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Reciprocity for Valuations of Theta Functions

Disclaimer

These are preliminary results, and have not yet appeared!

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Theta functions: the rough idea

Theta functions: Why you should care

- An extended exchange matrix B (or a more general seed $\mathfrak{s}$) determines a family of Laurent series called **theta functions**.
- The theta functions of B with finitely many terms span an algebra with positive structure constants.
- Every cluster monomial of B is a theta function of B.
- If the cluster algebra of B equals the upper cluster algebra, the finite theta functions form a **basis for the cluster algebra**.

For 90% of this talk, all that matters is that the theta functions are a particularly nice basis for any cluster algebra*.

But what actually *are* theta functions?

Given a seed $\mathfrak{s}$ in a lattice M, each $m \in M$ determines

$$\vartheta_{\mathfrak{s}}[m] := \sum_{m' \in M} c_{m,m'} x^{m'}$$

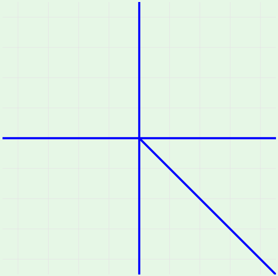
where $c_{m,m'}$ is the (weighted) count of broken lines in a scattering diagram $\mathcal{D}(\mathfrak{s})$ with initial and final derivatives $-m$ and $-m'$.

- I won't define a general seed, but any extended exchange matrix B determines an $\mathcal{A}$-type seed in $\mathbb{Z}^{\text{height}(B)}$.
- The basepoint will be assumed in the positive chamber.

A scattering diagram is a collection of **walls** in $\mathbb{R} \otimes M \simeq \mathbb{R}^r$.

Example: Everyone's first scattering diagram

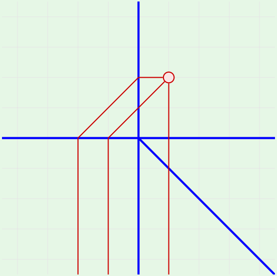
The scattering diagram of the $\mathcal{A}$-type seed of $B = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is



A **broken line** is a piecewise linear ray which can only bend at the walls, and only in certain ways.

Example: A simple theta function

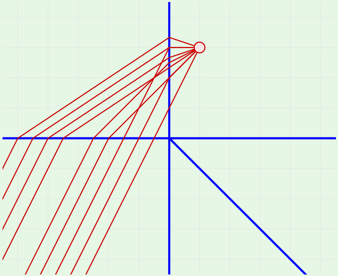
For $m = (0, -1)$, $\vartheta_B[m]$ counts the three broken lines below.



$$\vartheta_B[0, -1] = x^{(0,-1)} + x^{(-1,-1)} + x^{(-1,0)} = \frac{x_1 + 1 + x_2}{x_1 x_2}$$

Example: A more complicated theta function

For $m = (-1, -2)$, $\vartheta_{\mathrm{B}}[m]$ counts the nine broken lines below.



$$\vartheta[-1,-2] = x^{(-1,-2)} + x^{(-1,-1)} + 2x^{(-2,-2)} + 4x^{(-2,-1)} + 2x^{(-2,0)} + x^{(-3,-2)} + 3x^{(-3,-1)} + 3x^{(-3,0)} + x^{(-3,1)}$$

$$= \left(\frac{x_1 + 1 + x_2}{x_1 x_2}\right)^2 \left(\frac{1 + x_2}{x_1}\right)$$

Monomial Valuations

Let $N := \mathrm{Hom}(M, \mathbb{Z})$ be the dual lattice to M.

Definition: Monomial valuations

Given $n \in N$, the **monomial valuation** val_n on $\mathbb{Z}[x^M]$ is

$$\mathrm{val}_n\left(\sum_{m \in M} c_m x^m\right) := \min_{m | c_m \neq 0} (n \cdot m)$$

Here, $n \cdot m$ denotes image of m under $n \in N := \mathrm{Hom}(M, \mathbb{Z})$.

If $M \simeq \mathbb{Z}^d$, then $N \simeq \mathbb{Z}^d$ with pairing given by the dot product, and val_n is the minimum dot product of n with the exponents.

Equivalent to boundary valuations and integral tropical points.

