Wavefront sets of unipotent representations

Dan Ciubotaru

Mathematical Institute
University of Oxford

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Goal. Investigate the relation between characters of admissible representations and the local Langlands parameters. Applications to the construction of Arthur packets.

Joint work with L. Mason-Brown and E. Okada:

- The wave front sets of unipotent supercuspidal representations, `arXiv:2206.08628`;
- Some unipotent Arthur packets for reductive p-adic groups, `arXiv:2210.00251`;
- The wavefront sets of Iwahori-spherical representations of reductive p-adic groups, `arXiv:2112.14354v4` (2022).

The ideas also owe to papers by Lusztig, Barbasch-Moy, Waldspurger, and Okada.

The local character expansion

k: nonarchimedean local field of characteristic 0; $\mathbb{F}_q$: residue field;
$\bar{\mathsf{k}}$: algebraic closure; $K \subset \bar{\mathsf{k}}$: maximal unramified extension of k.
G: connected semisimple algebraic k-group. Later, assume G is k-split.
(π, X): smooth admissible (complex) $G(\mathsf{k})$-representation.
$\mathcal{H}(G(\mathsf{k}))$: Hecke algebra, smooth compactly-supported complex functions on $G(\mathsf{k})$ with convolution defined with respect to a Haar measure dg.

Harish-Chandra: the *character distribution*

$$\Theta_\pi : \mathcal{H}(G) \to \mathbb{C}, \quad f \mapsto \mathrm{tr}(\pi(f)),$$

is represented on $G(\mathsf{k})_{rs}$ (regular s.s. elements) by a locally constant function.

The Howe-Harish-Chandra local character expansion says that there exist $c_{\mathbb{O}}(\pi) \in \mathbb{C}$ such that

$$\Theta_\pi(\exp(\xi)) = \sum_{\mathbb{O}} c_{\mathbb{O}}(\pi)\ \hat{\mu}_{\mathbb{O}}(\xi),$$

for all $\xi \in \mathfrak{g}(\mathrm{k})$ in a sufficiently small neighbourhood U_π of 0. Here $\mathbb{O}$ ranges over the (finite) set of nilpotent orbits in $\mathfrak{g}(\mathrm{k})$, and $\hat{\mu}_{\mathbb{O}}$ is the Fourier transform of the nilpotent orbital integral for $\mathbb{O}$.

The neighbourhood U_π is known (under some assumptions) by the work of Waldspurger, DeBacker. It depends only on the Moy-Prasad depth of π. E.g., if π has *depth zero*, then it holds for all "topologically nilpotent elements" of $\mathfrak{g}(\mathrm{k})$.
Much less is known about the constants $c_{\mathbb{O}}(\pi)$.

Example $GL(n)$

Suppose $G = GL(n)$. The orbits $\mathbb{O}$ are in one-to-one correspondence with partitions λ of n (Jordan normal form). Howe showed that

$$\hat{\mu}_{\mathbb{O}_\lambda} = \Theta_{\mathrm{Ind}_{P_{\lambda^t}}^G(\mathbf{1})},$$

where P_{λ^t} is the parabolic subgroup whose Levi has blocks of size λ^t. Then, if π has depth zero, the local char expansion becomes:

$$\Theta_\pi = \sum_\lambda c_\lambda(\pi)\Theta_{\mathrm{Ind}_{P_{\lambda^t}}(\mathbf{1})},$$

on top. nilpotent elements. At least when π has Iwahori fixed vectors, this can be then interpreted as an "irreducible in terms of standard" character formula, i.e., $c_\lambda(\pi)$ can be expressed in terms of Kazhdan-Lusztig multiplicities.

The Wavefront set

Back to the general case. Let $\mathcal{N}_o(\mathsf{k})$ be the set of adjoint $G(\mathsf{k})$ orbits in $\mathfrak{g}(\mathsf{k})$, a poset with respect to the closure ordering. The *p-adic wave front set* is

$$\mathrm{WF}(\pi) = \max\{\mathbb{O} \mid c_{\mathbb{O}}(\pi) \neq 0\} \subset \mathcal{N}_o(\mathsf{k}).$$

One can also define a coarser invariant, the *algebraic wave front set*:

$$^{\bar{\mathsf{k}}}\mathrm{WF}(\pi) = \max\{G(\bar{\mathsf{k}}) \cdot \mathbb{O} \mid c_{\mathbb{O}}(\pi) \neq 0\} \subset \mathcal{N}_o(\bar{\mathsf{k}}).$$

Conjecture

(Mœglin-Waldspurger): For each irreducible admissible (π, X), $^{\bar{\mathsf{k}}}\mathrm{WF}(\pi)$ is a single orbit.

