{cusp}}(\mathrm{Bun}_{G,N}(\mathbb{F}_q),\overline{\mathbb{Q}_\ell})=\bigoplus_{\sigma}\mathfrak{H}_\sigma$$

is defined over $\overline{\mathbb{Q}}$ (instead of $\overline{\mathbb{Q}_\ell}$) and is independent on ℓ and on the embedding $\iota:\overline{\mathbb{Q}}\subset\overline{\mathbb{Q}_\ell}$. The question makes sense because in a recent article Drinfeld defined the set of Langlands parameters σ independently on ℓ and ι.
We could prove this is true if we knew how to construct the excursion operators in a motivic way (then the σ would be motivic Langlands parameters). In other words, we hope that the motive of Sht_I with coefficients in $\mathcal{F}_{I,W}$, equipped with the partial Frobenius morphisms, is equivalent to a sum of external products of motives over $X\setminus N$, but this seems completely out of reach.

A joint work with Alain Genestier.

In the theorem the canonical decomposition

$$\mathrm{Funct}_c^{\mathrm{cusp}}(\mathrm{Bun}_{G,N}(\mathbb{F}_q), \overline{\mathbb{Q}_\ell}) = \bigoplus_\sigma \mathfrak{H}_\sigma, \tag{0.3}$$

is preserved by all Hecke operators, including ramified Hecke operators (not defined in this talk) at closed points v in N.
The theorem gives the compatibility with the Satake isomorphism at closed points in $X \setminus N$ but does not say how the action on $\mathfrak{H}_\sigma$ of ramified Hecke operators at closed points v in N is related to σ.
In a joint work with Alain Genestier, we construct a local parameterization up to semisimplification and show a local-global compatibility at all closed points.
It implies that in the decomposition above, for any closed point $v \in N$, the semisimplification of $\sigma\big|_{\mathrm{Weil}(\overline{F_v}/F_v)}$ depends only on the character by which the center of the algebra of ramified Hecke operators at v acts on $\mathfrak{H}_\sigma$.

The Arthur-Kottwitz heuristics says that for every σ there exists a $\overline{\mathbb{Q}_\ell}$-linear representation $\mathfrak{A}_\sigma$ of its centralizer $S_\sigma \subset \widehat{G}$, so that for all I, W,

$$H_{I,W} \stackrel{?}{=} \bigoplus_\sigma \left(\mathfrak{A}_\sigma \otimes W_{\sigma^I}\right)^{S_\sigma}$$

where S_σ acts diagonally and W_{σ^I} is the representation of $(\mathrm{Weil}(X \setminus N))^I$ obtained by composition of the representation W of $\widehat{G}^I$ with the morphism
$\sigma^I : (\mathrm{Weil}(X \setminus N))^I \to \widehat{G}^I$.
Thanks to an idea of Drinfeld we can obtain something close to this heuristics.

A construction proposed by Drinfeld.

First we define a scheme $\mathcal{S}$ (locally of finite type over $\mathbb{Q}_\ell$) of morphisms $\sigma : \mathrm{Weil}(X \setminus N) \to \widehat{G}$, such that for any $\mathbb{Q}_\ell$-algebra R, the R-points of $\mathcal{S}$ are the morphisms $\sigma : \mathrm{Weil}(X \setminus N) \to \widehat{G}(R)$, such that for any morphism $\alpha : \widehat{G} \to GL_N$, $\alpha \circ \sigma : \mathrm{Weil}(X \setminus N) \to GL_N(R)$ gives an action of $\mathrm{Weil}(X \setminus N)$ on R^N such that R^N (seen as a $\mathbb{Q}_\ell$-vector space) is an inductive limit of continuous representations of $\pi_1^{\mathrm{geom}}(X \setminus N)$ on finite dimensional $\mathbb{Q}_\ell$-vector spaces.
Let Reg be the left regular representation of $\widehat{G}$ (i.e. the action by left translation of $\widehat{G}$ on the vector space of all algebraic functions on $\widehat{G}$). We can endow $H_{\{0\},\mathrm{Reg}}$ with

a) a structure of $\mathcal{O}$-module on $\mathcal{S}$,

b) an action of $\widehat{G}$ compatible with conjugation by $\widehat{G}$ on $\mathcal{S}$.

This gives rise to an $\mathcal{O}$-module on the algebraic stack $\mathcal{S}/\widehat{G}$ of Langlands parameters and $\mathfrak{A}_\sigma$ should be the fiber of this $\mathcal{O}$-module at σ. It would be equipped with an action of the centralizer S_σ, which is the automorphism group of σ in $\mathcal{S}/\widehat{G}$.
Xinwen Zhu and I prove this works over elliptic σ (which means that S_σ is finite).