Example: Monomial valuations of $\vartheta[0,-1]$

With B as before, we identify M and N with $\mathbb{Z}^2$. Then

$$\begin{aligned}\mathrm{val}_{(n_1,n_2)}(\vartheta_{\mathrm{B}}[0,-1]) &= \mathrm{val}_{(n_1,n_2)}(x^{(0,-1)} + x^{(-1,-1)} + x^{(-1,0)}) \\ &= \min(-n_2, -n_1 - n_2, -n_1)\end{aligned}$$

Example: Monomial valuations of $\vartheta[-1,-2]$

In $\mathrm{val}_{(n_1,n_2)}(\vartheta_{\mathrm{B}}[-1,-2])$, only 4 of the 9 monomials matter:

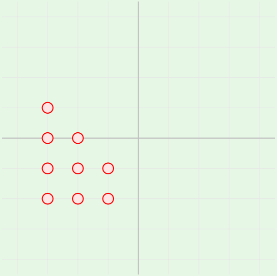
$$\min(-n_1 - 2n_2, -n_1 - n_2, -3n_1 + n_2, 3n_1 - 2n_2)$$

Only the Newton polytope matters

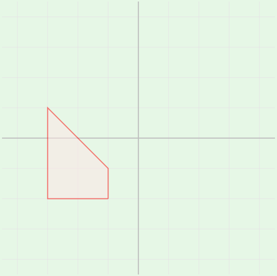
The monomial valuation $\mathrm{val}_n(\vartheta[m])$ only depends on the **Newton polytope** of $\vartheta[m]$: the convex hull of the exponents in $\vartheta[m]$.

Example: The Newton polytope of $\vartheta_{\mathrm{B}}[-1,-2]$

Exponent vectors



Newton polytope



Valuation as tropicalization

$\mathrm{val}_n(\vartheta[m])$ is given by plugging n into the **tropicalization** of $\vartheta[m]$:

$$+ \mapsto \oplus := \min$$
$$\times \mapsto \otimes := +$$
$$x^p \mapsto p \cdot n$$

Example

$$\vartheta_{\mathfrak{s}}[0,-1]) = x^{(0,-1)} + x^{(-1,-1)} + x^{(-1,0)}$$
$$\mathrm{val}_n(\vartheta_{\mathfrak{s}}[0,-1]) = ((0,-1)\cdot n) \oplus ((-1,-1)\cdot n) \oplus ((-1,0)\cdot n)$$
$$= \min(-n_2, -n_1 - n_2, -n_1)$$

Theta Reciprocity

To recap

For a fixed seed $\mathfrak{s}$ in M, we have...

- a family of theta functions $\vartheta[m]$ indexed by $m \in M$, and
- a family of monomial valuations val_n indexed by $n \in N$.

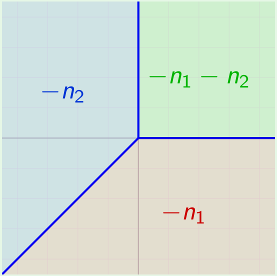
Let's refer to $\mathrm{val}_n(\vartheta_{\mathfrak{s}}[m])$ as the **theta pairing** between n and m.

Question

How does this pairing behave as a function of each argument?

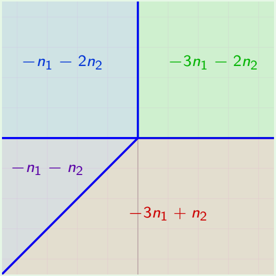
Example: Visualizing $\mathrm{val}_?(\vartheta[m])$

$\mathrm{val}_{(n_1,n_2)}(\vartheta_{\mathrm{B}}[0,-1])$



$=\min(-n_1, -n_1-n_2, -n_2)$

$\mathrm{val}_{(n_1,n_2)}(\vartheta_{\mathrm{B}}[-1,-2])$



$=\min(-n_1-2n_2, -n_1-n_2, -3n_1+n_2, 3n_1-2n_2)$

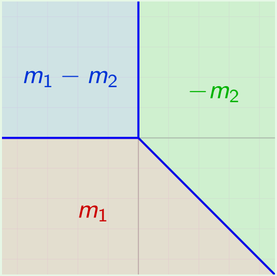
The function $\mathrm{val}_?(\vartheta_{\mathfrak{s}}[m])$ is always a **piecewise linear function**, since it is the minimum of a collection of linear functions.

What about $\mathrm{val}_n(\vartheta[?])$

This is much harder to compute directly! One needs a construction of $\vartheta[m]$ for all $m \in M$, which we only know for very nice seeds.