(Also, M-W related the leading coefficients $c_{\mathbb{O}}(\pi)$ to the dimension of degenerate Whittaker models for π.)

Deligne-Langlands-Lusztig correspondence

Restrict to *unipotent representations* (in the sense of Lusztig). In particular, this includes the class of representations with Iwahori fixed vectors.

Suppose G is split and adjoint for simplicity. Let $G^\vee$ be the complex Langlands dual group and $T^\vee$ a maximal torus. Kazhdan-Lusztig (Iwahori), Lusztig (unipotent) exhibited a natural correspondence between the set of irred. admissible $G(\mathsf{k})$-repns and $G^\vee$-orbits of triples (s, n, ρ), where:

- $s \in T^\vee$;
- $n \in \mathfrak{g}^\vee$, $\mathrm{Ad}(s)n = qn$;
- $\rho \in \widehat{A_{G^\vee}(s, n)}$, $\rho|_{Z(G^\vee)} = \mathrm{Id}$.

In this correspondence, $\pi(s, n, \rho)$ is tempered if and only if $s = s_c q^{h/2}$ where h is a neutral element for a Lie triple of n and $s_c \in T^\vee$ is compact.
$\pi(s, n, \rho)$ has I-fixed vectors if and only if ρ is "of Springer type".
$\pi(s, n, \rho)$ is spherical (i.e., has $G(\mathfrak{o})$-fixed vectors) if and only if $n = 0$, $\rho = \mathbf{1}$.
Say $\pi = \pi(s, n, \rho)$ has *(positive) real infinitesimal character* if $s \in T^\vee_{\mathbb{R}_{>0}}$ (in the polar decomposition).

Aubert-Zelevinsky duality

There is an involution on the Grothendieck group $R(G(\mathsf{k}))$ of smooth $G(\mathsf{k})$-reps:

$$\mathrm{AZ}(\pi) = \sum_Q (-1)^{r_Q} i_{Q(\mathsf{k})}^{G(\mathsf{k})}(r_{Q(\mathsf{k})}^{G(\mathsf{k})}(\pi),$$

where Q ranges over a set of parabolic k-subgroups of G containing a fixed Borel k-subgroup. This was defined in various equivalent forms by Zelevinsky, Aubert, Bernstein, Schneider-Stuhler. By Kato's results, it is equivalent to a variant of the Iwahori-Matsumoto involution for I-spherical representations.
It is the p-adic analogue of the Alvis-Curtis duality for representations of finite reductive groups.

AZ maps irreducibles to irreducibles (up to a sign), and it preserves Bernstein components, in particular, it preserves unipotent representations (and I-spherical representations).

Main results.

For $\pi = \pi(s, n, \rho)$, denote $\mathbb{O}^\vee_\pi = G^\vee \cdot n$.

Theorem (C.-Mason-Brown-Okada, 2022)

Let $\pi = \pi(s, n, \rho)$ be an irreducible unipotent representation. The algebraic wave front of π is

$$^{\bar{k}}\mathrm{WF}(\pi) = d(\mathbb{O}^\vee_{\mathrm{AZ}(\pi)})$$

when

- *π is supercuspidal (and G is inner to split);*
- *π is spherical (and G is split);*
- *π has Iwahori fixed vectors and real infinitesimal character (and G is split).*

This will also be true for all unipotent with real infinitesimal character (details soon), but it's more subtle for nonreal inf. character.

The duality d

What is d? This is the duality map for nilpotent orbits between G and $G^\vee$ over $\bar{\mathsf{k}}$ (or $\mathbb{C}$) defined by Spaltenstein (1982), Lusztig (1984), also Barbasch-Vogan (1985):

$$d : \mathcal{N}_o \to \mathcal{N}_o^\vee, \quad d : \mathcal{N}_o^\vee \to \mathcal{N}_o.$$

The image of d consists of the set of special nilpotent orbits (in the sense of Lusztig). There are two extensions of this map that are important for us:

$$\text{Sommers}: \ d_S : \mathcal{N}_{o,c} \twoheadrightarrow \mathcal{N}_o^\vee, \quad d_S : \mathcal{N}_{o,c}^\vee \twoheadrightarrow \mathcal{N}_o,$$

where $\mathcal{N}_{o,c} = \{(\mathbb{O}, C) : C \text{ conjugacy class in } A(\mathbb{O})\}$, and Achar's maps where $A(\mathbb{O})$ is replaced by the canonical quotient:

$$\text{Achar}: \ d_A : \mathcal{N}_{o,\bar{c}} \to \mathcal{N}_{o,\bar{c}}^\vee, \quad d_A : \mathcal{N}_{o,\bar{c}}^\vee \to \mathcal{N}_{o,\bar{c}}.$$