A reformulation of a) is the following structure. For any finite dimensional $\mathbb{Q}_\ell$-linear representation V of $\widehat{G}$, with underlying vector space $\underline{V}$, $H_{\{0\},\mathrm{Reg}} \otimes \underline{V}$ is equipped with an action of $\mathrm{Weil}(X \setminus N)$, such that it is an inductive limit of finite dimensional continuous representations of $\pi_1^{\mathrm{geom}}(X \setminus N)$. Moreover this structure is functorial in V and compatible with tensor products.

This structure is obtained in the following way. We have a $\widehat{G}$-equivariant isomorphism

$$\theta : \mathrm{Reg} \otimes \underline{V} \simeq \mathrm{Reg} \otimes V$$
$$f \otimes x \mapsto [g \mapsto f(g)g.x]$$

where $\widehat{G}$ acts diagonally on the RHS. This formula for θ makes sense when we consider the RHS as the vector space of algebraic functions $\widehat{G} \to V$. We deduce an isomorphism

$$H_{\{0\},\mathrm{Reg}} \otimes \underline{V} = H_{\{0\},\mathrm{Reg} \otimes \underline{V}} \xrightarrow[\sim]{\theta} H_{\{0\},\mathrm{Reg} \otimes V} \simeq H_{\{0\}\cup\{1\},\mathrm{Reg} \boxtimes V}$$

where the first equality is tautological and the last isomorphism is fusion. Then the action of $\mathrm{Weil}(X \setminus N)$ on the LHS is defined as the action of $\mathrm{Weil}(X \setminus N)$ on the RHS corresponding to the leg 1.

If V_1 and V_2 are two representations of $\widehat{G}$, the two actions of $\mathrm{Weil}(X \setminus N)$ on $H_{\{0\},\mathrm{Reg}} \otimes \underline{V_1} \otimes \underline{V_2}$ associated to actions of $\widehat{G}$ on V_1 and V_2 commute with each other and the diagonal action of $\mathrm{Weil}(X \setminus N)$ is equal to the action associated to the diagonal action of $\widehat{G}$ on $V_1 \otimes V_2$.
If V is as above, $x \in V$, $\xi \in V^*$, and f is the function on $\widehat{G}$ defined as the matrix coefficient $f(g) = \langle \xi, g.x \rangle$, and $\gamma \in \mathrm{Weil}(X \setminus N)$ we see that $F_{f,\gamma} : \sigma \mapsto f(\sigma(\gamma))$ is a function on $\mathcal{S}$. Its action on $H_{\{0\},\mathrm{Reg}}$ by the structure a) of $H_{\{0\},\mathrm{Reg}}$ as an $\mathcal{O}$-module on $\mathcal{S}$ (which we want to construct) is defined as the composition

$$H_{\{0\},\mathrm{Reg}} \xrightarrow{\mathrm{Id} \otimes x} H_{\{0\},\mathrm{Reg}} \otimes \underline{V} \xrightarrow{\gamma} H_{\{0\},\mathrm{Reg}} \otimes \underline{V} \xrightarrow{\mathrm{Id} \otimes \xi} H_{\{0\},\mathrm{Reg}} \tag{0.4}$$

where the arrow in the middle is the action of $\gamma \in \mathrm{Weil}(X \setminus N)$ on $H_{\{0\},\mathrm{Reg}} \otimes \underline{V}$ defined in the previous slide.

Any function f on $\widehat{G}$ can be written as a matrix coefficient. There is a continuity property : for any f, any $\gamma_0 \in \pi_1(X \setminus N)$ and any $c \in H_{\{0\},\mathrm{Reg}}$, the space generated by the $\{F_{f,\gamma} c \mid \gamma \in \pi_1^{\mathrm{geom}}(X \setminus N)\gamma_0\}$ is of finite dimension and $\gamma \mapsto F_{f,\gamma} c$ is continuous. The property of the previous slide with V_1 and V_2 implies the relations between the $F_{f,\gamma}$, namely that

$$F_{f,\gamma_1\gamma_2} = \sum_{\alpha} F_{f_1^\alpha,\gamma_1} F_{f_2^\alpha,\gamma_2} \tag{0.5}$$

if the image of f by the coproduct is $\sum_\alpha f_1^\alpha \otimes f_2^\alpha$.