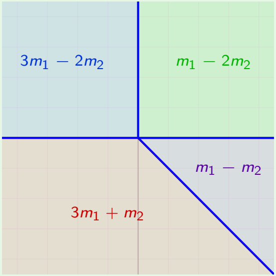
Example: Visualizing $\mathrm{val}_n(\vartheta[?])$

$\mathrm{val}_{(0,-1)}(\vartheta_{\mathrm{B}}[m])$



$=\min(-m_2, m_1 - m_2, m_1)$

$\mathrm{val}_{(1,-2)}(\vartheta_{\mathrm{B}}[m])$



$=\min(m_1 - 2m_2, m_1 - m_2, 3m_1 + m_2, 3m_1 - 2m_2)$

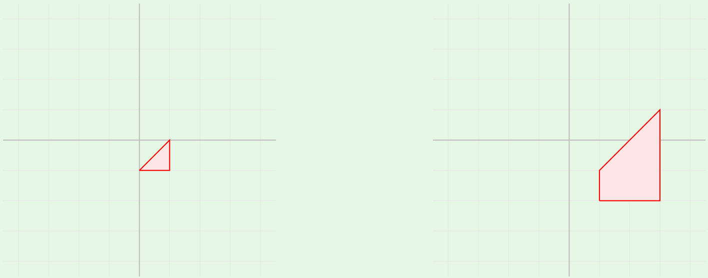
Gosh, these are also tropicalizations of Laurent polynomials!

Example

Both $\mathrm{val}_{(0,-1)}(\vartheta_{\mathrm{B}}[m])$ and $\mathrm{val}_{(1,-2)}(\vartheta_{\mathrm{B}}[m])$ can be realized as

$$\mathrm{val}_m(f)$$

for any Laurent polynomial f with respective Newton polytopes:



Can these fs be realized as theta functions in some other seed?

Theta Reciprocity [Cheung-Mandel-Magee-M, to appear]

Let $\mathfrak{s}$ be a seed on M. For all $m \in M$ and $n \in N$,

$$\mathrm{val}_n(\vartheta_{\mathfrak{s}}[m]) = \mathrm{val}_m(\vartheta_{\mathfrak{s}^\vee}[n])$$

where $\mathfrak{s}^\vee$ is the mirror dual seed to $\mathfrak{s}$.

Some remarks

- Conjectured in [GHKK, Remark 9.11], who proved it when one of $\vartheta_{\mathfrak{s}}[m]$ and $\vartheta_{\mathfrak{s}^\vee}[n]$ is a cluster variable.
- This theorem extends to infinite theta functions by defining $\mathrm{val}_n(\vartheta[m])$ to be the infimum of n over the support.
- This implies reciprocity also holds for any basis with the same Newton polytopes as the theta basis.
- The skew-symmetrizable case involves considerably more machinery, and may wait until a second paper.

I'll tell you what the mirror dual is, in the case of cluster algebras.

Mirror dual theta functions of $\mathcal{A}$-type seeds

If $\mathfrak{s}$ is the $\mathcal{A}$-type seed of an exchange matrix B, then

$$\vartheta_{\mathfrak{s}^\vee}[n] = y^n F_{\mathrm{B}^\top}[\mathrm{B}^\top n]$$

where $F_{\mathrm{B}^\top}[\mathrm{B}^\top n]$ is the F-polynomial of $\vartheta_{\mathrm{B}^\top}[\mathrm{B}^\top n]$.

Theta reciprocity and F-polynomials

Let B be an exchange matrix. For any $m, n \in \mathbb{Z}^{\mathrm{height(B)}}$,

$$\begin{aligned}\mathrm{val}_n(\vartheta_{\mathrm{B}}[m]) &= \mathrm{val}_m(y^n F_{\mathrm{B}^\top}[\mathrm{B}^\top n]) \\ &= m \cdot n + \mathrm{val}_m(F_{\mathrm{B}^\top}[\mathrm{B}^\top n])\end{aligned}$$

Example

Recall the formula from earlier:

$$\mathrm{val}_{(0,1)}(\vartheta_{\mathrm{B}}[m]) = \min(-m_2, m_1 - m_2, m_1)$$

If $n = (0, -1)$, then $\mathrm{B}^\top n = (-1, 0)$.