Compatibility:

$$d_S(\mathbb{O}, 1) = d(\mathbb{O}), \quad d_A(\mathbb{O}, \bar{C}) = (d_S(\mathbb{O}, C), \bar{C}'), \text{ for some } \bar{C}'.$$

The formula for the wave front set should be compared to

1. Lusztig's theorem (1992) relating the Kawanaka wave front set of a unipotent representation of a finite reductive group to the dual Springer support. In fact our results rely on the Barbasch-Moy test functions which are lifts of Kawanaka Gelfand-Graev representations for finite reductive groups (quotients of parahoric subgroups), and the bridge to the dual Langlands group is given by Lusztig's results.
2. Waldspurger's computation of algebraic wave front sets of tempered and AZ-dual of tempered ireducible representations when G is inner to split for $SO(2n+1)$.
3. The calculation of the wave front sets of Arthur unipotent representations of complex and real reductive groups (Adams, Barbasch, Vogan...).

An important ingredient is the *canonical unramified wave front set* (Okada, 2021), ${}^{K}\mathrm{WF}(\pi)$. This is an equivalence class of K-orbits (K the unramified extension of k) which sits in between $\mathrm{WF}(\pi)$ and ${}^{\bar{\mathrm{k}}}\mathrm{WF}(\pi)$. What we prove in fact is the stronger version:

$$ {}^{K}\mathrm{WF}(\pi) = d_A(\mathbb{O}^{\vee}_{\mathrm{AZ}(\pi)}, 1). $$

What's important here is a classification of the unramified nilpotent orbits (so for $G(K)$ due to Okada (2021), based on DeBacker's parametrization and McNinch-Sommers' results: there is a "Bala-Carter bijection" between unramified nilpotent orbits and $\mathcal{N}_{o,c}$, i.e., pairs

$$ (\mathbb{O}, C), \quad \mathbb{O} \text{ nilpotent orbit in } \mathfrak{g}(\bar{\mathrm{k}}), \quad C \subset A(\mathbb{O}) \text{ conj. class.} $$

Moreover, we prove a general lower bound for the wave front set.

Theorem (C-MB-O)

Suppose $\pi = \pi(q^{h^\vee/2}, n, \rho)$*, where* $h^\vee$ *is a neutral element for a fixed nilpotent orbit* $\mathbb{O}^\vee$*. Then*

$$d_A(\mathbb{O}^\vee, 1) \leq_A {}^K\mathrm{WF}(\pi).$$

In particular,

$$d(\mathbb{O}^\vee) \leq {}^{\bar{\mathsf{k}}}\mathrm{WF}(\pi).$$

Moreover, $d_A(\mathbb{O}^\vee, 1) = {}^K\mathrm{WF}(\pi)$ *if and only if* $\mathrm{AZ}(\pi)$ *is tempered.*

Applications: Langlands params for unipotent cuspidal reps

Let (π, X) be an irred. unipotent supercuspidal repn of $G(\mathsf{k})$. It is known that X occurs as a direct summand of a compactly induced $\mathrm{ind}_P^{G(\mathsf{k})}(\sigma)$, where P is a maximal parahoric subgroup of $G(\mathsf{k})$ and σ is a cuspidal (Deligne-Lusztig) unipotent repn of $M(\mathbb{F}_q) = \overline{P}$.
Lusztig parameterized the unipotent representations of $M(\mathbb{F}_q)$ in terms of nilpotent orbits for the dual group $M^\vee$ (and equivariant vector bundles on the canonical quotient group). Suppose σ is parameterized by $\mathbb{O}_\sigma^\vee$ in $\mathfrak{m}^\vee$.

Theorem (C-MB-O)

The unipotent part of the Langlands parameter for X is given by

$$\mathbb{O}_X^\vee := d(G(\bar{\mathsf{k}})) \cdot d_M(\mathbb{O}_\sigma^\vee)) \subset \mathfrak{g}^\vee.$$

d_M is the Spaltenstein duality $M^\vee \to M$ and d is the duality $G \to G^\vee$.
Here $G(\bar{\mathsf{k}})) \cdot d_M(\mathbb{O}_\sigma^\vee)$ is exactly the geometric wave front set of X.