In fact $\mathcal{S}$ is essentially defined as the Spec of the algebra generated by the $F_{f,\gamma}$, with these relations, and taking into account this continuity property (it can also be derived, as in articles by Arinkin, Gaitsgory, Kazhdan, Raskin, Rozenblyum, Varshavsky and by Xinwen Zhu). Thus we have established structure a) above.
The structure b) is the action of $\widehat{G}$ on $H_{\{0\},\mathrm{Reg}}$ associated to the right action of $\widehat{G}$ on Reg. The conjugation $gF_{f,\gamma}g^{-1}$ for $g \in \widehat{G}$ is then equal to the action of $F_{f^g,\gamma}$ where $f^g(h) = f(g^{-1}hg)$. This proves that structure b) is compatible with conjugation by $\widehat{G}$ on $\mathcal{S}$.
Remark : The $F_{f,\gamma}$ are functions on $\mathcal{S}$, and the excursion operators correspond to functions on $\mathcal{S}/\widehat{G}$. The excursion operators are exactly the combinations of products of $F_{f,\gamma}$ which are invariant under $\widehat{G}$. More precisely, for any finite set I, any function f on $\widehat{G}^I$, and $(\gamma_i)_{i\in I}$, we can define $F_{f,(\gamma_i)_{i\in I}}$, which is equal to $\prod_i F_{f_i,\gamma_i}$ when $f = \boxtimes_i f_i$ (and any f is a sum of $\boxtimes_i f_i$). Then the excursion operators are exactly the $F_{f,(\gamma_i)_{i\in I}}$ when f is a function on $\widehat{G}\backslash\widehat{G}^I/\widehat{G}$.

For any morphism $\sigma : \mathrm{Weil}(X \setminus N) \to \widehat{G}(\overline{\mathbb{Q}_\ell})$, we define $\mathfrak{A}_\sigma$ as the biggest quotient of $H_{\{0\},\mathrm{Reg}} \otimes_{\mathbb{Q}_\ell} \overline{\mathbb{Q}_\ell}$ such that, for any f and γ, $F_{f,\gamma}$ acts on it by the multiplication by the scalar $f(\sigma(\gamma))$. We see that S_σ acts on $\mathfrak{A}_\sigma$.
We say that σ is elliptic if S_σ is finite (to simplify we assumed G semisimple). For example when $\widehat{G} = SL_n$ this is equivalent to say that σ is irreducible.
Then, in a joint work with Xinwen Zhu, we prove that, denoting by $(H_{I,W})_\sigma$ the generalized eigenspace for the excursion operators with the eigenvalues associated to σ, it is also the true eigenspace, and we have

$$(H_{I,W})_\sigma = \left(\mathfrak{A}_\sigma \otimes W_{\sigma^I}\right)^{S_\sigma}$$

All terms in the above formula are of finite dimension (thanks to the work of Cong Xue establishing that $H_{I,W}$ is of finite type over a Hecke algebra, which is written only in the split case).

Link with the geometric Langlands program (1)

For the moment, this relation is limited to the unramified case (when N is empty). In an astonishing recent series of works, Arinkin, Gaitsgory, Kazhdan, Raskin, Rozenblyum and Varshavsky introduce a new geometric object, the "stack" $\mathrm{Locsys}_{\hat{G}}^{\mathrm{restr}}$ of local systems on a curve X with restricted variation ; this stack makes sense in any sheaf-theoretic context (ℓ-adic, de Rham, etc.), and is the only one which exists in the ℓ-adic setting, where X is a smooth projective curve over $\mathbb{F}_q$ as in this lecture. This "stack" is derived and is formal in some directions. The authors formulate the categorical geometric Langlands conjecture for the DG-category $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ of ℓ-adic sheaves on Bun_G with singular support in some nilpotent cone, and they prove :

(a) $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ admits a spectral decomposition over $\mathrm{Locsys}_{\hat{G}}^{\mathrm{restr}}$,

(b) the categorical trace of Frobenius on $\mathrm{Shv}_{\mathrm{Nilp}}(\mathrm{Bun}_G)$ is canonically isomorphic to $\mathrm{Funct}_c(\mathrm{Bun}_G(\mathbb{F}_q), \overline{\mathbb{Q}_\ell})$; the categorical trace of Frobenius composed with Hecke functors is canonically isomorphic to $H_{I,W}$.

Link with the geometric Langlands program (2)

They define the stack of arithmetic local systems on X as the Frobenius invariants (in a derived sense) :

$$\mathrm{Locsys}_{\hat{G}}^{\mathrm{arithm}} := \left(\mathrm{Locsys}_{\hat{G}}^{\mathrm{restr}}\right)^{Frob}.$$

They show that $\mathrm{Locsys}_{\hat{G}}^{\mathrm{arithm}}$ is a quasi-compact, algebraic derived stack, locally almost of finite type. The classical stack associated to it by forgetting the derived structure is $\mathcal{S}/\widehat{G}$. They construct an object Drinf in a DG-category of quasi-coherent sheaves on $\mathrm{Locsys}_{\hat{G}}^{\mathrm{arithm}}$ which allows to recover all the $H_{I,W}$, not only as cohomology groups with action of $\mathrm{Weil}(X)^I$, but as objects in a DG-category of lisse sheaves on X^I (equipped with partial Frobenius morphisms).
For each elliptic σ as above there is an embedding of a connected component $pt/S_\sigma \to \mathrm{Locsys}_{\hat{G}}^{\mathrm{arithm}}$ and the restriction of Drinf to pt/S_σ is exactly $\mathfrak{A}_\sigma$.