$$\begin{aligned}
\vartheta_{\mathrm{B}^\top}[-1,0] &= \frac{x_1 + 1 + x_2}{x_1 x_2} = x^{(-1,0)} + x^{(-1,-1)} + x^{(0,-1)} \\
&= x^{(-1,0)}(1 + x^{\mathrm{B}^\top(1,0)} + x^{\mathrm{B}^\top(1,1)}) \\
F_{\mathrm{B}^\top}[-1,0] &= 1 + y^{(1,0)} + y^{(1,1)} \\
\vartheta_{\mathfrak{s}^\vee}[0,-1] &= y^{(0,-1)}(1 + y^{(1,0)} + y^{(1,1)}) \\
&= y^{(0,-1)} + y^{(1,-1)} + y^{(1,0)} \\
\mathrm{val}_m(\vartheta_{\mathfrak{s}^\vee}[0,-1]) &= \min(-m_2, m_1 - m_2, m_1)
\end{aligned}$$

Theta Reciprocity: an intrinsic description

Let Θ and $\Theta^\vee$ denote the theta bases of $\mathfrak{s}$ and $\mathfrak{s}^\vee$, respectively. Given $(\vartheta, \vartheta^\vee) \in \Theta \times \Theta^\vee$, we can define two numbers:

- Apply the valuation parameterizing ϑ to $\vartheta^\vee$.
- Apply the valuation parameterizing $\vartheta^\vee$ to ϑ.

Theta Reciprocity implies these two numbers are the same, and so we have a well-defined **theta pairing**:

$$\Theta \times \Theta^\vee \to \mathbb{Z}$$

Such a pairing was conjectured in 2003 by Fock and Goncharov.

Example

For a marked surface, Θ can be identified with simple multicurves, $\Theta^\vee$ can be identified with certain laminations, and the theta pairing is (a multiple of) the number of intersections.

What we can say with a Λ-matrix

Definition: Compatible pairs

A **compatible pair** (Λ, B) consists of

- an extended exchange matrix B, and
- a skew-symmetric matrix Λ,

such that $\Lambda \mathrm{B} = \begin{bmatrix} \mathrm{D} & 0 \end{bmatrix}^\top$ for some diagonal matrix D.
The pair is **positive** if the diagonal entries of D are positive.

Example

For any integers $b, c > 0$, we have a positive compatible pair:

$$\mathrm{B} = \begin{bmatrix} 0 & -c \\ b & 0 \end{bmatrix} \qquad \Lambda = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

We can use a Λ matrix to reformulate Theta Reciprocity.

Lambda-Theta Reciprocity

Let (Λ, B) be a positive compatible pair. Then for all $m, m' \in M$,

$$\mathrm{val}_{-\Lambda m'}(\vartheta_{\mathrm{B}}[m]) = \mathrm{val}_{\Lambda m}(\vartheta_{-\mathrm{B}}[m'])$$

Note $\vartheta_{\mathrm{B}}[m]$ and $\vartheta_{-\mathrm{B}}[m']$ lie in the same cluster algebra*.

Example

If $m' = (-1, 0)$, then $-\Lambda m' = (0, -1)$ and so

$$\begin{aligned} \mathrm{val}_{(0,-1)}(\vartheta_{\mathrm{B}}[m]) &= \mathrm{val}_{\Lambda m}(\vartheta_{-\mathrm{B}}[-1, 0]) \\ &= \mathrm{val}_{(m_2,-m_1)}(x^{(-1,0)} + x^{(-1,-1)} + x^{(0,-1)}) \\ &= \min(-m_2, m_1 - m_2, m_1) \end{aligned}$$

We can also use Λ to reinterpret valuations of theta functions.

Definition: Λ-momentum

If (Λ, B) is a compatible pair and Γ is a broken line in $\mathfrak{D}(\mathrm{B})$, then

$$\Gamma(t) \cdot \Lambda\Gamma'(t)$$

is independent of t. We call this quantity the **Λ-momentum** of Γ.

Example/Etymology

For the compatible pair

$$\mathrm{B} = \begin{bmatrix} 0 & -c \\ b & 0 \end{bmatrix} \qquad \Lambda = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

the Λ-momentum of a broken line is its **angular momentum** counter-clockwise around the origin.

Theorem: Tropical theta functions and Λ-momentum

If (Λ, B) is a positive compatible pair and $m, m' \in M$, then

$$\mathrm{val}_{\Lambda m'}(\vartheta_{\mathrm{B}}[m]) = \text{minimum } \Lambda\text{-momentum of a broken line with initial derivative } -m \text{ and endpoint } m'$$

Interactive Example

Let's play around with these broken lines!