Applications: Arthur packets

Arthur: to each parameter $\psi : W_{\mathsf{k}}' \times \mathrm{SL}(2,\mathbb{C}) \to G^\vee$, there should be attached a finite packet of irreducible unitary representations Π_ψ^{A} of $G(\mathsf{k})$-representations satisfying certain stability/endoscopic properties. Here $W_{\mathsf{k}}' = W_{\mathsf{k}} \ltimes \mathbb{C}$ is the Weil-Deligne group and $\psi|_{W_{\mathsf{k}}'}$ should be tempered. Moreover, if

$$\phi_\psi(w) = \psi(w, \begin{pmatrix} \|w\|^{1/2} & 0 \\ 0 & \|w\|^{-1/2} \end{pmatrix}), \qquad w \in W_{\mathsf{k}}',$$

we want the Langlands packet $\Pi_{\phi_\psi}^{\mathsf{L}}$ to be a subset of Π_ψ^{A}.
Alternatively, we have a "simplified" Arthur parameter:

$$\widetilde{\psi} : W_{\mathsf{k}} \times \mathrm{SL}(2,\mathbb{C}) \times \mathrm{SL}(2,\mathbb{C}) \to G^\vee.$$

It is conjectured that AZ should flip the SL(2)'s in $\widetilde{\psi}$.

Consider ψ such that $\psi|_{W_k}$ = trivial. These should be the unipotent Arthur representations with "real infinitesimal character". These are just the same as maps

$$\widetilde{\psi} : \mathrm{SL}(2)_A \times \mathrm{SL}(2)_L \to G^\vee.$$

If $\psi|_{\mathrm{SL}(2)_A}$ =trivial, then $\Pi^A_\psi = \Pi^L_{\phi_\psi}$ is the tempered L-packet. So the AZ-dual should be an A-packet, the *anti-tempered A-packet*.

Theorem (C-MB-O)

Let $\mathbb{O}^\vee$ be a nilpotent adjoint $G^\vee$-orbit and let $\psi_{\mathbb{O}^\vee}$ be the associated anti-tempered Arthur parameter. Then $\Pi^A_{\psi_{\mathbb{O}^\vee}}(\mathbf{G}(\mathrm{k}))$ is the set of irreducible representations X with unipotent cuspidal support such that

(i) *The infinitesimal character of X is $q^{\frac{1}{2}h^\vee}$.*

(ii) *The canonical unramified wavefront set ${}^K\mathrm{WF}(X)$ is minimal subject to (i).*

In (ii), ${}^K\mathrm{WF}(X) = d_A(\mathbb{O}^\vee, 1)$ in fact.

Weak Arthur packets

Motivated by the real groups ABV picture, define:

$$\Pi^{\mathrm{Weak}}_{\psi_{\mathbb{O}^\vee}}(G(\mathsf{k})) := \{X = X(q^{\frac{1}{2}h^\vee}, n, \rho) \mid {}^{\bar{\mathsf{k}}}\mathrm{WF}(X) \leq d(\mathbb{O}^\vee)\}.$$

Of course, $\Pi^{A}_{\psi_{\mathbb{O}^\vee}}(G(\mathsf{k})) \subseteq \Pi^{\mathrm{Weak}}_{\psi_{\mathbb{O}^\vee}}(G(\mathsf{k}))$.

Because of the Theorem on wavefront sets, it follows:

Theorem (C-MB-O)

The weak Arthur packet $\Pi^{\mathrm{Weak}}_{\mathbb{O}^\vee}(G(\mathsf{k}))$ is the set of irreducible representations $\mathrm{AZ}(X(q^{\frac{1}{2}h^\vee}, n, \rho))$, where n belongs to the special piece (in the sense of Spaltenstein) of $\mathbb{O}^\vee$.

Conjecture.

$\Pi^{\mathrm{Weak}}_{\psi_{\mathbf{O}^\vee}}(G(\mathsf{k}))$ is the union of Arthur packets, in particular, all of the representations in it are unitary.

We know in this setting that AZ preserves unitarity (Barbasch-Moy+), so this indicates that $X(q^{\frac{1}{2}h^\vee}, n, \rho)$, where n belongs to the special piece of $\mathbb{O}^\vee$, must be unitary. I believe these are either tempered or endpoints of complementary series (but I only checked some examples).