In [CMMM], we first prove Theta Reciprocity in terms of Λ-momenta, and then derive the original statement.

Application: Lifting theta functions

Definition: Polynomial in a cluster variable x

A theta function is **polynomial in x** if its Laurent expansion in some cluster containing x has no negative powers of x.

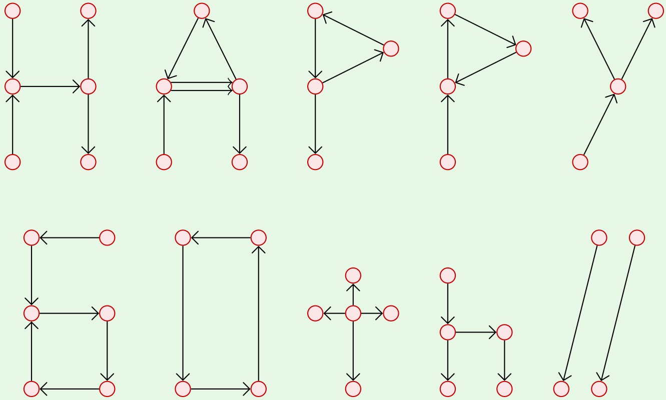
[Cao-Li, 2018]: If this holds in one cluster, it holds in all of them.

Theorem: Polynomial lifting

If ϑ is a theta function which is polynomial in a set of frozen variables, ϑ remains a theta function when they are unfrozen.

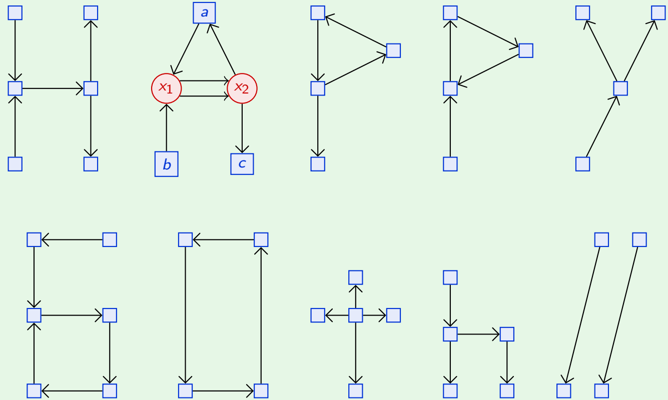
An example chosen at random

Imagine that you come across a seed with the following quiver.



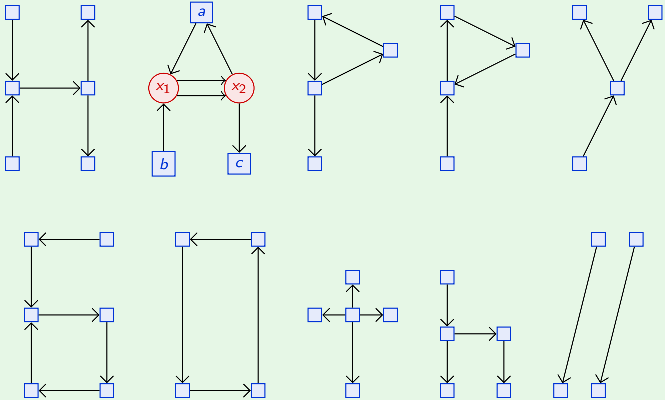
An example chosen at random

Freeze every vertex except the two ends of the double arrow:



An example chosen at random

Then $\dfrac{bx_1^2 + abc + cx_2^2}{x_1 x_2}$ is a theta function in the original seed.



Construction: The loop element of a double arrow

Let B be an extended exchange matrix, and let i, j be mutable indices such that $\mathrm{B}_{i,j} = 2$. Define $c \in \mathbb{Z}^m$ by

$$c_k := \max(\mathrm{B}_{k,i}, -\mathrm{B}_{k,j}, 0)$$

Then the following **loop element**

$$\ell := \frac{x^{c+\mathrm{B}e_j} + x^c + x^{c-\mathrm{B}e_i}}{x_i x_j}$$

is a theta function of B, as are all Chebyshev polynomials in ℓ.

This construction yields all closed simple loops (and their bracelets) in the cluster algebra of a marked surface of genus 0.

Open questions

- Which good bases for cluster algebras have the same Newton polytopes as the theta basis?
- For which cluster algebras can every theta function be lifted from a rank 2 freezing?
- What does the theta pairing look like when the theta bases are parameterized by some interesting class of objects?