An example in F_4

	$G^\vee n$	ρ	I-spherical?	Tempered?	Unitary?	AZ	^{K}WF
X_1	$F_4(a_3)$	(4)	yes	yes	yes	X_{20}	$(F_4, 1)$
X_2	$F_4(a_3)$	(31)	yes	yes	yes	X_{19}	$(F_4(a_1), (12))$
X_3	$F_4(a_3)$	(2^2)	yes	yes	yes	X_{17}	$(F_4(a_1), 1)$
X_4	$F_4(a_3)$	(21^2)	yes	yes	yes	X_{13}	$(C_3, 1)$
X_5	$F_4(a_3)$	(1^4)	no	yes	yes	X_5	$(F_4(a_3), 1)$
X_6	$C_3(a_1)$	(2)	yes	no	yes	X_{15}	$(F_4(a_2), 1)$
X_7	$C_3(a_1)$	(1^2)	yes	no	yes	X_9	$(F_4(a_3), (1234)$
X_8	$A_1 + \widetilde{A}_2$	(1)	yes	no	yes	X_8	$(F_4(a_3), (123))$
X_9	$\widetilde{A}_1 + A_2$	(1)	yes	no	yes	X_7	$(F_4(a_3), (12))$
X_{10}	B_2	(2)	yes	no	yes	X_{18}	$(F_4(a_1), 1)$
X_{11}	B_2	(1^2)	yes	no	yes	X_{11}	$(F_4(a_3), (12)(34))$
X_{12}	A_2	(2)	yes	no	yes	X_{14}	$(B_3, 1)$
X_{13}	A_2	(1^2)	yes	no	yes	X_4	$(F_4(a_3), 1)$
X_{14}	$\widetilde{A}_2$	(1)	yes	no	yes	X_{12}	$(C_3, 1)$
X_{15}	$A_1 + \widetilde{A}_1$	(1)	yes	no	yes	X_6	$(F_4(a_3), (12))$
X_{16}	$A_1 + \widetilde{A}_1$	(1)	yes	no	no	X_{16}	$(F_4(a_2), 1)$
X_{17}	$\widetilde{A}_1$	(2)	yes	no	yes	X_3	$(F_4(a_3), 1)$
X_{18}	$\widetilde{A}_1$	(1^2)	yes	no	yes	X_{10}	$(F_4(a_3), (12)(34))$
X_{19}	A_1	(1)	yes	no	yes	X_2	$(F_4(a_3), 1)$
X_{20}	0	(1)	yes	no	yes	X_1	$(F_4(a_3), 1)$

Table: Irreducible representations of split F_4 with infinitesimal character $q^{h^\vee/2}$ for $\mathbb{O}^\vee = F_4(a_3)$.

The Arthur packet attached to $\mathbb{O}^\vee = F_4(a_3)$ is the set

$$\Pi^{\mathsf{Art}}_{\mathbb{O}^\vee}(G(\mathsf{k})) = \{X_5, X_{13}, X_{17}, X_{19}, X_{20}\}$$

This is precisely the set of anti-tempered representations.

On the other hand, the *weak Arthur packet* attached to $F_4(a_3)$ is the larger set

$$\Pi^{\mathsf{Weak}}_{\mathbb{O}^\vee}(G(\mathsf{k})) = \{X_5, , X_7, X_8, X_9, X_{11}, X_{13}, X_{15}, X_{17}, X_{18}, X_{19}, X_{20}\}$$

There are 10 Arthur parameters $\widetilde{\psi} : \mathrm{SL}(2,\mathbb{C})_{\mathsf{L}} \times \mathrm{SL}(2,\mathbb{C})_{\mathsf{A}} \to G^\vee$ with inf. char. $\frac{1}{2}h^\vee$. The corresponding pairs $(\mathbb{O}^\vee_{\mathsf{L}}, \mathbb{O}^\vee_{\mathsf{A}})$ are

$(0, F_4(a_3)),\ (A_1, C_3(a_1)),\ (\widetilde{A}_1, B_2),\ (A_1+\widetilde{A}_1, A_1+\widetilde{A}_2),\ (\widetilde{A}_1+A_2, \widetilde{A}_1+A_2).$

and their ‘flips’ .

A naive guess for the Arthur packets contained in the weak Arthur packet is:

$\mathbb{O}_L^\vee$	$\mathbb{O}_A^\vee$	Arthur packet
0	$F_4(a_3)$	$\{X_5, X_{13}, X_{17}, X_{19}, X_{20}\}$
$\widetilde{A}_1$	B_2	$\{X_5, X_{17}, X_{18}, X_{11}\}$
$A_1 + \widetilde{A}_1$	$A_1 + \widetilde{A}_2$	$\{X_5, X_{15}, X_8\}$
$\widetilde{A}_1 + A_2$	$\widetilde{A}_1 + A_2$	$\{X_5, X_9, X_7\}$

This guess is compatible with the containment of the Langlands packet, AZ-duality, and the occurrence of the supercuspidals in A-packets.